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ON A PROPERTY OF NILPOTENT GROUPS

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ABSTRACT. Let g be an element of a group G and $[g,G] = \langle g^{-1}a^{-1}ga \mid a \in G\rangle$. We prove that if G is locally nilpotent then for each $g,t \in G$ either $g[g,G] = t[t,G]$ or $g[g,G] \cap t[t,G] = \emptyset$. The converse is true if G is finite.

1. **Introduction.** For a group G and a fixed $g \in G$ we denote by $[g,G]$ the subgroup of G generated by all commutators $[g,a] = g^{-1}a^{-1}ga$ $(a \in G)$. Obviously, $[g,G]$ is normal in G, since $[g,a]^b = [g,b]^{-1}[g,ab]$ for each $a,b \in G$ (here $a^b = b^{-1}ab$).

It is an elementary but useful property of a nilpotent group G of class 2 that G is divided into disjoint sets of the form $g[g,G]$. We intend to generalize this fact for arbitrary locally nilpotent groups and at the same time to obtain a criterion of nilpotency in the case of finite groups.

DEFINITION. We say that a group G satisfies condition (X) if for all $g,t \in G$ either $g[g,G] = t[t,G]$ or $g[g,G] \cap t[t,G] = \emptyset$.

The purpose of the present paper is to prove the following theorems.

THEOREM A. *A locally nilpotent group satisfies condition (X).*

THEOREM B. *A finite group G is nilpotent if and only if G satisfies condition (X).*

Thus, condition (X) can be considered as a generalization of nilpotency for groups.

In the case when G is nilpotent metabelian, Theorem A has been applied in [1] for an investigation of torsion units in integral group rings. Note, that in that case for each $g \in G$ and $t \in g[g,G]$ the order of t equals to the order of g (see [1], Lemma 2.6).

We use the following notation: $G^{(n)}$ is the n-th derived subgroup of a group G, G_n is the n-th term of the lowest central series of G, $C_G(H)$ the centralizer of a subset H in G, $\langle H\rangle$ the subgroup of G, generated by a subset $H \subseteq G$.

2. **Proof of Theorem A.**

LEMMA 2.1. *Let G be an arbitrary group. If $b \in a[a,G]$ $(a,b \in G)$ then $b[b,G] \subseteq a[a,G]$.*

PROOF. Let c be an arbitrary element from $b[b,G]$. There exist elements $h_1 \in [a,G]$ and $h_2 = \prod_{i=1}^m [b,g_i]^{\varepsilon_i} \in [b,G]$ $(\varepsilon_i = \pm 1)$ such that $b = h_1 a$ and $c = h_2 b$. We have

$$h_2 = \prod_{i=1}^m [h_1 a, g_i]^{\varepsilon_i} = \prod_{i=1}^m ([h_1,g_i]^a [a,g_i])^{\varepsilon_i} \in [a,G]$$

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since $[a, G]$ is normal in G. Consequently, $c = h_2 h_1 a \in a[a, G]$ as desired.

Now we prove the Theorem A.

Let G be a locally nilpotent group. Suppose that $c \in a[a, G] \cap b[b, G]$ for some $a, b, c \in G$. We have to show that $a[a, G] = b[b, G]$. By Lemma 2.1

$$c[c, G] \subseteq a[a, G] \cap b[b, G]. \tag{2.2}$$

Clearly, $c = h^{-1}a$ for some $h = \prod_{i=1}^{m} [a, g_i]^{\varepsilon_i} \in [a, G]$ ($g \in G, \varepsilon_i = \pm 1$). Let $G_1 = \langle a, g_1, \ldots, g_m \rangle$. Since $a = hc$, $h \equiv \prod_{i=1}^{m} [h, g_i]^{\varepsilon_i}$ modulo $[c, G_1]$, that is, $h = \prod_{i=1}^{m} [h, g_i]^{\varepsilon_i}$ in $G_1 / [c, G_1]$. However, since the latter group is nilpotent it follows that $h = 1$ in $G_1 / [c, G_1]$ and $h \in [c, G_1]$. Therefore, $a = hc \in c[c, G]$ and in view of Lemma 2.1 $a[a, G] \subseteq c[c, G]$. It follows from (2.2) that $a[a, G] = c[c, G]$.

Similarly, $b[b, G] = c[c, G]$ and, consequently, $a[a, G] = b[b, G]$, proving Theorem A.

3. **Some elementary properties of groups which satisfy condition (X).**

LEMMA 3.1. *The following conditions are equivalent:*

(i) G satisfies (X),

(ii) for any $a \in G$

$$a[a, G] \subseteq b[b, G] \Rightarrow a[a, G] = b[b, G],$$

(iii) for each $a, b \in G$

$$h \in [a, G] \Rightarrow [ah, G] = [a, G].$$

PROOF. Clearly, (i) $\Rightarrow$ (ii). Applying Lemma 2.1 we get (ii) $\Rightarrow$ (iii).

Suppose that (iii) holds and $c \in a[a, G] \cap b[b, G]$. Then $c = ah$ for suitable $h \in [a, G]$ and $[c, G] = [a, G]$. Hence, $c[c, G] = ah[a, G] = a[a, G]$. Similarly, $c[c, G] = b[b, G]$, so that $a[a, G] = b[b, G]$, which completes the proof.

COROLLARY 3.2. *Let G be a group satisfying condition (X). Then any factor group of G also satisfies (X).*

PROOF. The corollary immediately follows Lemma 3.1 since for any normal subgroup $H \subseteq G$ and $a \in G$, $\nu \in [a, G]$

$$[a\nu, G] = [a, G] \Rightarrow [a\nu, G]H = [a, G]H.$$

Note that the symmetric group $S_3 = \langle a, b \mid a^3 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ does not satisfy condition (X) since $1 \in a[a, G]$. Therefore by Corollary 3.2 the free group of rank 2 does not satisfy condition (X). It shows, for example, that the class of groups satisfying condition (X) does not contain the class of residually nilpotent groups.

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COROLLARY 3.3. *The direct product of an arbitrary set of groups satisfying condition (X) is also a group which satisfies (X).*

PROOF. Let $G_\alpha(\alpha \in J)$ be a set of groups which satisfy condition (X) and G be the direct product of $G_\alpha(\alpha \in J)$. Suppose that $a[a,G] \subseteq b[b,G]$ for some $a, b \in G$. Denoting by a_α the projection of a on G_α we get $a_\alpha[a_\alpha, G_\alpha] \subseteq b_\alpha[b_\alpha, G_\alpha]$ for each $\alpha \in J$. By Lemma 3.1 $a_\alpha[a_\alpha, G_\alpha] = b_\alpha[b_\alpha, G_\alpha]$ $(\alpha \in J)$ and, consequently, $a[a,G] = b[b,G]$ proving the corollary.

LEMMA 3.4. *A finite group which satisfies condition (X) is soluble.*

PROOF. Note first that a finite group G satisfying condition (X) contains a soluble normal subgroup. Indeed, if not then there is a noncommutative simple subgroup P in G (see [3], § 61). Clearly, for an element $1 \neq g \in P$ we have $g \in P = [g,P]$ and therefore $1 \in g[g,G]$ which is impossible.

Denote by $\mathrm{Sol}(G)$ the largest soluble normal subgroup of G. If $\mathrm{Sol}(G) \neq G$ then $\mathrm{Sol}\big(G/\,\mathrm{Sol}(G)\big) = 1$ and by the above $G/\,\mathrm{Sol}(G)$ cannot satisfy condition (X). The contradiction with Corollary 3.2 proves the lemma.

4. **Proof of Theorem B.** We need the following lemma [4, p. 149].

LEMMA 4.1. *Let A be a normal Abelian subgroup of a group G. Suppose that A has exponent p^n and G acts by conjugation on A as a finite p-group of automorphisms. Then*

$$[A, G, G, \ldots, G] = [A, {}_lG] = 1$$

for a suitable l.

Now, we can prove Theorem B.

Let G be a finite group which satisfies condition (X). According to Lemma 3.4 G is soluble and $G^{(n)} \neq 1$, $G^{(n+1)} = 1$ for some $n \in N$. By Corollary 3.2 $G/G^{(n)}$ satisfies condition (X) and using induction on n we can assume that $G/G^{(n)}$ is nilpotent. Let m be the nilpotency class of $G/G^{(n)}$ and $p_1, p_2, \ldots, p_k$ be all the prime divisors of $|G_{m+1}|$. Denote by S_i the Sylow p_i-subgroup of G_{m+1}. Since $G_{m+1} \subseteq G^{(n)}$ and $G^{(n)}$ is Abelian, each S_i is an Abelian characteristic subgroup of G_{m+1}.

Fix an $i \in \{1, \ldots, k\}$. We claim that for $g \in G$

$$\big(o(g), p_i\big) = 1 \Rightarrow g \in C_G(S_i). \tag{4.2}$$

Indeed, suppose that there exists an element $g \in G$ such that $\big(o(g), p_i\big) = 1$ and $g \notin C_G(S_i)$. Regarding $\Omega_1(S_i) = \langle a \in S_i \mid a^{p_i} = 1\rangle$ as a $K\langle g\rangle$-module, where K is the field with p_i elements and applying Mashke's theorem we obtain

$$\Omega_1(S_i) = A_1 \times \cdots \times A_r,$$

where A_i $(i = 1, \ldots, r)$ are irreducible $K\langle g\rangle$-modules. By Theorem 5.2.4 [2], $\langle g\rangle$ acts non-trivially on $\Omega_1(S_i)$ and consequently, $g \notin C_G(A_j)$ for some $j \in \{1, \ldots, r\}$. Thus,

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there exists an element $a \in A_j$ such that $[a, g] \neq 1$. Using the identity $[a, g^{i_1}]^{g^{i_2}} = [a, g^{i_2}]^{-1}[a, g^{i_1+i_2}]$ we conclude that the subgroup $[a, \langle g \rangle]$ is a $K\langle g \rangle$-submodule of A_j. Since A_j is irreducible, $[a, \langle g \rangle] = A_j$ and $a \in [a, \langle g \rangle]$, so that $I \in a[a, G]$. The contradiction with condition (X) proves (4.2).

It is easy to see that

$$G_{m+1+l} = [S_1, {}_lG] \times [S_2, {}_lG] \times \cdots \times [S_k, {}_lG].$$

In view of (4.2) G acts as a finite p_i-group of automorphisms on S_i $(i = 1, \ldots, k)$ and by Lemma 4.1 we can choose a number l such that $[S_i, {}_lG] = 1$ for each $i \in \{1, \ldots, k\}$. Hence $G_{m+1+l} = 1$ and G is nilpotent which completes the proof.

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REFERENCES

1. M. A. Dokuchaev, *Torsion units in integral group rings of nilpotent metabelian groups*, Commun. Algebra (2) **20**(1992), 423–435.
2. D. Gorenstein, *Finite groups*, Harper and Row, New York, 1968.
3. A. G. Kurosh, *The theory of groups, II*, Chelsea Publ. Co., New York, 1956.
4. S. K. Sehgal, *Topics in group rings*, Marcel Dekker, New York, 1978.

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