ON PACKINGS OF SPHERES IN HILBERT SPACE

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A point $\mathbf{x}$ in real Hilbert space is represented by an infinite sequence $(x_1, x_2, x_3, \ldots)$ of real numbers such that

$$\| \mathbf{x} \|^2 = \sum_{r=1}^{\infty} x_r^2$$

is convergent. The unit " sphere " S consists of all points $\mathbf{x}$ for which $\| \mathbf{x} \| \leqslant 1$. The sphere of radius a and centre $\mathbf{y}$ is denoted by $S_a(\mathbf{y})$ and consists of all points $\mathbf{x}$ for which $\| \mathbf{x} - \mathbf{y} \| \leqslant a$.

From the results obtained in a recent paper (**1**) the following theorem can easily be deduced.

THEOREM. *An infinity of spheres $S_a(\mathbf{y})$ can be placed in S without overlapping if and only if $a \leqslant \sqrt{2} - 1$. If $\sqrt{2} - 1 < a \leqslant 1$ the maximum number of spheres which may be placed in S without overlapping is*

$$\left[\frac{2a^2}{a^2 + 2a - 1} \right].$$

(*The square brackets denote the integral part*).

That infinitely many spheres $S_a(\mathbf{y})$ can be placed in S without overlapping when $a \leqslant \sqrt{2} - 1$ follows at once by considering the system of spheres $S_a(\mathbf{y}_n)$ $(n = 1, 2, 3, \ldots)$, where the co-ordinates y_{nk} of $\mathbf{y}_n$ satisfy

$$y_{nk} = 0 \quad (n \neq k), \quad y_{nn} = 2 - \sqrt{2}.$$

For then, if $m \neq n$,

$$\| \mathbf{y}_m - \mathbf{y}_n \| = \sqrt{2}(2 - \sqrt{2}) \geqslant 2a,$$

while, if $\mathbf{x} \in S_a(\mathbf{y}_n)$,

$$\| \mathbf{x} \| = \| (\mathbf{x} - \mathbf{y}_n) + \mathbf{y}_n \| \leqslant a + 2 - \sqrt{2} \leqslant 1.$$

If $\frac{1}{2} < a \leqslant 1$ every sphere $S_a(\mathbf{y})$ in S contains $\mathbf{0} = (0, 0, 0, \ldots)$ as an interior point. For, if $\mathbf{y} \neq \mathbf{0}$, $\mathbf{z} = \mathbf{y} + a\mathbf{y}/\| \mathbf{y} \| \in S_a(\mathbf{y}) \subseteq S$, and so

$$\| \mathbf{y} \| = \| \mathbf{y} \| + a - a = \| \mathbf{z} \| - a \leqslant 1 - a < a.$$

This remains true for $\mathbf{y} = \mathbf{0}$. Thus only one sphere can be placed in S and the theorem is therefore true in this case.

Suppose that $\sqrt{2} - 1 < a \leqslant \frac{1}{2}$ and write

$$\alpha = \sin^{-1} \frac{a}{1 - a},$$

so that $\frac{1}{4}\pi < \alpha \leqslant \frac{1}{2}\pi$. Also let

$$N = \left[\frac{2a^2}{a^2 + 2a - 1} \right] = \left[\frac{2 \sin^2 \alpha}{2 \sin^2 \alpha - 1} \right],$$

so that

$$\alpha_N = \tfrac{1}{4}\pi + \tfrac{1}{2} \sin^{-1} \frac{1}{N} < \alpha \leqslant \tfrac{1}{4}\pi + \tfrac{1}{2} \sin^{-1} \frac{1}{N - 1} = \alpha_{N-1}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now, by Theorem 1 (ii) of (**1**), N spherical caps of angular radius α can be placed on the surface of an $(N-1)$-dimensional unit sphere S_{N-1}. Their centres may be represented by the N points of Hilbert space

$$\mathbf{x}_k = (x_{k1}, x_{k2}, x_{k3}, \ldots) \qquad (k = 1, 2, \ldots, N),$$

where $x_{kn}=0$ for $n \geqslant N$. Then, if $\mathbf{y}_k=(1-a)\mathbf{x}_k$, the N spheres $S_a(\mathbf{y}_k)$ lie in S and do not overlap, since, for $j \neq k$,

$$\begin{aligned} \| \mathbf{x}-\mathbf{y}_j \| + \| \mathbf{x}-\mathbf{y}_k \| \geqslant \| \mathbf{y}_j-\mathbf{y}_k \| &= (1-a) \| \mathbf{x}_j - \mathbf{x}_k \| \\ &= (1-a)\left\{ \sum_{i=1}^{N-1} (x_{ji}-x_{ki})^2 \right\}^{\frac{1}{2}} \\ &\geqslant (1-a)\{2-2\cos 2\alpha\}^{\frac{1}{2}} \\ &= 2(1-a)\sin\alpha = 2a. \end{aligned}$$

On the other hand, if there existed $N+1$ non-overlapping spheres $S_a(\mathbf{y}_k)$ $(k=1, 2, \ldots, N+1)$ in S we could, by a suitable orthogonal transformation, choose the coordinates y_{kn} $(n=1, 2, \ldots)$ of the centres $\mathbf{y}_k$ such that

$$y_{kn}=0 \quad \text{for } n > N+1.$$

We should then have $N+1$ non-overlapping $(N+1)$-dimensional spheres of radius a in S_{N+1}, the unit sphere in $N+1$ dimensions, with centres at the points

$$(y_{k1}, y_{k2}, \ldots, y_{k,N+1}) \qquad (k=1, 2, \ldots, N+1).$$

This point we continue to call $\mathbf{y}_k$ and write $r_k = \| \mathbf{y}_k \|$. For $1 \leqslant i < j \leqslant N+1$ we write θ_{ij} for the angle subtended by $\mathbf{y}_i$ and $\mathbf{y}_j$ at the centre of S_{N+1}, and we have

$$4a^2 \leqslant \sum_{m=1}^{N+1} (y_{im}-y_{jm})^2 = r_i^2 + r_j^2 - 2r_ir_j\cos\theta_{ij}.$$

Hence

$$2r_ir_j\cos\theta_{ij} \leqslant r_i^2+r_j^2-4a^2 \leqslant 2(1-a)^2-4a^2 = -2(a^2+2a-1) < 0.$$

Thus

$$-\cos\theta_{ij} \geqslant \frac{a^2+2a-1}{r_ir_j} \geqslant \frac{a^2+2a-1}{(a-1)^2} = -\cos 2\alpha > 0.$$

It follows that $\theta_{ij} \geqslant 2\alpha$ and we can therefore place $N+1$ non-overlapping spherical caps of angular radius α on the surface of S_{N+1}. By (1) and Theorem 1 (ii) of (1) we deduce that

$$N+1 \leqslant \left[\frac{2\sin^2\alpha}{2\sin^2\alpha - 1} \right] = N,$$

which is a contradiction.

This completes the proof of the theorem. It is clear that the theorem can also be stated in terms of packings of spherical caps on the surface of the Hilbert unit sphere.

By similar methods it can be shown that the presence of a single sphere $S_a(\mathbf{y})$ of radius $a > \surd 2 - 1$ inside S is enough to prevent an infinity of spheres $S_b(\mathbf{y})$ of radii $b \geqslant \surd 2 - 1$ from being placed in S without overlapping.

REFERENCE

(1) R. A. Rankin, The closest packing of spherical caps in n dimensions, *Proc. Glasgow Math. Assoc.* **2**(1955), 139-144.

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