RECONSTRUCTION OF CACTI

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Following the work of Kelly (**8**), Harary and Palmer (**5**), and Bondy (**1**) on the reconstruction of trees, and of Manvel (**10**) on the reconstruction of connected graphs with a single cycle, it was a natural step to attempt to solve the reconstruction problem for cacti. The solution of this problem, presented here, assumes both Kelly's Theorem and the result of Manvel in (**10**).

Any definitions not given here can be found in (**2**).

1. Introduction. Let graph G have point set $V = \{v_1, v_2, \ldots, v_p\}$ and line set $X = \{x_1, x_2, \ldots, x_q\}$. For each $v_i \in V$, $G_i = G - v_i$ is the maximal subgraph of G which does not contain v_i and is formed by deleting v_i and all lines incident with it from G. If $x = uv$ is a line of G between the points u and v, then the subgraph $G - x$ is formed by deleting x from G. A *homeomorph* of G is obtained by introducing points of degree 2 into the lines of G.

A *block* of a graph G is a maximal 2-connected (non-separable) subgraph of G. A *cutpoint* of G is a point v such that $G - v$ is disconnected. An *endblock* of G is a block containing only one cutpoint of G. In the case of endblocks which are lines or cycles, we will speak of *endlines* and *endcycles*.

A *cactus* (formerly called Husimi tree (**6**; **7**)) is a connected graph each of whose blocks is either a cycle or a line. Clearly, trees and cycles are special types of cacti.

In (**11**), Ulam posed a now famous conjecture which we will state in the stronger form proposed by Harary (**3**).

The Reconstruction Conjecture. A graph G with at least three points is uniquely determined up to isomorphism by the subgraphs $G_i = G - v_i$.

A useful heuristic device is to consider that the collection $\{G_i\}$ forms a deck of cards, one graph to a card, and the problem is to reconstruct the graph G from this deck.

The first attack on the conjecture was made by Kelly (**8**), who solved the problem in the affirmative for trees. Harary and Palmer (**5**) and Bondy (**1**) showed that it is not in fact necessary to use the entire deck $\{G_i\}$ to reconstruct a tree. In (**3**), Harary solved the reconstruction problem for disconnected graphs, and showed how such parameters as the number of lines, independent cycles, blocks and cutpoints, as well as the degree sequence and the connectivity, could be obtained from the G_i. Finally, Manvel (**10**), verified

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reconstruction for connected graphs with exactly one cycle (see also (**4**)). Cacti appeared to be the next reasonable class of graphs for which the reconstruction problem might be solved.

2. Reconstruction of cacti. The first step in our problem is to find a method of determining, for a deck $\{G_i\}$ which belongs to some graph G, whether or not G is a cactus. Since the reconstruction problems for trees and cycles have been settled, we may assume that we have found that the deck $\{G_i\}$ represents neither of these. Then, two criteria will suffice. First, since a cactus can be alternately characterized as a connected graph containing no homeomorph of $K_4 - x$, the unique graph with four points and five lines, it is clear that none of the G_i can have such a subgraph. This condition does not distinguish between cacti and homeomorphs of $K_4 - x$, so that we must add the further observation that if G is a cactus, at least one of the G_i is disconnected. Except for cycles, which have been already handled, these two conditions identify when the deck $\{G_i\}$ belongs to a cactus.

THEOREM. *A cactus G is uniquely determined by the collection $\{G - v_i\}$.*

Proof. There are two cases to consider. Note first that we know the number of endlines of G since we know its degree sequence.

Case I: *G has no endline.* In this case, all endblocks of G are cycles and if at least one of these endblocks is a triangle, there will be some connected G_j which has exactly one endline uv, where $\deg v = 1$. Then, in G, v_j is adjacent to exactly the points u and v, so that G can be easily reconstructed. If all endblocks are cycles with at least four points, then in some connected G_j we can find two paths $vu_1u_2 \ldots u_{m-1}u_m$ and $vu_1'u_2' \ldots u_{n'-1}u_n'$ emanating from the same point v, such that u_m and u_n' are endpoints and all the rest of the u_i and u_k' have degree 2. Such a G_j results by removing from an endcycle a point v_j which is not adjacent to a cutpoint. Then in G, v_j must be adjacent to exactly u_m and u_n'.

Case II: *G has endlines.* If Z is a cycle of G such that at most one component of $G - X(Z)$ has cycles, then Z *has level* 1, or is a *level*-1 *cycle.* Recursively, if Z is a cycle such that $G - X(Z)$ has at least two components each of which contains a cycle having level $i - 1$ but no cycle of higher level, then Z *has level i.*

Analogous to the well-known result that every tree is either centred or bicentred (**9**, p. 65), it is clear that G has either one or two cycles of maximum level. Such cycles are *central*, and the set of all central cycles will be called the *centre* of G. Since G has endlines, among all connected G_i there will be a G_j resulting from the removal of an endpoint v_j which will exhibit the cycle structure of G; in particular, G and G_j have the same centre. Let C be the centre of G, and define the level of C to be the level of its cycles. The components of $G - X(C)$ are *appendages* of G. If we indicate for each appendage of G its intersection with C, then each appendage will be either a rooted cactus or, in the case of the appendage between the central cycles in a bicentred G, a doubly rooted tree.

For any graph H let $\epsilon(H)$ be the number of endlines in H. Note that $\epsilon = \epsilon(G)$ is known, since we know the degree sequence of G; it is also clear that ϵ is the number of connected G_i with the same cycle structure as G.

We proceed to reconstruct the set of appendages of G from the G_i. To this end, we first determine whether the number $k(G)$ of appendages of G with endlines is 1 or greater than 1. This can be determined quite easily if $\epsilon = 1$ or if $\epsilon \geqq 3$. The first case is trivial, while, in the second, we consider all G_j resulting from deletion of an endpoint, and note that $k(G) = 1$ if and only if for each G_j, $k(G_j) = 1$.

Suppose that $\epsilon = 2$. We wish to decide whether or not both endlines belong to the same appendage. Let $\mathfrak{S}$ be the statement that both endlines lie in the same appendage. Usually the truth of $\mathfrak{S}$ can be determined by one of the following two tests:

T1. If there is a connected G_j with $\epsilon(G_j) = 2$ for which $\mathfrak{S}$ fails, then $\mathfrak{S}$ fails for G;

T2. If $\mathfrak{S}$ is true for some connected G_j with $\epsilon(G_j) = 2$, then $\mathfrak{S}$ is true for G.

If none of the G_i satisfy either test, then when either endline of G is deleted, the resulting G_j has only one endline. In this case, suppose that $\mathfrak{S}$ holds. If the endlines lie in an appendage which contains a cycle Z, then either some connected G_i has at least two endlines, all of which lie in the same appendage, or some G_i with exactly one isolated point contains only one appendage with an endline. The latter case occurs when the length of Z is equal to 3 and the endlines are at different points. Note that once the two tests have failed, neither of the above possibilities is consistent with $\bar{\mathfrak{S}}$. However, when the two endlines belong to an appendage with no cycle, they must have a common point, for otherwise, if $u_k v_k$ are the endlines and v_k endpoints, for $k = 1, 2$, then for at least one k, $G - v_k$ has u_k as an endpoint and hence $G - v_k$ has two endlines on the same appendage, a situation covered by T2. However, the two endlines have a common point if and only if some G_j has two isolated points. Thus we can always identify when $\mathfrak{S}$ or, by elimination, $\bar{\mathfrak{S}}$ holds.

Now among all the connected G_i there will be a non-empty subcollection $\{G_i'\}$ each of whose members results from the deletion of an endpoint from G. Then each of the G_i will have the same centre as G. Let $\mathbf{A}$ be the collection of appendages (with roots) of these G_i'. We know that $\mathbf{A}$ is non-empty since G has endlines.

If at least two appendages of G have endlines, then each appendage A will appear $k[\epsilon - \epsilon(A)]$ times in $\mathbf{A}$, where k is the number of copies of A which are appendages of G. However, some elements of $\mathbf{A}$ will be false appendages, which result from the removal of endpoints from true appendages to form some of the G_i'. These extraneous members of $\mathbf{A}$ can easily be eliminated. Choose any $A \in \mathbf{A}$ with the largest number of points, and for each endpoint $u_i \in A$ remove

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$A - u_i$ from **A**. Proceed to the next largest appendage which remains in **A** and repeat the process. At each step the appendage under consideration must be an appendage of G, since there is no larger A from which it could have been formed by endpoint deletion. When the procedure terminates, **A** will consist of all appendages of G in the proper multiplicity.

On the other hand, if only one appendage A of G has endlines, then, if there is some $u \in A$ such that $A - u$ has endlines, we claim that we find all appendages of G except A from $G - u$. This is certainly clear if removing u does not change the centre. Assume that G is centred. The bicentred case is similar and will be omitted. If $G - u$ is centred, then clearly the centres are the same. Thus we investigate when there is no point u of A such that $G - u$ is centred and has endlines. In particular, for any point u of any level-1 cycle of A, if $G - u$ is connected, then it is bicentred. But then, if the centre has level t, G has two appendages with cycles of level $t - 1$. Thus we can remove a non-cutpoint u from a level-1 cycle of A and note that of the two central cycles in $G - u$, the one containing the root of the unique appendage with endlines is the centre of G. Thus, if A is the only appendage of G with endlines, we can find all other appendages from $G - u$. We then find A by deleting a point from a level-1 cycle in some other appendage.

Finally, if no point u may be deleted from A such that $A - u$ has endlines, then A itself is an endline, and the other appendages can be easily found from the unique connected G_i with no endlines. Hence, we have determined in every case the collection **A** of rooted appendages of G with the proper multiplicities.

If A is a singly rooted (and hence non-central) appendage of G, and u is a point of A such that the component of $A - u$ containing the root is not itself an appendage of G, then, since we know the appendages of G, we can tell exactly which point has been deleted from G to obtain $G - u$, and hence we can reconstruct G. We thus suppose that for every appendage $A \in$ **A** and any $u \in A$, the rooted component of $A - u$ is in **A**. We now proceed to reconstruct G.

(1) *G is bicentral.* Let the centre C consist of cycles C_1 and C_2, and let H_i be the subgraph of G consisting of C_i and all singly rooted appendages with roots on C_i, and with the root of the doubly rooted appendage indicated. Let the doubly rooted appendage be denoted K. An appendage consisting of only a root will be called *trivial.*

If K has an endpoint v_j which is not a root, then the corresponding G_j will display the entire structure of G except for the location of one endline. Furthermore, we know K, so that we have only to decide how it is oriented between H_1 and H_2. Without loss of generality, suppose that $|V(H_1)| \leqq |V(H_2)|$. If we can delete a point from H_1 which leaves G bicentred, our proof is complete. Also, if we can delete a point v from H_2 which leaves G bicentred, our proof is complete unless $H_1 = H_2 - v$. Suppose, therefore, that each of these possi-

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bilities fails. Then H_1 is a "chain" of cycles, perhaps connected by paths, and H_2 is just H_1 with an extra endline. But then we can delete a point from the level-1 cycle of H_2 to obtain a centred G_j with endlines in only one appendage, and since that appendage will contain both K and C_2 (except in the trivial case where C_2 has level 1), the orientation of K with respect to the centre of G will be clear.

If K has no endline, there are two possibilities. Suppose first that both central cycles have appendages with endlines. This can be recognized as follows: For any bicentred G_j let $H_{1,j}$ and $H_{2,j}$ be the subgraphs each consisting of a central cycle and all singly rooted appendages on that cycle; recall that H_1 and H_2 have been similarly defined for G. Then both central cycles have endlines if and only if there are graphs G_j and G_k such that, with proper labeling, $H_{1,j}$ and $H_{1,k}$ each have endlines, and $H_{2,j} \cong H_{1,k} - u$ and $H_{2,k} \cong H_{1,j} - u'$ for judicious choice of u and u'. Then $H_{1,j}$ and $H_{1,k}$ are, with possible reordering, H_1 and H_2, and their placement with respect to K is clear.

Finally, if only C_1, say, has an appendage with an endline, then each of the G_i which has been obtained by deleting an endpoint will clearly display H_2. If we can find a point v_j such that G_j is connected and bicentral, but neither $H_{1,j}$ nor $H_{2,j}$ is H_2, then, unless $H_1 \cong H_2 - u$, the proof is complete. If for some point u, $H_1 \cong H_2 - u$, then some bicentred G_j will have $H_{1,j} \cong H_{2,j} \cong H_1$, and the proof is complete.

However, it may be impossible to find any connected bicentral G_i which is not the result of deleting a point of H_1. In this case, C_2 can have only one appendage A, and this appendage can have only one cycle of each level, so that it must be a "chain" of cycles, perhaps connected by paths. When we delete a point v_j from a level-1 cycle of A, we will have G_j centred at C_1 and displaying H_1, and an appendage A^* consisting of K and $H_2 - v_j$. Confusion can occur only if some appendages of C_1 are isomorphic to A^*. If A^* appears k times as an appendage of C_1, then each connected bicentred G_i containing $k - 1$ copies of A^* results from deletion of a point of one of these appendages of C_1. We already have a picture of G which is complete up to distinguishing the point at which H_1 attaches to the central appendage, and any one of these latter G_i will suffice to complete the picture, and hence reconstruct G.

(2) *G is central.* We define a map f assigning integers to all appendages of G^*, including the trivial ones, so that for any two appendages A_1 and A_2:

(i) $f(A_1) = f(A_2)$ if and only if $A_1 \cong A_2$,

(ii) if A_1 is isomorphic to the rooted component of $A_2 - v$, then $f(A_1) < f(A_2)$,

(iii) if A_1 is trivial and A_2 is an endline, then $f(A_1) = 0$ and $f(A_2) = 1$, and

(iv) if A_1 has a cycle of level s, but not $s + 1$, and A_2 has a cycle of level $t > s$, then $f(A_2) > f(A_1)$.

Let f^* be the largest integer assigned to any appendage of G, plus 1.

Consider an arbitrary orientation of C and choose any appendage A_1 on C.

If $A_2, A_3, \ldots, A_t$ are the other appendages on C in cyclic order, proceeding in one direction from A_1, then we can associate with this choice of orientation and appendage the number $f(A_1)\, f(A_2) \ldots f(A_t)$ in the base f^*. Among all numbers which can be assigned to C there will be a maximum, μ. Since we have assumed that for any point u of any appendage $A \in \mathbf{A}$, the rooted component of $A - u$ is also in $\mathbf{A}$, it is clear that numbers can be assigned in a similar fashion to the centres of the connected central G_i.

The connected central G_j, if one exists, with the largest assigned number must be the result of removing a point from the appendage A_μ corresponding to the last non-zero digit of μ. (Such a G_j will exist unless G has exactly two non-trivial appendages.) To see this, it is necessary only to note that if $\mu = d_1 d_2 \ldots d_t$, then $d_t \leqq d_2$, for otherwise $d_1 d_t \ldots d_2$ is a larger number. Of course, the two appendages with the largest values under f have cycles with level one less than that of C. Thus, unless these are the only two non-trivial appendages, the removal of a non-cutpoint v_j from A_μ leaves G centred. However, from G_j and $\mathbf{A}$ it is easy to reconstruct G, unless A_μ is an endline and μ ends with 00, 01, or 10. For then the number associated with G_j ends in all zeros, and we cannot be sure exactly which digit should be a 1 in μ.

Thus there are two problem cases. If G has only two non-trivial appendages A_1 and A_2, with $f(A_1) \leqq f(A_2)$, then for any non-cutpoint $u \in A_1$, $A_1 - u \in \mathbf{A}$, and hence $A_1 - u$ is trivial. Thus A_1 must be an endline and, since G is centred, A_2 is either an endline or a path of length 2. Since G is unicyclic, the proof is complete.

In the final case, A_μ is an endline and we must decide just which of the several terminal zeros was produced by deletion of A_μ. To do this we look at the connected central G_k with the next largest number. The sequence of appendages around the centre of this graph will correspond to that around the centre of the G_j with the largest assigned number except directly before the string of zeros ending the largest number and one place within that string. The placement of the endline within that string in G_k will show us where to replace appendage A_μ in G_k, and complete our picture of G.

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