A Simple Proof of a Theorem of Landau

By E. M. Wright

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Let $\sigma_k(x)$ be the number of integers $n \leqslant x$ which are the product of just k prime factors, so that

$$n = p_1 p_2 \dots p_k \tag{1}$$

and let $\pi_k(x)$ be the number of such n for which all the p_i are different. The behaviour of $\pi_k(x)$ and $\sigma_k(x)$ as $x \to \infty$ is given by

Theorem: $$\pi_k(x) \sim \sigma_k(x) \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\log x}.$$

In 1791 Gauss [1] conjectured this result for $k = 1, 2, 3$ "*et sic in inf.*" The case $k = 1$ is the celebrated Prime Number Theorem, first proved by Hadamard [2] and de la Vallée Poussin [3] in 1896. For $k \geqslant 2$, the theorem was first proved by Landau [4] in 1900.

Subsequently, Landau [5] found asymptotic expansions for $\pi_k(x)$ and $\sigma_k(x)$ with error $O(x \log^{-m} x)$ for any m. More recently, S. M. Shah [6] and S. Selberg [7] have obtained similar results by more elementary methods.

A. Selberg [8] recently found an elementary proof of

$$\vartheta(x) \equiv \sum_{p \leqslant x} \log p \sim x, \tag{2}$$

which is equivalent to the Prime Number Theorem. Here I present an elementary deduction of our theorem for $k \geqslant 2$ from (2) and the well-known elementary result

$$\sum_{p \leqslant x} \frac{1}{p} \sim \log\log x. \tag{3}$$

So far as I am aware, my method is substantially simpler than any of the earlier methods.

We write c_n for the number of ways of expressing n in the form (1), order being relevant. Clearly $c_n = 0$, unless n is a product of just k prime factors; in this case $c_n = k!$ or $1 \leqslant c_n < k!$ according as the k primes are

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or are not, all different. We write

$$\Pi_k(x) = \sum_{n \leqslant x} c_n = \sum_{p_1 p_2 \ldots p_k \leqslant x} 1,$$

and so have

$$k!\, \pi_k(x) \leqslant \Pi_k(x) \leqslant k!\, \sigma_k(x). \tag{4}$$

Again, there are just $\sigma_k(x) - \pi_k(x)$ values of $n \leqslant x$, each of which is representable in the form (1) with two at least of the p_i equal. We may take $p_{k-1} = p_k$ and so

$$\sigma_k(x) - \pi_k(x) \leqslant \sum_{p_1 p_2 \ldots p_{k-2} p_{k-1}^2 \leqslant x} 1 \leqslant \sum_{p_1 \ldots p_{k-1} \leqslant x} 1 = \Pi_{k-1}(x). \tag{5}$$

We write $\Omega_0(x) = 1$ and, for $k \geqslant 1$,

$$\Omega_k(x) = \sum_{n \leqslant x} \frac{c_n}{n} = \sum_{p_1 \ldots p_k \leqslant x} \frac{1}{p_1 \cdots p_k},$$

so that

$$\Omega_k(x) = \sum_{p_1 \leqslant x} \frac{1}{p_1} \sum_{p_2 \ldots p_k \leqslant x/p_1} \frac{1}{p_2 \cdots p_k} = \sum_{p_1 \leqslant x} \frac{1}{p_1} \Omega_{k-1}\left(\frac{x}{p_1}\right).$$

We also write

$$\vartheta_k(x) = \sum_{n \leqslant x} c_n \log n = \sum_{p_1 \ldots p_k \leqslant x} \log(p_1 p_2 \ldots p_k),$$

so that

$$k\vartheta_{k+1}(x) = \sum_{p_1 \ldots p_{k+1} \leqslant x} \{\log(p_2 p_3 \ldots p_{k+1}) + \log(p_1 p_3 \ldots p_{k+1}) + \ldots + \log(p_1 p_2 \ldots p_k)\}$$

$$= (k+1) \sum_{p \leqslant x} \vartheta_k\left(\frac{x}{p}\right).$$

Hence, if

$$\phi_k(x) = \vartheta_k(x) - kx\Omega_{k-1}(x),$$

we have

$$k\phi_{k+1}(x) = (k+1) \sum_{p \leqslant x} \phi_k\left(\frac{x}{p}\right) \quad (k \geqslant 1).$$

If, for some fixed $k \geqslant 1$,

$$\phi_k(x) = o\{(\log\log x)^{k-1}\}, \tag{6}$$

it follows that

$$|\phi_{k+1}(x)| \leqslant x(\log\log x)^{k-1} \sum_{p \leqslant x} \frac{1}{p} f\left(\frac{x}{p}\right),$$

where, for any $\epsilon > 0$,

$$0 < f(x) \leqslant A \quad (x \geqslant 1), \quad f(x) < \epsilon \quad \big(x \geqslant x_0 = x_0(\epsilon)\big).$$

Hence

$$\sum_{p \leqslant x} \frac{1}{p} f\left(\frac{x}{p}\right) \leqslant \epsilon \sum_{p \leqslant x/x_0} \frac{1}{p} + A \sum_{x/x_0 \leqslant p \leqslant x} \frac{1}{p}$$

$$\leqslant \epsilon \log\log\left(\frac{x}{x_0}\right) + A \log\left(\frac{\log x}{\log x - \log x_0}\right) + O(1)$$

$$\leqslant 2\epsilon \log\log x$$

for $x \geqslant x_1 \geqslant x_0$, and so

$$\phi_{k+1}(x) = o\{x(\log\log x)^k\},$$

which is (6) with $k+1$ for k. But, for $k=1$, (6) is equivalent to (2). Hence (6) is true for all $k \geqslant 1$.

Next we have

$$\left(\sum_{p \leqslant \sqrt[k]{x}} \frac{1}{p}\right)^k \leqslant \Omega_k(x) \leqslant \left(\sum_{p \leqslant x} \frac{1}{p}\right)^k$$

and so, by (3),

$$\Omega_k(x) \sim (\log\log x)^k.$$

Hence, by (6),

$$\vartheta_k(x) \sim kx(\log\log x)^{k-1}.$$

Trivially

$$\vartheta_k(x) = \sum_{n \leqslant x} c_n \log n \leqslant \Pi_k(x) \log x \tag{7}$$

and, if $X = \dfrac{x}{\log x}$,

$$\vartheta_k(x) \geqslant \sum_{X < n \leqslant x} c_n \log n \geqslant \{\Pi_k(x) - \Pi_k(X)\} \log X.$$

But $\log X \sim \log x$ and, for $k \geqslant 2$,

$$\Pi_k(X) = O(X) = O\left(\frac{x}{\log x}\right) = o\left(\frac{\vartheta_k(x)}{\log x}\right) = o\left(\Pi_k(x)\right)$$

by (7). Hence

$$\Pi_k(x) \sim \frac{\vartheta_k(x)}{\log x} \sim \frac{kx(\log\log x)^{k-1}}{\log x}$$

and so, by (4) and (5),

$$\pi_k(x) \sim \sigma_k(x) \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\,\log x} \quad (k \geqslant 2).$$

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DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF ABERDEEN.