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# ISOMORPHISM CLASSES OF SOLENOIDAL ALGEBRAS I

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ABSTRACT. Each $g \in \mathbb{Z}[x]$ defines a homeomorphism of a compact space $\hat{\Lambda}_g$. We investigate the isomorphism classes of the $C^*$-crossed product algebra $B_g$ associated with the dynamical system $(\hat{\Lambda}_g, \mathbb{Z})$. An isomorphism invariant $E_g$ of the algebra $B_g$ is shown to determine the algebra $B_g$ up to $*$ or $*$ anti-isomorphism if degree $g \leq 1$ and 1 is not a root of $g$ or if degree $g = 2$ and $g$ is irreducible. It is also observed that the entropy of the dynamical system $(\hat{\Lambda}_g, \mathbb{Z})$ is equal to the growth rate of the periodic points if $g$ has no roots of unity as zeros. This slightly extends the previously known equality of these two quantities under the assumption that $g$ has no zeros on the unit circle.

**1.** To each $g \in \mathbb{Z}[x]$ associate a $C^*$-algebra $B_g$, the crossed product algebra $C(\hat{\Lambda}_g) \rtimes_\alpha \mathbb{Z}$. Here $\hat{\Lambda}_g$ denotes the dual group of the discrete abelian group $\Lambda_g = \mathbb{Z}[x, x^{-1}]/(g)$ where $(g)$ is the principal ideal generated by $g$ in the ring $\mathbb{Z}[x, x^{-1}]$. The action $\alpha$ of $\mathbb{Z}$ on $C(\hat{\Lambda}_g)$ is defined by the action of $\mathbb{Z}$ on $\Lambda_g$ given by multiplication by $x$. Note that if $g, h \in \mathbb{Z}[x]$ with $g(x) = x^n h(x)$ for some $n \in \mathbb{N}$ then $B_g = B_h$; so assume henceforth that 0 is not a root of $g$. It is also evident that $B_g$ is $*$ isomorphic to $B_{-g}$. If $g$ is irreducible, degree $g > 0$ and $g$ has a positive real root $a \in \mathbb{R}$ then $B_g$ is just the dilation algebra $B_a$ introduced in [1].

Define $E_g(n)$ to be the cardinality of the set of points in $\hat{\Lambda}_g$ fixed by $\alpha^n$, $(n \in \mathbb{N})$. The sequence $\{E_g(n) \mid n \in \mathbb{N}\}$ is a sequence of isomorphism invariants of the $C^*$-algebra $B_g$ [3], [1]. The proof of Proposition 3 in [1] yields the following result (*cf.* [5]).

PROPOSITION 1.1. *Let $g \in \mathbb{Z}[x]$ and $n \in \mathbb{N}$. Let $\{r_1, \ldots, r_d\}$ be the (complex) roots of $g$, $a_d$ the leading coefficient of $g$ and $f_m$ the $m$-th cyclotomic polynomial.*

*(a) If $f_m$ does not divide $g$ for all $m$ with $m|n$ then*

$$E_g(n) = \Big|\prod_{j=1}^{n} g\big(\exp(2\pi i j n^{-1})\big)\Big| = |a_d|^n \prod_{k=1}^{d} |1 - r_k^n|.$$

*(b) If $f_m|g$ for some $m|n$ then $E_g(n)$ is infinite.*

If $g$ has no roots of unity as a zero then by Proposition 1.1 a) the growth rate of the periodic points $G(g) = \lim_{n\to\infty} n^{-1} \log\big(E_g(n)\big)$ exists and equals $\log|a_d| + \sum_{|r_k|>1} \log|r_k|$. This latter number is known to be the topological entropy of the automorphism $\alpha$ of the compact group $\hat{\Lambda}_g$, [2, Section 12]. If $\hat{\Lambda}_g$ is connected, *i.e.*, a solenoid, this fact

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(namely, the entropy of $\alpha$ is $\log|a_d|+\sum_{|r_k|>1}\log|r_k|$) is due to Yuzvinski, [6]. It is straightforward to see that $\hat{\Lambda}_g$ is connected (equivalently that $\Lambda_g$ is torsion free) if and only if the content of $g$, cont($g$), is one. In fact, the quotient of $\Lambda_g$ by its subgroup of torsion elements is $\Lambda_{g_0}$ where $g = \text{cont}(g)g_0$.

For the dynamical systems considered here we have shown that the entropy of $\alpha$ equals the growth rate of the periodic points if $g$ has no roots of unity as zeros. This extends what was known previously, namely, that this was true if $\alpha$ was expansive or, equivalently, if $g$ had no zeros on the unit circle, [4], [5].

This may be extended further. Each $g \in \mathbb{Z}[x]$ has only finitely many zeros, so $f_m|g$ for finitely many $m$, say $m_1,\ldots,m_r$. Since $\mathbb{Z}\setminus(\bigcup_{i=1}^r m_i\mathbb{Z})$ is an infinite set if $m_i \neq 1$, one has $\liminf_{n\to\infty} n^{-1}\log\big(E_g(n)\big) = \log|a_d| + \sum_{|r_k|>1}\log|r_k|$ which is the topological entropy of $\alpha$. Thus, if 1 is not a zero of $g$ (equivalently, if $\alpha$ has at most a finite number of fixed points), the growth rate of the finite numbers of periodic points of $\alpha$ is equal to the entropy of $\alpha$.

For $g = \sum_{j=0}^d a_jx^j \in \mathbb{Z}[x]$ (with 0 not a root of $g$) define $g^0 \in \mathbb{Z}[x]$, the opposite of $g$, by $g^0 = \sum_{j=0}^d a_{d-j}x^j$. Thus, $g^0(x) = x^dg(x^{-1})$ and if $\{r_1,\ldots,r_d\}$ are the roots of $g$ then $g^0$ has roots $\{r_1^{-1},\ldots,r_d^{-1}\}$. Since $|g(0)| = |a_d\prod_{k=1}^d r_k|$ it follows that $E_g = E_{g^0}$, so the invariant $E_g$ cannot distinguish between the $C^*$-algebras $B_g$ and $B_{g^0}$. These algebras are, however, $*$ anti-isomorphic.

PROPOSITION 1.2. *For $g \in \mathbb{Z}[x]$, the $C^*$-algebra $(B_g)^{\text{op}}$ is $*$ isomorphic to $B_{g^0}$.*

PROOF. The map $x \longrightarrow x^{-1}$ defines a ring automorphism of $\mathbb{Z}[x,x^{-1}]$ mapping the ideal $(g)$ to the ideal $(g^0)$. This yields a group isomorphism of $\Lambda_g$ with $\Lambda_{g^0}$ which intertwines the $\mathbb{Z}$ action defined by $\alpha$ on $\Lambda_g$ with the $\mathbb{Z}$ action defined by $\alpha^{-1}$ on $\Lambda_{g^0}$. The $C^*$-algebras $B_g$ and $C(\hat{\Lambda}_{g^0}) \rtimes_{\alpha^{-1}} \mathbb{Z}$ are thus $*$ isomorphic. The latter is, however, $*$ anti-isomorphic to the $C^*$-algebra $C(\hat{\Lambda}_{g^0}) \rtimes_\alpha \mathbb{Z} = B_{g^0}$ via a map $\psi$ with $\psi|_{C(\hat{\Lambda}_{g^0})}$ the identity map and $\psi(U) = U^{-1}$ where $U$ is the unitary in $B_{g^0}$ implementing the automorphism $\alpha$ on $C(\hat{\Lambda}_{g^0})$. ∎

**2.** If $g \in \mathbb{Z}[x]$ has degree 0, *i.e.*, $g \in \mathbb{Z}$ then $E_g(m) = |g|^m$, ($m \in \mathbb{N}$). Thus, $B_g$ is $*$ isomorphic to $B_h$ for $h \in \mathbb{Z}$ if and only if $|g| = |h|$. Note that $G(g) = \log|g|$ in this case.

Consider $g \in \mathbb{Z}[x]$ of degree 1, so $g(x) = ax+b$. The only possible roots of modulus 1 for $g$ are either 1 (if $a = -b$) or $-1$ (if $a = b$). In the first case $E_g$ is always infinite valued; so, for different values of $m = (a,b)$, the algebras $B_g$ cannot be distinguished by means of the invariant $E$. In the second case, $E_g(2n)$ is infinite for $n \in \mathbb{N}$ and $E_g(1) = 2|a|$. Thus, if $g,h \in \mathbb{Z}[x]$ with degree $g$ = degree $h = 1$ and $-1$ is a root of $g$ then $B_g$ is $*$ isomorphic to $B_h$ if and only if $g = h$ or $g = -h$.

If $g \in \mathbb{Z}[x]$ has degree one with no zeros of modulus 1 then $G(g) = \max\{\log|a|, \log|b|\}$ and $E_g(1) = |a+b|$. Define $h \in \mathbb{Z}[x]$ by $h(x) = Ax + B$ with $A = \exp\big(G(g)\big) = \max\{|a|,|b|\}$ and $B$ the unique element in $\mathbb{Z}$ with $-A < B < A$ and $|A+B| = E_g(1)$. Then $g = \pm h$ or $g = \pm h^0$.

We have shown that if $g, h \in \mathbb{Z}[x]$ with degree $g$ and degree $h \leq 1$ and 1 is not a zero of $g$ then $B_g$ is $*$ isomorphic or $*$ anti-isomorphic to $B_h$ if and only if $g = \pm h$ or $g = \pm h^0$.

The following result uses the simple observation that knowledge of $E_g(n)$ for $n = 1, 2, 4$ is equivalent to knowledge of the values $|g(1)|$, $|g(-1)|$ and $|g(i)|$. Since $g(1)$ and $g(-1) \in \mathbb{R}$, knowledge of $E_g(1)$ and $E_g(2)$ leaves only two possible values for each of $g(1)$ and $g(-1)$.

THEOREM 2.1. *If $g, l \in \mathbb{Z}[x]$ are degree two and irreducible then $B_g$ is $*$ isomorphic or $*$ anti-isomorphic to $B_l$ if and only if $l = \pm g$ or $l = \pm g^0$.*

PROOF. Proposition 1(b) shows that the invariant $E$ distinguishes the algebras $B_g$, $g$ a cyclotomic polynomial. Assume therefore that $g$ has no roots of unity as zeros. If $g(x) = a_2x^2 + a_1x + a_0 = a_2(x - r_1)(x - r_2)$ then

$$\exp\big(G(g)\big) = \begin{cases} |a_2 r_1 r_2| = |a_0| & \text{if } |r_1|, |r_2| > 1 \\ |a_2| & \text{if } |r_1|, |r_2| \leq 1 \\ |a_2 r_1| & \text{if } |r_1| > 1 \text{ and } |r_2| \leq 1 \\ |a_2 r_2| & \text{if } |r_1| \leq 1 \text{ and } |r_2| > 1. \end{cases}$$

Since $g$ has real coefficients, the roots, if non-real, are a complex conjugate pair and so are equal in modulus. Thus, $\exp\big(G(g)\big) = |a_2 r_1|$ (or $|a_2 r_2|$) only if the roots are real and (since $g$ is irreducible) non rational. Thus, either $\exp\big(G(g)\big) \in \mathbb{N}$ or $\exp\big(G(g)\big) \notin \mathbb{Q}$, the latter occuring if and only if the roots of $g$ are real and 1 lies between their moduli. Consider these two cases separately.

First assume $\exp\big(G(g)\big) \in \mathbb{N}$. Since $|a_2, r_1 r_2| = |a_0|$, $\exp G(g) = \max\{|a_0|, |a_2|\}$. Let $A_0 = \exp\big(G(g)\big)$, $|g(1)| = \alpha$ and $|g(-1)| = \beta$. Since a degree two polynomial is uniquely specified by three points on its graph, there are eight possible degree two polynomials $l$ with constant term $\pm A_0$, $|l(1)| = \alpha$ and $|l(-1)| = \beta$. We have $l(x) = a_l x^2 + b_l x \pm A_0$ where $a_l = \big(l(1) + l(-1)\big)2^{-1} \mp A_0$ and $b_l = \big(l(1) - l(-1)\big)2^{-1}$. Note that the possible polynomials with constant term $A_0$ are, after multiplying by $-1$, just those possible polynomials with constant term $-A_0$; so it is enough to consider the four with constant term $A_0$.

Thus, $|l(i)|^2 = (A_0 - a_l)^2 + b_l^2 = 4A_0^2 + (\alpha^2 + \beta^2)2^{-1} - 2A_0\big(l(1) + l(-1)\big)$ and since $|l(i)|, A_0$, $\alpha$ and $\beta$ are all determined by $E_g$, so is $\big(l(1) + l(-1)\big)$. This specifies exactly one polynomial $l$ unless $\alpha = \beta$, in which case there are two possibilities, namely, $l(1) = \alpha$ and $l(-1) = -\alpha$ or $l(1) = -\alpha$ and $l(-1) = \alpha$. Thus $l(x) = -A_0x^2 - \alpha x + A_0$ or $l(x) = -A_0x^2 + \alpha x + A_0$. However, one of these is minus the opposite of the other, so the only possible $l$ with $E_g = E_l$ are $\pm g$ or $\pm g^0$.

Now consider the case $\exp\big(G(g)\big) \notin \mathbb{Q}$ which occurs if and only if the roots of $g$ are real and 1 lies between their moduli. Since the roots of $g$ have different moduli, $a_1 \neq 0$. Again, if $\alpha$, $\beta$ denote the (non-zero) natural numbers $|g(1)|$ and $|g(-1)|$ respectively, we have $l|(1)| = \alpha$ and $|l(-1)| = \beta$. Since any polynomial $l$ of degree two may be written as $l(x) = a_l x^2 + [l(1) - l(-1)]2^{-1}x + \Big[\big(l(1) + l(-1)\big)2^{-1} - a_l\Big]$, one has $|l(i)|^2 = (\alpha^2 + \beta^2)2^{-1} + 2\Big[2a_l^2 - a_l\big(l(1) + l(-1)\big)\Big]$. Note that $l(1) - l(-1) \neq 0$ since the roots of $l$ have different moduli.

Denoting $\exp G(g)$ by $e$ we have $|a_l\lambda| = \exp G(l) = e$ where $\lambda$ is the real root of $l$ of modulus larger than 1. Substitute the two possible values (in terms of the coefficients of $l$) of the roots of $l$ in $e = \pm a_l\lambda$ to obtain

$$4e \pm \big(l(1) - l(-1)\big) = \Big[\big(l(1) - l(-1)\big)^2 - 8a_l\big(l(1) + l(-1) - 2a_l\big)\Big]^{1/2}.$$

Square both sides to obtain

$$2e^2 \pm e\big(l(1) - l(-1)\big) = 2a_l^2 - a_l\big(l(1) + l(-1)\big) \tag{1}$$

and substitute this into the above expression for $|l(i)|^2$ to conclude the non-zero value

$$\pm\big(l(1) - l(1)\big) = \big[|l(i)|^2 - (\alpha^2 + \beta^2)2^{-1} - 4e^2\big]2^{-1}e^{-1}. \tag{2}$$

Since $|l(1)| = \alpha$ and $|l(-1)| = \beta$, there are only four different possible values for $\pm\big(l(1) - l(-1)\big)$, namely, $\alpha + \beta$, $\alpha - \beta$, $-\alpha + \beta$ and $-\alpha - \beta$. (If $\alpha = \beta$, one has only two different possibilities, $\alpha + \beta$ and $-\alpha - \beta$, since $l(1) - l(-1)$ must be non zero.) Since the right hand side of equation (2) is completely determined in terms of information contained in $E_g$, one of these four possibilities is determined by $E_g$. We choose one of these four values and show that there are only four possibilities for $l$, namely, $\pm g$ and $\pm g^0$ (the other three cases resolve themselves in an analogous manner).

Suppose the value determined by $E_g$ is $-\alpha + \beta$, so $l(1) - l(-1) = -\alpha + \beta$ or $-\big(l(1) - l(-1)\big) = -\alpha + \beta$. Note that the quadratic (1) allows two possibilities for $a_l$ (and thus for $l$) once the values of $l(1)$ and $l(-1)$ are fixed. If $l(1) - l(-1) = -\alpha + \beta$, the two possibilities for $l$ are

$$(-\alpha - \beta + \gamma)4^{-1}x^2 + (-\alpha + \beta)2^{-1}x + (-\alpha - \beta - \gamma)4^{-1}$$

and

$$(-\alpha - \beta - \gamma)4^{-1}x^2 + (-\alpha + \beta)2^{-1}x + (-\alpha - \beta + \gamma)4^{-1}$$

where $\gamma = \sqrt{\big[(\alpha + \beta)^2 + 8\big(2e^2 + (-\alpha + \beta)e\big)\big]}$. If $-\big(l(1) - l(-1)\big) = -\alpha + \beta$ the possibilities for $l$ are

$$(\alpha + \beta + \gamma)4^{-1}x^2 + (\alpha - \beta)2^{-1}x + (\alpha + \beta - \gamma)4^{-1}$$

and

$$(\alpha + \beta - \gamma)4^{-1}x^2 + (\alpha - \beta)2^{-1}x + (\alpha + \beta + \gamma)4^{-1}.$$

Since one of these four is $g$ we conclude that $l$ is $\pm g$ or $\pm g^0$. ∎

**3.** The above result for degree two irreducible polynomials in $\mathbb{Z}[x]$ made use only of $E_g(n)$ with $n = 1, 2, 4$ and $G(g)$. Perhaps for higher degree irreducible polynomials more of the information in $E_g$ could be used to show a similar result. The techniques employed here are inadequate for this, however, and the results obtained here should mainly be viewed as evidence for a more general result.

Since $E_{hk} = E_hE_k$, it follows that $E$ cannot distinguish between $B_g$ and $B_l$ where $g = hk$ and $l = h^0k$, $h, k \in \mathbb{Z}[x]$. For $h, k$ coprime in $\mathbb{Z}[x]$ I have shown that these algebras are $*$ isomorphic (and thus also $*$ anti-isomorphic), lending some weight to the possibility that $E$ is sufficient to distinguish the algebras $B_g$ (for $g$ not divisible by cyclotomic polynomials).

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