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Symmetric Sequence Subspaces of $C(\alpha)$, II

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Abstract. If α is an ordinal, then the space of all ordinals less than or equal to α is a compact Hausdorff space when endowed with the order topology. Let $C(\alpha)$ be the space of all continuous real-valued functions defined on the ordinal interval $[0,\alpha]$. We characterize the symmetric sequence spaces which embed into $C(\alpha)$ for some countable ordinal α. A hierarchy (E_α) of symmetric sequence spaces is constructed so that, for each countable ordinal α, E_α embeds into $C(\omega^{\omega^\alpha})$, but does not embed into $C(\omega^{\omega^\beta})$ for any $\beta < \alpha$.

Let α be an ordinal. The ordinal interval $[0,\alpha]$ is a compact Hausdorff space in the order topology. The space of all continuous real-valued functions on $[0,\alpha]$ is commonly denoted by $C(\alpha)$. In [4], the symmetric sequence spaces which embed into $C(\omega^\omega)$ are characterized. This paper, which is a continuation of [4], gives a characterization of the symmetric sequence spaces which embed into $C(\alpha)$ for some countable ordinal α. In [4], it is shown that any Orlicz sequence space which embeds into $C(\alpha)$ for some countable ordinal α already embeds into $C(\omega^\omega)$. Here, we construct a hierarchy of symmetric sequence spaces $(E_\alpha)_{\alpha<\omega_1}$ such that, for each countable ordinal α, E_α embeds into $C(\omega^{\omega^\alpha})$, but does not embed into $C(\omega^{\omega^\beta})$ for any $\beta < \alpha$. Since, according to Bessaga and Pełczynski [2], if $\alpha < \beta$ are countable infinite ordinals, then $C(\alpha)$ and $C(\beta)$ are isomorphic if and only if $\beta < \alpha^\omega$, (E_α) is a full hierarchy of mutually non-isomorphic symmetric sequence spaces which embed into $C(\alpha)$ for some countable ordinal α. The authors thank the referee for pointing out some errors in an earlier version of the paper, and for various suggestions for improving the exposition.

For terms and notation concerning ordinal numbers and general topology, we refer to [3]. The first infinite ordinal, respectively, the first uncountable ordinal, is denoted by ω, respectively, ω_1. Any ordinal is either 0, a successor, or a limit. If α is a successor ordinal, denote its immediate predecessor by $\alpha - 1$. If K is a compact Hausdorff space, $C(K)$ denotes the space of all continuous real-valued functions on K. It is a Banach space under the norm $\|f\| = \sup_{t\in K} |f(t)|$. If K is a topological space, its *derived set* $K^{(1)}$ is the set of all of its limit points. A transfinite sequence of derived sets may be defined as follows. Let $K^{(0)} = K$. If α is an ordinal, let $K^{(\alpha+1)} = (K^{(\alpha)})^{(1)}$. Finally, for a limit ordinal α, we define $K^{(\alpha)} = \bigcap_{\beta<\alpha} K^{(\beta)}$. The cardinality of a set A is denoted by $|A|$. By $\mathcal{P}_\infty(\mathrm{N})$, respectively, $\mathcal{P}_{<\infty}(\mathrm{N})$, we mean the collection of all infinite, respectively, finite, subsets of N. These are subsets of 2^{N}, and consequently inherit the product topology. If A and B are nonempty subsets of N, we say that $A < B$ if $\max A < \min B$. We also allow that $\varnothing < A$ and $A < \varnothing$ for any $A \subseteq \mathrm{N}$.

We follow standard Banach space terminology, as may be found in the book [5]. We say that a Banach space is a *sequence space* if it is a vector subspace of the space of all real sequences. Such is the case, for instance, when a Banach space E has a *(Schauder) basis* (e_k),

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i.e., every element $x \in E$ has a unique representation $x = \sum a_k e_k$ for some sequence of scalars (a_k). Naturally, we identify every $x \in E$ with the sequence (a_k) used in its representation. If (e_k) is a basis of a Banach space E, there is a unique sequence of bounded linear functionals (e'_k) on E such that $\langle e_j, e'_k \rangle = 1$ if $j = k$, and 0 otherwise. The sequence (e'_k) is called the sequence of *biorthogonal functionals* to the sequence (e_k). It is a well known fact that every $x' \in E'$, the dual space of E, has a unique representation $x' = \sum a_k e'_k$, where the sum converges in the weak* topology on E'. Therefore, E' may also be regarded as a sequence space. If (e'_k) is a basis of E' (so that the foregoing sum actually converges in norm for every $x' \in E'$), then the basis (e_k) is said to be *shrinking*. If x is an element of a sequence space, let $\operatorname{supp} x$ be the set of all coordinates k at which x is nonzero. The vector space consisting of all finitely supported real sequences is denoted by c_{00}. Given a real null sequence $a = (a_n)$, let $a^* = (a^*_n)$ be the decreasing rearrangement of $(|a_n|)$. A basis (e_k) of a Banach space is *unconditional* if $\sum \varepsilon_k a_k e_k$ converges for every choice of signs (ε_k) whenever $\sum a_k e_k$ converges. A basis (e_k) is *subsymmetric* if it is unconditional and $\sum a_j e_{k_j}$ converges for every subsequence (e_{k_j}) whenever $\sum a_k e_k$ converges. It is *symmetric* if $\sum a_k e_{\pi(k)}$ converges for every permutation π on N whenever $\sum a_k e_k$ converges. A symmetric basis is necessarily unconditional [5, Section 3a]. We say that it is 1-symmetric (respectively, 1-subsymmetric) if $\|\sum \varepsilon_k a_k e_{\pi(k)}\| = \|\sum a_k e_k\|$ for every choice of signs (ε_k), and every permutation π on N (respectively, every increasing function $\pi \colon \mathrm{N} \to \mathrm{N}$). Examples of Banach spaces with 1-symmetric bases are ℓ^p $(1 \le p < \infty)$, and c_0. These norms are defined by

$$\|(a_k)\|_p = \left(\sum |a_k|^p\right)^{\frac{1}{p}} \quad \text{and} \quad \|(a_k)\|_\infty = \sup |a_k|$$

respectively. A sequence (x_k) in a Banach space is *normalized* if $\|x_k\| = 1$ for all k. Given two sequences (x_k) and (y_k) in possibly different Banach spaces, we say that they are *equivalent* if there is a finite positive constant C such that

$$C^{-1}\Big\|\sum a_k x_k\Big\| \le \Big\|\sum a_k y_k\Big\| \le C\Big\|\sum a_k x_k\Big\|$$

for every finitely supported sequence (a_k). Two Banach spaces E and F are said to be *isomorphic* if they are linearly homeomorphic. We say that E embeds into F, $E \hookrightarrow F$, if E is isomorphic to a subspace of F.

Throughout the rest of the paper, for each countable limit ordinal α, fix a sequence of ordinals (α_n) which strictly increases to α. In [4], the family $(\mathcal{A}^f_\alpha)$ of subsets of $\mathcal{P}_{<\infty}(\mathrm{N})$ is introduced. If $f \colon \mathrm{N} \to \mathrm{N}$ is strictly increasing, let

$$\mathcal{A}^f_0 = \{A \subseteq \mathrm{N} : \max A \le f(\min A)\} \cup \{\varnothing\}.$$

For a countable ordinal α, let

$$\mathcal{A}^f_{\alpha+1} = \{A = \bigcup_{i=1}^{n} A_i : A_1 < \cdots < A_n,\ A_i \in \mathcal{A}^f_\alpha,\ n \le f(\min A)\}.$$

If $\alpha < \omega_1$ is a limit ordinal, recall the sequence (α_n) chosen above. Set

$$\mathcal{A}^f_\alpha = \{A : \text{there exists } n \le f(\min A) \text{ such that } A \in \mathcal{A}^f_{\alpha_n}\}.$$

The results in [4] yield the following fact.

Proposition 1 *Let E be a Banach space with an unconditional basis (e_n) such that E embeds into $C(\omega^{\omega^\alpha})$ for some $\alpha < \omega_1$. Then there exist an increasing function $f\colon \mathrm{N} \to \mathrm{N}$, and a constant $K < \infty$ such that for all $x = \sum a_n e_n \in E$,*

$$\|x\| \le K \sup\Big\{ \Big\| \sum_{n \in A} a_n e_n \Big\| : A \in \mathcal{A}_\alpha^f \Big\}.$$

The definition of the family $(\mathcal{A}_\alpha^f)$ is modelled on the definition of the well known Schreier family $(\mathcal{S}_\alpha^f)$ [1], [7]. The Schreier set $\mathcal{S}_0^f = \{A \subseteq \mathrm{N} : |A| \le 1\}$. The inductive steps defining $\mathcal{S}_\alpha^f$ are exactly the same as in the definition for $(\mathcal{A}_\alpha^f)$ (with $\mathcal{A}$ replaced by $\mathcal{S}$). We will need a slight modification of Proposition 1. The next lemma is easily proved by induction.

Lemma 2 *Let α be a countable ordinal, and let $f\colon \mathrm{N} \to \mathrm{N}$ be an increasing function. If $h\colon \mathrm{N} \to \mathrm{N}$ is an increasing function such that $h(n+1) > f\big(h(n)\big)$ for all n, then $A \cap h(\mathrm{N}) \in \mathcal{S}_\alpha^f$ for all $A \in \mathcal{A}_\alpha^f$.*

Proposition 3 *Let E be a Banach space with a 1-subsymmetric basis (e_n) such that E embeds into $C(\omega^{\omega^\alpha})$ for some $\alpha < \omega_1$. Then there exist an increasing function $f\colon \mathrm{N} \to \mathrm{N}$, and $K < \infty$ such that for all $x = \sum a_n e_n \in E$,*

$$\|x\| \le K \sup\Big\{ \Big\| \sum_{n \in A} a_n e_n \Big\| : A \in \mathcal{S}_\alpha^f \Big\}.$$

Proof By Proposition 1, there exist an increasing function $\tilde{f}\colon \mathrm{N} \to \mathrm{N}$, and a constant $K < \infty$ such that for all $x = \sum a_n e_n \in E$,

$$\|x\| \le K \sup\Big\{ \Big\| \sum_{n \in A} a_n e_n \Big\| : A \in \mathcal{A}_\alpha^{\tilde{f}} \Big\}.$$

Let $h\colon \mathrm{N} \to \mathrm{N}$ be an increasing function such that $h(n+1) > \tilde{f}\big(h(n)\big)$ for all n. Define $y = \sum a_n e_{h(n)}$. Then

$$\begin{aligned}
\|x\| = \|y\| &\le K \sup\Big\{ \Big\| \Big(\sum a_n e_{h(n)}\Big) \chi_A \Big\| : A \in \mathcal{A}_\alpha^{\tilde{f}} \Big\} \\
&= K \sup\Big\{ \Big\| \Big(\sum a_n e_{h(n)}\Big) \chi_{h(\mathrm{N}) \cap A} \Big\| : A \in \mathcal{A}_\alpha^{\tilde{f}} \Big\} \\
&\le K \sup\Big\{ \Big\| \sum_{n \in A} a_n e_n \Big\| : h(A) \in \mathcal{S}_\alpha^{\tilde{f}} \Big\} \quad \text{by Lemma 2} \\
&\le K \sup\Big\{ \Big\| \sum_{n \in A} a_n e_n \Big\| : A \in \mathcal{S}_\alpha^{\tilde{f} \circ h} \Big\}.
\end{aligned}$$

The proposition follows by taking $f = \tilde{f} \circ h$. ■

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1 Norming sets

In this section, we show that if E is a symmetric sequence space which embeds into some $C(\omega^{\omega^\alpha})$, then the norm on E can be isomorphically generated by a norming subset of E' of a particular type. Recall that a subset W of E' is *isomorphically norming* if W is bounded and there exists $K > 0$ such that $K\|x\| \leq \sup_{x' \in W} |\langle x, x'\rangle|$ for all $x \in E$. We begin with the following definitions. Let $g : \mathbb{N} \to \mathbb{R}_+$ be a nondecreasing function such that $\lim_{n\to\infty} g(n) = \infty$. Define

$$\mathcal{C}_0^g = \{x \in c_{00} : \|x\|_\infty \leq 1, |\operatorname{supp} x| \leq 1\}.$$

If α is a successor ordinal, let

$$\mathcal{C}_\alpha^g = \Big\{x = \sum_{i=1}^n x_i : x_i \in \mathcal{C}_{\alpha-1}^g,\ (x_i) \text{ pairwise disjoint, and } g(n)\|x\|_\infty \leq 1\Big\},$$

If α is a limit ordinal, recall the sequence (α_n) chosen in the introduction. Define $\mathcal{C}_\alpha^g = \{x : x \in \mathcal{C}_{\alpha_n}^g, g(n)\|x\|_\infty \leq 1\}$. It is easy to see that $\mathcal{C}_\alpha^g$ is a symmetric set, *i.e.*, it is invariant under permutations of the coordinates.

Let E be a sequence space that admits a normalized 1-symmetric shrinking basis which is not equivalent to the unit vector basis of c_0. We represent both E and E' naturally as spaces of real sequences. Denote the (closed) unit balls of E and E' by U_E and $U_{E'}$ respectively.

Lemma 4 *Given an increasing function $f : \mathbb{N} \to \mathbb{N}$ and numbers δ, η such that $0 < \delta < 1$, $\eta > 0$, there exists a nondecreasing function $g : \mathbb{N} \to \mathbb{R}_+$, $\lim_{n\to\infty} g(n) = \infty$, such that if $a = a^* = (a_n) \in U_E$, $b = (b_n) \in U_{E'}$, $|\langle a, b\chi_A\rangle| \geq \eta$ for some $A \in \mathcal{P}_{<\infty}(\mathbb{N})$, there exists c such that $|c| \leq |b\chi_A|$, $\|c\|_\infty g\big(f(\min A)\big) \leq 1$ and $|\langle a, c\rangle| \geq \delta|\langle a, b\chi_A\rangle|$.*

Proof Define $\lambda(n) = \|(\overbrace{1, 1, \dots, 1}^{n})\|_E$ and $\mu(n) = \|(\overbrace{1, 1, \dots, 1}^{n})\|_{E'}$. Since the basis for E is shrinking but not equivalent to the c_0-basis, $\lambda(n) \to \infty$ and $\mu(n) \to \infty$ as $n \to \infty$. Therefore, there exists a nondecreasing function $g : \mathbb{N} \to \mathbb{R}_+$, $\lim_{n\to\infty} g(n) = \infty$, such that for every $k \in \mathbb{N}$,

$$g\big(f(k)\big) \leq \begin{cases} 1 & \text{if } \lfloor (1-\delta)\eta\lambda(k)\rfloor = 0 \\ \frac{1}{2}\mu\big(\lfloor (1-\delta)\eta\lambda(k)\rfloor\big) & \text{otherwise,} \end{cases}$$

where $\lfloor\cdot\rfloor$ is the greatest integer function. Let a, b and A be given that satisfy the hypotheses, and let $m = \min A$. If $\lfloor (1-\delta)\eta\lambda(m)\rfloor = 0$, let $\varepsilon = 1$; otherwise, let $\varepsilon = 2/\mu\big(\lfloor (1-\delta)\eta\lambda(m)\rfloor\big)$. Consider $B = \{n \in A : |b_n| > \varepsilon\}$. In the first case, $B = \varnothing$. In the second case,

$$1 \geq \|b\| \geq \|b\chi_B\| \geq \varepsilon\mu(|B|),$$

which implies that

$$\mu(|B|) \leq \frac{1}{\varepsilon} = \frac{1}{2}\mu\big(\lfloor (1-\delta)\eta\lambda(m)\rfloor\big) < \mu\big(\lfloor (1-\delta)\eta\lambda(m)\rfloor\big).$$

Consequently, $|B| < (1-\delta)\eta\lambda(m)$ as μ is nondecreasing. Also,

$$1 \geq \|a\| \geq \|(\overbrace{|a_m|, \ldots, |a_m|}^{m})\| = |a_m|\lambda(m).$$

Therefore, $\|a\chi_A\|_\infty = |a_m| \leq \frac{1}{\lambda(m)}$. Let $c = b\chi_{A\setminus B}$. Then $|c| \leq |b\chi_A|$ and $\|c\|_\infty g\big(f(\min A)\big) \leq \varepsilon g\big(f(m)\big)$. If $\lfloor(1-\delta)\eta\lambda(m)\rfloor = 0$, then $\varepsilon = 1$ and $g\big(f(m)\big) \leq 1$; hence $\varepsilon g\big(f(m)\big) \leq 1$. Otherwise,

$$\varepsilon g\big(f(m)\big) \leq \frac{2}{\mu\big(\lfloor(1-\delta)\eta\lambda(m)\rfloor\big)} \cdot \frac{1}{2}\mu\big(\lfloor(1-\delta)\eta\lambda(m)\rfloor\big) = 1.$$

Finally,

$$\begin{aligned}|\langle a, c\rangle| &\geq |\langle a, b\chi_A\rangle| - |\langle a, b\chi_B\rangle| \\ &\geq |\langle a, b\chi_A\rangle| - \|a\chi_B\|_\infty\,|B| \\ &\geq |\langle a, b\chi_A\rangle| - \|a\chi_A\|_\infty\,|B| \\ &\geq |\langle a, b\chi_A\rangle| - \frac{1}{\lambda(m)}\cdot(1-\delta)\eta\lambda(m) \\ &\geq \delta|\langle a, b\chi_A\rangle|.\end{aligned}$$

■

Lemma 5 *Let h, and g_n, $n \in \mathbb{N}$, be nondecreasing functions from $\mathbb{N}$ into $\mathbb{R}_+$ such that $\lim_{k\to\infty} h(k) = \lim_{k\to\infty} g_n(k) = \infty$ for all n. There exists a nondecreasing function $g\colon \mathbb{N} \to \mathbb{R}_+$, $\lim_{k\to\infty} g(k) = \infty$, such that $g \leq h$, and $d \in \mathcal{C}^g_\alpha$ whenever $\alpha < \omega_1$, and $d \in \mathcal{C}^{g_n}_\alpha$ for some n satisfying $\|d\|_\infty h(n) \leq 1$.*

Proof There exist $0 = m_0 < m_1 < m_2 < \cdots \in \mathbb{N}$ and a nondecreasing function $g'\colon \mathbb{N} \to \mathbb{R}_+$ such that $\lim_{k\to\infty} g'(k) = \infty$, and $g'(k) \leq \min\{g_1(k), \ldots, g_i(k)\}$ whenever $m_{i-1} < k \leq m_i$, $i \in \mathbb{N}$. Now choose a nondecreasing function $g\colon \mathbb{N} \to \mathbb{R}_+$ such that $\lim_{k\to\infty} g(k) = \infty$, $g \leq g'$, and $g(m_i) \leq h(i)$ for all $i \in \mathbb{N}$. Clearly, $g \leq h$.

We claim that the function g satisfies the remaining condition of the lemma. The proof is by induction on α. If $\alpha = 0$, there is nothing to prove. Suppose the claim is true for some $\alpha < \omega_1$. Assume that $d \in \mathcal{C}^{g_n}_{\alpha+1}$, and $\|d\|_\infty h(n) \leq 1$. We can write $d = d_1 + \cdots + d_l$, where $d_1, \ldots, d_l$ are pairwise disjoint elements of $\mathcal{C}^{g_n}_\alpha$, and $\|d\|_\infty g_n(l) \leq 1$. Since $\|d_j\|_\infty h(n) \leq \|d\|_\infty h(n) \leq 1$, $d_j \in \mathcal{C}^g_\alpha$ by the inductive hypothesis. Choose i so that $m_{i-1} < l \leq m_i$, then $g(l) \leq \min\{g_1(l), \ldots, g_i(l)\}$. If $n \leq i$, then $\|d\|_\infty g(l) \leq \|d\|_\infty g_n(l) \leq 1$. Otherwise, $i < n$; hence $\|d\|_\infty g(l) \leq \|d\|_\infty g(m_i) \leq \|d\|_\infty h(i) \leq \|d\|_\infty h(n) \leq 1$. Therefore $d \in \mathcal{C}^g_{\alpha+1}$.

Finally, suppose that α is a limit ordinal and that the claim holds for all ordinals $\beta < \alpha$. Assume that $d \in \mathcal{C}^{g_n}_\alpha$, and $\|d\|_\infty h(n) \leq 1$. Let (α_j) be the sequence used to define $\mathcal{C}^{g_n}_\alpha$ and $\mathcal{C}^g_\alpha$. By definition, $d \in \mathcal{C}^{g_n}_\alpha$ implies $d \in \mathcal{C}^{g_n}_{\alpha_j}$ for some j such that $\|d\|_\infty g_n(j) \leq 1$. By the inductive hypothesis, $d \in \mathcal{C}^g_{\alpha_j}$. Choose i such that $m_{i-1} < j \leq m_i$. If $n \leq i$, then

$$\|d\|_\infty g(j) \leq \|d\|_\infty g'(j) \leq \|d\|_\infty g_n(j) \leq 1.$$

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On the other hand, if $i < n$, then

$$\|d\|_\infty g(j) \le \|d\|_\infty g(m_i) \le \|d\|_\infty h(i) \le \|d\|_\infty h(n) \le 1.$$

Hence $d \in \mathcal{C}^g_{\alpha_j}$, and $\|d\|_\infty g(j) \le 1$. Consequently, $d \in \mathcal{C}^g_\alpha$. This completes the proof of the claim. ■

Proposition 6 *Given an increasing function $f\colon \mathrm{N} \to \mathrm{N}$, $0 < \delta < 1$, $\eta > 0$, and a countable ordinal α, there exists a nondecreasing function $g\colon \mathrm{N} \to \mathrm{R}_+$, $\lim_{n\to\infty} g(n) = \infty$, such that if $a = a^* = (a_n) \in U_E$, $b = (b_n) \in U_{E'}$, $A \in \mathcal{S}^f_\alpha$, and $|\langle a, b\chi_A\rangle| \ge \eta$, then there is a $c \in \mathcal{C}^g_\alpha$, $|c| \le |b\chi_A|$, $\|c\|_\infty g\big(f(\min A)\big) \le 1$ and $|\langle a, c\rangle| \ge \delta|\langle a, b\chi_A\rangle|$.*

Proof We will prove the proposition by induction on α. Consider first the case when $\alpha = 0$. Choose g by Lemma 4. If a, b and A are given as in the hypothesis, there exists c such that $|c| \le |b\chi_A|$, $\|c\|_\infty g\big(f(\min A)\big) \le 1$ and $|\langle a, c\rangle| \ge \delta|\langle a, b\chi_A\rangle|$. Since $A \in \mathcal{S}^f_0$, $|A| \le 1$. Thus, it follows from the fact that $|c| \le |b\chi_A|$ that $|\operatorname{supp} c| \le 1$. As $\|b\|_\infty \le 1$, the same inequality also shows that $\|c\|_\infty \le 1$. Therefore, $c \in \mathcal{C}^g_0$, as desired.

Suppose the proposition holds for some $\alpha < \omega_1$. Choose a function h by Lemma 4 corresponding to f, $\sqrt[3]{\delta}$, and η. For each $n \in \mathrm{N}$, choose a function g_n by the inductive hypothesis (for the ordinal α) corresponding to f, $\sqrt[3]{\delta}$, and $\eta\sqrt[3]{\delta}\,(1 - \sqrt[3]{\delta}\,)/n$. Finally, apply Lemma 5 with the functions h and (g_n) to obtain a function g. Let $a = a^* \in U_E$, $b \in U_{E'}$ and $A \in \mathcal{S}^f_{\alpha+1}$ be given so that $|\langle a, b\chi_A\rangle| \ge \eta$. There exists c, $|c| \le |b\chi_A|$, $\|c\|_\infty h\big(f(\min A)\big) \le 1$, such that $|\langle a, c\rangle| \ge \sqrt[3]{\delta}\,|\langle a, b\chi_A\rangle|$. Let $n = f(\min A)$. Since $A \in \mathcal{S}^f_{\alpha+1}$, $A = A_1 \cup \cdots \cup A_k$, where $A_1 < \cdots < A_k$, $A_1, \ldots, A_k \in \mathcal{S}^f_\alpha$, and $k \le n$. Let I be the set of the indices i such that $|\langle a, c\chi_{A_i}\rangle| \ge \eta\sqrt[3]{\delta}\,(1 - \sqrt[3]{\delta}\,)/n$, and let $B = \bigcup_{i\in I} A_i$. Then

$$\begin{aligned}|\langle a, c\chi_B\rangle| &\ge |\langle a, c\chi_A\rangle| - (k - |I|)\eta\sqrt[3]{\delta}\,(1 - \sqrt[3]{\delta}\,)/n\\ &\ge \sqrt[3]{\delta}\,|\langle a, b\chi_A\rangle| - \sqrt[3]{\delta}\,(1 - \sqrt[3]{\delta}\,)\eta \ge \delta^{2/3}|\langle a, b\chi_A\rangle|.\end{aligned}$$

By choice of g_n, for each $i \in I$, there exists $d_i \in \mathcal{C}^{g_n}_\alpha$, $|d_i| \le |c\chi_{A_i}|$, $\|d_i\|_\infty g_n\big(f(\min A_i)\big) \le 1$, and $|\langle a, d_i\rangle| \ge \sqrt[3]{\delta}\,|\langle a, c\chi_{A_i}\rangle|$. Define $d = \sum_{i\in I} \operatorname{sgn}\langle a, d_i\rangle d_i$. Now $\operatorname{sgn}\langle a, d_i\rangle d_i \in \mathcal{C}^{g_n}_\alpha$, and

$$\|\operatorname{sgn}\langle a, d_i\rangle d_i\|_\infty h(n) \le \|c\|_\infty h(n) \le 1.$$

Hence $\operatorname{sgn}\langle a, d_i\rangle d_i \in \mathcal{C}^g_\alpha$ by the choice of the function g. Note that

$$\|d\|_\infty g(k) \le \|d\|_\infty g(n) \le \|c\|_\infty g(n) \le \|c\|_\infty h(n) \le 1.$$

In particular, $d \in \mathcal{C}^g_{\alpha+1}$. Clearly, $|d| \le |b\chi_A|$. Also,

$$\begin{aligned}|\langle a, d\rangle| &= \sum_{i\in I} |\langle a, d_i\rangle|\\ &\ge \sqrt[3]{\delta}\,\sum_{i\in I} |\langle a, c\chi_{A_i}\rangle|\\ &\ge \sqrt[3]{\delta}\,|\langle a, c\chi_B\rangle|\\ &\ge \delta|\langle a, b\chi_A\rangle|.\end{aligned}$$

Finally, suppose that $\alpha < \omega_1$ is a limit ordinal and the proposition holds for all $\beta < \alpha$. Let (α_n) be the sequence used in defining $\mathcal{C}_\alpha^g$ and $\mathcal{S}_\alpha^f$. Apply Lemma 4 with f, $\sqrt{\delta}$, and η to obtain a function h. Then, for each n, apply the inductive hypothesis with f, $\sqrt{\delta}$, $\sqrt{\delta}\,\eta$, and the ordinal α_n to obtain a function g_n. Again, choose a function g corresponding to h and (g_n) by Lemma 5.

Let a, b, and A be given satisfying the hypothesis of the proposition for the ordinal α. By definition, $A \in \mathcal{S}_\alpha^f$ implies that $A \in \mathcal{S}_{\alpha_n}^f$ for some $n \le f(\min A)$. By the choice of the function h, we can find a c such that $|c| \le |b\chi_A|$, $\|c\|_\infty h\big(f(\min A)\big) \le 1$ and $|\langle a, c\rangle| \ge \sqrt{\delta}\,|\langle a, b\chi_A\rangle|$. Similarly, because of the choice of the function g_n, there exists d such that $|d| \le |c\chi_A|$, $\|d\|_\infty g_n\big(f(\min A)\big) \le 1$, $d \in \mathcal{C}_{\alpha_n}^{g_n}$ and $|\langle a, d\rangle| \ge \sqrt{\delta}\,|\langle a, c\chi_A\rangle| \ge \delta|\langle a, b\chi_A\rangle|$. Then $|d| \le |c\chi_A| \le |b\chi_A|$. Since $d \in \mathcal{C}_{\alpha_n}^{g_n}$, and $\|d\|_\infty h(n) \le \|c\|_\infty h\big(f(\min A)\big) \le 1$, it follows from the choice of g that $d \in \mathcal{C}_{\alpha_n}^g$. Observe that

$$\|d\|_\infty g(n) \le \|d\|_\infty h(n) \le \|c\|_\infty h\big(f(\min A)\big) \le 1.$$

Therefore, $d \in \mathcal{C}_\alpha^g$. Finally,

$$\|d\|_\infty g\big(f(\min A)\big) \le \|d\|_\infty h\big(f(\min A)\big) \le 1.$$

This proves the proposition. ■

Theorem 7 *Let E be a Banach space with a normalized 1-symmetric basis. Suppose E embeds into $C(\omega^{\omega^\alpha})$ for some $\alpha < \omega_1$. Then there exists a nondecreasing function $g\colon \mathbb{N} \to \mathbb{R}_+$, $\lim_{n\to\infty} g(n) = \infty$ such that $U_{E'} \cap \mathcal{C}_\alpha^g$ is an isomorphically norming subset of E'.*

Proof If $E = c_0$, the result is obvious; hence we may assume that $E \ne c_0$. Since E embeds into $C(\omega^{\omega^\alpha})$, any normalized 1-symmetric basis of E must be shrinking. According to Proposition 3, there exist an increasing function $f\colon \mathbb{N} \to \mathbb{N}$, and a finite constant K such that for all $x \in E$,

$$\|x\|_E \le K \sup\{\|x\chi_A\|_E : A \in \mathcal{S}_\alpha^f\}.$$

Given $x \in E \cap c_{00}$, $\|x\|_E = 1$, pick $A \in \mathcal{S}_\alpha^f$ such that

$$1 = \|x\|_E = \|x^*\|_E \le K\|x^*\chi_A\|_E.$$

Now choose $x' \in U_{E'}$ such that $1 \le K|\langle x^*, x'\chi_A\rangle|$. Let g be the function given by applying Proposition 6 with the function f, $\delta = 1/2$, and $\eta = 1/K$. It follows that there exists a y', $y' \in \mathcal{C}_\alpha^g$, $|y'| \le |x'\chi_A|$, and

$$|\langle x^*, y'\rangle| \ge \frac{1}{2}|\langle x^*, x'\chi_A\rangle| \ge \frac{1}{2K}.$$

Since $x' \in U_{E'}$ and $|y'| \le |x'\chi_A|$, we see that $y' \in U_{E'}$. Thus $y' \in U_{E'} \cap \mathcal{C}_\alpha^g$. Since $U_{E'} \cap \mathcal{C}_\alpha^g$ is a symmetric set, this proves that $U_{E'} \cap \mathcal{C}_\alpha^g$ is an isomorphically norming subset of E', as desired. ■

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2 A characterization theorem

In this section, we prove the converse of Theorem 7 (see Theorem 16). Given a nondecreasing function $g\colon \mathbb{N} \to \mathbb{R}_+$ such that $\lim_{n\to\infty} g(n) = \infty$, and a pointwise compact subset $\mathcal{F}$ of c_{00} such that $|x|\chi_A \in \mathcal{F}$ whenever $x \in \mathcal{F}$ and $A \in \mathcal{P}_{<\infty}(\mathbb{N})$, define

$$g(\mathcal{F}) = \Big\{ x = \sum_{i=1}^{n} x_i : x_i \in \mathcal{F}, (x_i) \text{ pairwise disjoint, and } g(n)\|x\|_\infty \le 1 \Big\}.$$

If $x = \sum_{i=1}^n x_i$ as in the foregoing definition, we say that $\sum_{i=1}^n x_i$ is an *admissible representation* of x.

Lemma 8 *The set $\mathcal{G} = g(\mathcal{F})$ is pointwise compact.*

Proof It suffices to show that $\mathcal{G}$ is pointwise closed. Let (x_j) be a sequence in $\mathcal{G}$ converging pointwise to some nonzero x. By the definition of $g(\mathcal{F})$, for each j, there exists a pairwise disjoint sequence $(x_{j,i})_{i=1}^{n_j}$ in $\mathcal{F}$ such that $x_j = \sum_{i=1}^{n_j} x_{j,i}$, and $g(n_j)\|x_j\|_\infty \le 1$. Now $\liminf \|x_j\|_\infty \ge \|x\|_\infty$. Therefore, $\limsup g(n_j) \le 1/\|x\|_\infty$. In particular, it follows that (n_j) is a bounded sequence. By using a subsequence, we may assume that there is a constant n such that $n_j = n$ for all j. As a result, we may represent x_j as

$$x_j = \sum_{i=1}^{n} x_{j,i}.$$

Since $x_{j,i} \in \mathcal{F}$ and $\mathcal{F}$ is compact, we may assume that $\lim_{j\to\infty} x_{j,i} = z_i \in \mathcal{F}$ exists. Then $x = \sum_{i=1}^n z_i$. It is clear that $(z_i)_{i=1}^n$ is a pairwise disjoint sequence. It follows from the above that $g(n)\|x\|_\infty \le 1$. Hence $x \in \mathcal{G}$, as required. ∎

The proof of Lemma 8 shows the following:

Lemma 9 *Let (x_j) be a sequence in $\mathcal{G}$ converging to a nonzero vector x. Suppose each x_j has an admissible representation $\sum_{i=1}^{n_j} x_{j,i}$. Then there exist $M \in \mathcal{P}_\infty(\mathbb{N})$ and $n \in \mathbb{N}$ such that $n_j = n$ for all $j \in M$, $z_i = \lim_{j\in M} x_{j,i}$ exists for $1 \le i \le n$, and $x = \sum_{i=1}^n z_i$ is an admissible representation of x.*

Definition 10 For $x \in \mathcal{F}$, define the *degree* of x by

$$\deg(x) = \sup\{\beta : x \in \mathcal{F}^{(\beta)}\}.$$

If α is an ordinal, it can be expressed uniquely in its Cantor canonical form $\alpha = \omega^{\alpha_1}\cdot m_1 + \cdots + \omega^{\alpha_k}\cdot m_k$, where $\alpha_1 > \cdots > \alpha_k$, and $m_1, \ldots, m_k \in \mathbb{N}$. We say that the α_i-th component of α is m_i, $1 \le i \le k$; whereas the γ-th component of α is 0 if $\gamma \notin \{\alpha_1, \ldots, \alpha_k\}$. If α and β are two ordinals, let $\alpha \oplus \beta$ be the unique ordinal each of whose γ-th component is the sum of the γ-th components of α and β. The operation '$\oplus$' may be extended to any finite number of ordinals in an obvious fashion. It is clear that $\alpha \oplus \beta < \omega_1$ if both α and β are countable. The proof of the next proposition is left to the reader.

Proposition 11 *Let $(\alpha_i)_{i=1}^n$ and $(\beta_i)_{i=1}^n$ be two sequences of ordinals. If $\alpha_i \geq \beta_i$ for each i, and $\bigoplus_{i=1}^n \beta_i \geq \bigoplus_{i=1}^n \alpha_i$, then $\alpha_i = \beta_i$ for every i.*

Proposition 12 *Suppose that x is a nonzero vector in $\mathcal{G}^{(\alpha)}$ for some $\alpha < \omega_1$. If $x = \sum_{i=1}^n z_i$ is an admissible representation of x, then $\bigoplus_{i=1}^n \deg(z_i) \geq \alpha$.*

Proof If $\alpha = 0$, there is nothing to prove. Suppose that the proposition is true for all ordinals less than some $\alpha < \omega_1$. First consider the case when α is a successor ordinal. Let x be a nonzero vector in $\mathcal{G}^{(\alpha)}$. There exists a sequence $(x_j) \subseteq \mathcal{G}^{(\alpha-1)} \setminus \{x\}$ that converges to x. By Lemma 9, we may assume that there exists $n \in \mathbb{N}$ such that each x_j has an admissible representation $x_j = \sum_{i=1}^n x_{j,i}$, that $\lim_j x_{j,i} = z_i$ for each i, and $x = \sum_{i=1}^n z_i$ is an admissible representation of x. By taking a subsequence if necessary, we may further assume that $\lim_j \deg(x_{j,i}) = \alpha_i$ exists for each i, and $\deg(x_{j,i}) \leq \alpha_i$ for all j. Since $x_{j,i} \to z_i$, $\deg(z_i) \geq \alpha_i$. Now $\bigoplus_{i=1}^n \deg(x_{j,i}) \geq \alpha - 1$ by the inductive hypothesis. Of course, $\bigoplus_{i=1}^n \deg(z_i) \geq \bigoplus_{i=1}^n \alpha_i \geq \alpha - 1$. Suppose that $\bigoplus_{i=1}^n \deg(z_i) = \alpha - 1 \leq \bigoplus_{i=1}^n \deg(x_{j,i})$. By Proposition 11, $\deg(z_i) = \deg(x_{j,i})$ for all i, j. But since $\lim_j x_{j,i} = z_i$, $\deg(x_{j,i}) = \deg(z_i)$ would imply that $x_{j,i} = z_i$ for all large j. Consequently, $x_j = x$ for all large j, which is a contradiction. Therefore $\bigoplus_{i=1}^n \deg(z_i) \geq \alpha$.

Finally, consider the case when α is a limit ordinal. Let $x \in \mathcal{G}^{(\alpha)} = \bigcap_{\beta<\alpha} \mathcal{G}^{(\beta)}$. Suppose $x = \sum_{i=1}^n z_i$ is an admissible representation of x. Since $x \in \mathcal{G}^{(\beta)}$ for all $\beta < \alpha$, $\bigoplus_{i=1}^n \deg(z_i) \geq \beta$ for all $\beta < \alpha$ by the inductive hypothesis. Consequently, $\bigoplus_{i=1}^n \deg(z_i) \geq \alpha$. ■

We now proceed to apply the foregoing analysis to the sets $\mathcal{C}_\alpha^g$ defined in Section 1.

Lemma 13 *Let $\alpha < \omega_1$, then $\mathcal{C}_\alpha^g$ is pointwise compact.*

Proof The assertion is clear for $\alpha = 0$. Suppose that the lemma has been proved for all ordinals less than some $\alpha < \omega_1$. If α is a successor ordinal, a glance at the definitions shows that $\mathcal{C}_\alpha^g = g(\mathcal{C}_{\alpha-1}^g)$. It follows from Lemma 8 that $\mathcal{C}_\alpha^g$ is pointwise compact.

Suppose that α is a limit ordinal, and let (α_n) be the sequence of ordinals used in defining $\mathcal{C}_\alpha^g$. Let (x_k) be a sequence in $\mathcal{C}_\alpha^g$ converging pointwise to a vector x. If $x = 0$, then certainly $x \in \mathcal{C}_\alpha^g$. Thus we may assume that $x \neq 0$. For each k, let $n_k \in \mathbb{N}$ be such that $x_k \in \mathcal{C}_{\alpha_{n_k}}^g$, and $g(n_k)\|x_k\|_\infty \leq 1$. Since $\liminf_k \|x_k\|_\infty \geq \|x\|_\infty$, $\limsup_k g(n_k) \leq 1/\|x\|_\infty$.

This implies that (n_k) is bounded. By taking a subsequence if necessary, we may assume that $n_k = n$ for all k. Then $x_k \in \mathcal{C}_{\alpha_n}^g$ for all k. Since $\mathcal{C}_{\alpha_n}^g$ is compact, $x \in \mathcal{C}_{\alpha_n}^g$. Moreover, as $g(n)\|x\|_\infty = g(n_k)\|x\|_\infty \leq 1$, we conclude that $x \in \mathcal{C}_\alpha^g$. ■

Lemma 14 *Suppose $\alpha < \omega_1$ is a limit ordinal, and let (α_n) be the sequence of ordinals used to define $\mathcal{C}_\alpha^g$. Then for any ordinal $\beta < \omega_1$,*

$$(\mathcal{C}_\alpha^g)^{(\beta)} \subseteq \{x : x \in (\mathcal{C}_{\alpha_n}^g)^{(\beta)} \textit{ for some } n \textit{ such that } g(n)\|x\|_\infty \leq 1\} \cup \{0\}.$$

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Proof The proof is by induction on β. If $\beta = 0$, there is nothing to prove. Suppose the lemma is true for some ordinal $\beta < \omega_1$. Let $x \in (\mathcal{C}^g_\alpha)^{(\beta+1)}$, $x \neq 0$. Then there exists a sequence $(x_k) \subseteq (\mathcal{C}^g_\alpha)^{(\beta)} \setminus \{x\}$ converging pointwise to x. By the inductive hypothesis, $x_k \in (\mathcal{C}^g_{\alpha_{n_k}})^{(\beta)}$ for some n_k such that $g(n_k)\|x_k\|_\infty \leq 1$. Now $\liminf \|x_k\|_\infty \geq \|x\|_\infty$. Therefore,

$$1 \geq \limsup_k g(n_k)\|x_k\|_\infty \geq \limsup_k g(n_k)\|x\|_\infty.$$

Hence (n_k) is bounded. By going to a subsequence, we may assume that $n_k = n$ for all k, and $g(n)\|x\| \leq 1$. Since $(x_k) \subseteq (\mathcal{C}^g_{\alpha_n})^{(\beta)} \setminus \{x\}$ and (x_k) converges to x, $x \in (\mathcal{C}^g_{\alpha_n})^{(\beta+1)}$.

Suppose $\beta < \omega_1$ is a limit ordinal and the lemma holds for all $\beta' < \beta$. Let $x \in (\mathcal{C}^g_\alpha)^{(\beta)}$, $x \neq 0$, and let (β_n) be a sequence of ordinals strictly increasing to β. Choose a sequence (x_n) such that $x_n \in (\mathcal{C}^g_\alpha)^{(\beta_n)}$ for each n, and $\lim_{n\to\infty} x_n = x$ in the topology of pointwise convergence. By the inductive hypothesis, $x_n \in (\mathcal{C}^g_{\alpha_{k_n}})^{(\beta_n)}$, where $g(k_n)\|x_n\|_\infty \leq 1$. As before, we may assume without loss of generality that $k_n = k$ and $g(k)\|x\|_\infty \leq 1$. Then $x_n \in (\mathcal{C}^g_{\alpha_k})^{(\beta_n)}$ for all n. Since $\lim_{n\to\infty} x_n = x$ and $(\beta_n) \nearrow \beta$, $x \in (\mathcal{C}^g_{\alpha_k})^{(\beta)}$. This completes the induction. ■

Proposition 15 *If* $\alpha < \omega_1$, *then* $(\mathcal{C}^g_\alpha)^{(\omega^\alpha)} \subseteq \{0\}$.

Proof It is easy to verify that the proposition is true for $\alpha = 0$. We now suppose the proposition has been proved for all $\alpha < \beta$, where $\beta < \omega_1$. Consider first the case when β is a successor. Let $x \in (\mathcal{C}^g_\beta)^{(\omega^\beta)} = \big(g(\mathcal{C}^g_{\beta-1})\big)^{(\omega^\beta)}$, $x \neq 0$. Applying Proposition 12 with $\mathcal{F} = \mathcal{C}^g_{\beta-1}$, x has an admissible representation $x = \sum_{i=1}^n z_i$ such that $\bigoplus_{i=1}^n \deg(z_i) \geq \omega^\beta$. But by the inductive hypothesis, $(\mathcal{C}^g_{\beta-1})^{(\omega^{\beta-1})} \subseteq \{0\}$; hence $\deg(z_i) \leq \omega^{\beta-1}$ for all i. Consequently,

$$\omega^\beta \leq \bigoplus_{i=1}^n \deg(z_i) \leq \omega^{\beta-1} \cdot n,$$

which is a contradiction.

Suppose that β is a limit ordinal. Let (β_n) be the sequence used to define $\mathcal{C}^g_\beta$. By Lemma 14,

$$(\mathcal{C}^g_\beta)^{(\omega^\beta)} \subseteq \{x : x \in (\mathcal{C}^g_{\beta_n})^{(\omega^\beta)} \text{ for some } n \text{ such that } g(n)\|x\|_\infty \leq 1\} \cup \{0\}.$$

But $(\mathcal{C}^g_{\beta_n})^{(\omega^\beta)} = \varnothing$ by the inductive hypothesis. Hence $(\mathcal{C}^g_\beta)^{(\omega^\beta)} \subseteq \{0\}$. ■

Theorem 16 *Let E be a Banach space with a normalized* 1*-symmetric basis. Then E embeds into $C(\omega^{\omega^\alpha})$ for some $\alpha < \omega_1$ if and only if there exists a nondecreasing function $g\colon \mathbb{N} \to \mathbb{R}_+$, $\lim_{n\to\infty} g(n) = \infty$, such that $U_{E'} \cap \mathcal{C}^g_\alpha$ is an isomorphically norming subset of E'.*

Proof Suppose that such a function g exists. Since $U_{E'} \cap \mathcal{C}_\alpha^g$ is pointwise compact, and $(U_{E'} \cap \mathcal{C}_\alpha^g)^{(\omega^\alpha)} \subseteq (\mathcal{C}_\alpha^g)^{(\omega^\alpha)} \subseteq \{0\}$ by Proposition 15, $U_{E'} \cap \mathcal{C}_\alpha^g$ is homeomorphic to an ordinal interval $[0, \beta]$ for some $\beta \leq \omega^{\omega^\alpha}$. Now $U_{E'} \cap \mathcal{C}_\alpha^g$ is isomorphically norming. Therefore,

$$E \hookrightarrow C(U_{E'} \cap \mathcal{C}_\alpha^g) \hookrightarrow C(\beta) \hookrightarrow C(\omega^{\omega^\alpha}).$$

The converse is precisely Theorem 7 in Section 1. ■

3 A family of examples

The aim of this section is to construct a full complement of mutually non-isomorphic 1-symmetric sequence spaces which embed into $C(\alpha)$ for some $\alpha < \omega_1$. Let us define the following terms and operations on finite sequences of natural numbers. If $\mathsf{m} = (m_1, \ldots, m_i)$ and $\mathsf{n} = (n_1, \ldots, n_j)$ are finite sequences of natural numbers, let

1. $\varphi(\mathsf{m}) = m_1$ (the leading term of the sequence),
2. $\mathsf{m} \smile \mathsf{n} = (m_1, \ldots, m_i, n_1, \ldots, n_j)$ (the concatenation of m and n).

Also, we say that $\mathsf{m} \ll \mathsf{n}$ if $2m_i \leq n_1$, and that m is *at least doubling* if $2m_l \leq m_{l+1}$, $1 \leq l < i$. Now define $\mathcal{M}_1 = \{(m) : m \in \mathrm{N}\}$. For $1 \leq \alpha < \omega_1$, let

$$\mathcal{M}_{\alpha+1} = \{\mathsf{m}_1 \smile \cdots \smile \mathsf{m}_k : \mathsf{m}_1, \ldots, \mathsf{m}_k \in \mathcal{M}_\alpha,\ \mathsf{m}_1 \ll \cdots \ll \mathsf{m}_k,\ \text{and } k \leq \varphi(\mathsf{m}_1)\}.$$

If $\alpha < \omega_1$ and α is a limit ordinal, recall the sequence (α_n) chosen in the introduction. Define

$$\mathcal{M}_\alpha = \{\mathsf{m} : \text{there exists } n \in \mathrm{N}, n \leq \varphi(\mathsf{m}) \text{ such that } \mathsf{m} \in \mathcal{M}_{\alpha_n}\}.$$

It is easily verified that any $\mathsf{m} \in \mathcal{M}_\alpha$, $1 \leq \alpha < \omega_1$, is at least doubling.

Definition 17 Let $1 \leq \alpha < \omega_1$, if $\mathsf{m} = (m_1, \ldots, m_l)$ is a finite sequence of integers, we let $\mathcal{X}_\mathsf{m}$ be the set of all $x \in c_{00}$ such that there exist pairwise disjoint sets $A_1, \ldots, A_l \subseteq \mathrm{N}$, $|A_i| = m_i$, $1 \leq i \leq l$, and

$$x = \sum_{i=1}^{l} \frac{1}{\sqrt{m_i}} \chi_{A_i}.$$

Moreover, define $\mathcal{G}_\alpha = \cup\{\mathcal{X}_\mathsf{m} : \mathsf{m} \in \mathcal{M}_\alpha\}$.

Lemma 18 *Let $g \colon \mathrm{N} \to \mathrm{R}_+$ be defined by $g(n) = \sqrt{n}$. Then $\mathcal{G}_\alpha \subseteq \mathcal{C}_\alpha^g$ for $1 \leq \alpha < \omega_1$.*

Proof The proof is by induction on α. Suppose $x \in \mathcal{G}_1$. There exist $(m) \in \mathcal{M}_1$, $A \subseteq \mathrm{N}$, $|A| = m$ such that $x = \frac{1}{\sqrt{m}} \chi_A$. Let $x_i = \frac{1}{\sqrt{m}} \chi_{\{n_i\}}$, $1 \leq i \leq m$, where $A = \{n_1, \ldots, n_m\}$. Then $x = \sum_{i=1}^{m} x_i$ and $x_i \in \mathcal{C}_0^g$. Moreover, $g(m)\|x\|_\infty = \sqrt{m}\, \frac{1}{\sqrt{m}} = 1$. Hence $x \in \mathcal{C}_1^g$.

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Suppose now that $\mathcal{G}_\alpha \subseteq \mathcal{C}^g_\alpha$ for some $1 \le \alpha < \omega_1$. Let $x \in \mathcal{G}_{\alpha+1}$. There exist $\mathsf{m} = (m_1, \dots, m_l) \in \mathcal{M}_{\alpha+1}$, and pairwise disjoint sets $A_1, \dots, A_l \subseteq \mathbb{N}$, $|A_i| = m_i$, such that

$$x = \sum_{i=1}^{l} \frac{1}{\sqrt{m_i}} \chi_{A_i}$$

Since $\mathsf{m} \in \mathcal{M}_{\alpha+1}$, we may write $\mathsf{m} = \mathsf{r}_1 \smile \cdots \smile \mathsf{r}_n$ for some $\mathsf{r}_1, \dots, \mathsf{r}_n \in \mathcal{M}_\alpha$ such that $n \le \varphi(\mathsf{r}_1) = m_1$. Let $I_j = \{i : m_i \text{ is a coordinate of } \mathsf{r}_j\}$. Then since $\mathsf{r}_j \in \mathcal{M}_\alpha$, $x_j = \sum_{i \in I_j} \frac{1}{\sqrt{m_i}} \chi_{A_i} \in \mathcal{G}_\alpha$. Now $(x_j)_{j=1}^n$ is pairwise disjoint and $x_j \in \mathcal{C}^g_\alpha$ by the inductive hypothesis. Note that $x = \sum_{i=1}^n x_j$ and $\|x\|_\infty = \frac{1}{\sqrt{m_1}}$. Therefore, $g(n)\|x\|_\infty = \frac{g(n)}{\sqrt{m_1}} \le \frac{g(n)}{\sqrt{n}} = 1$. Hence $x \in \mathcal{C}^g_{\alpha+1}$.

Finally, suppose that $\alpha < \omega_1$ is a limit ordinal and $\mathcal{G}_\beta \subseteq \mathcal{C}^g_\beta$ for all $\beta < \alpha$. Let (α_n) be the sequence used in defining $\mathcal{G}_\alpha$ and $\mathcal{C}^g_\alpha$. Suppose $x \in \mathcal{G}_\alpha$, then there exists $\mathsf{m} = (m_1, \dots, m_l) \in \mathcal{M}_\alpha$ such that $x \in \mathcal{X}_{\mathsf{m}}$. Since $\mathsf{m} \in \mathcal{M}_\alpha$, there exists $n \le \varphi(\mathsf{m})$ such that $\mathsf{m} \in \mathcal{M}_{\alpha_n}$. Thus $x \in \mathcal{G}_{\alpha_n}$ and consequently, $x \in \mathcal{C}^g_{\alpha_n}$. As $n \le \varphi(\mathsf{m}) = m_1$, we see that $g(n)\|x\|_\infty \le g(m_1)\frac{1}{\sqrt{m_1}} = 1$. Hence $x \in \mathcal{C}^g_\alpha$, as required ■

Lemma 19 *Given $1 \le \alpha < \omega_1$, define a norm on c_{00} by*

$$\|y\|_\alpha = \sup\{\langle |y|, x\rangle : x \in \mathcal{G}_\alpha\}.$$

Then $\|\cdot\|_\alpha$ is a 1-symmetric norm on c_{00}, and $\|(1,0,0,\dots)\|_\alpha = 1$.

Proof By definition, $\mathcal{G}_\alpha$ is invariant under permutation of the coordinates. Therefore, $\|\cdot\|_\alpha$ is 1-symmetric. Also, every element of $\mathcal{G}_\alpha$ has ℓ^∞-norm at most 1. Hence $\|(1,0,0,\dots)\|_\alpha \le 1$. On the other hand, the singleton (1) lies in $\mathcal{M}_\alpha$ for every $1 \le \alpha < \omega_1$. Thus $(1,0,0,\dots) \in \mathcal{G}_\alpha$. Consequently,

$$\|(1,0,0,\dots)\|_\alpha \ge \langle (1,0,0,\dots), (1,0,0,\dots)\rangle = 1. \qquad ■$$

Lemma 20 *Given $n \in \mathbb{N}$, let $1_n = (\overbrace{1,\dots,1}^{n},0,0,\dots)$. Then for every $1 \le \alpha < \omega_1$, $\|1_n\|_\alpha \le 5\sqrt{n}$.*

Proof Suppose $x \in \mathcal{G}_\alpha$. There exist $\mathsf{m} = (m_1, \dots, m_l) \in \mathcal{M}_\alpha$, and pairwise disjoint sets $A_1, \dots, A_l \subseteq \mathbb{N}$, $|A_i| = m_i$, such that

$$x = \sum_{i=1}^{l} \frac{1}{\sqrt{m_i}} \chi_{A_i}.$$

Choose l_1 such that $m_1 + \cdots + m_{l_1 - 1} < n \le m_1 + \cdots + m_{l_1}$, and let $k = n - (m_1 + \cdots + m_{l_1-1})$. Then

$$\langle 1_n, x\rangle \le \langle 1_n, x^*\rangle \le \sqrt{m_1} + \sqrt{m_2} + \cdots + \sqrt{m_{l_1-1}} + \frac{k}{\sqrt{m_{l_1}}}.$$

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Since m is at least doubling,

$$\begin{aligned}\langle 1_n, x\rangle &\le \sqrt{m_{l_1-1}}\left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2^2}}+\cdots\right)+\frac{k}{\sqrt{m_{l_1}}}\\ &\le \frac{\sqrt{2n}}{\sqrt{2}-1}+\frac{\sqrt{kn}}{\sqrt{m_{l_1}}}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\langle 1_n, x\rangle}{\sqrt{n}} &\le \frac{\sqrt{2}}{\sqrt{2}-1}+\sqrt{\frac{k}{m_{l_1}}}\\ &\le \frac{\sqrt{2}}{\sqrt{2}-1}+1\le 5.\end{aligned}$$

As a result, $\|1_n\|_\alpha = \sup\{\langle 1_n, x\rangle : x \in \mathcal{G}_\alpha\} \le 5\sqrt{n}$. ■

The next proposition is due to Odell, Tomczak-Jaegermann, and Wagner [7, Proposition 3.2a].

Proposition 21 *Given $\beta \le \alpha < \omega_1$, and an increasing function $f : \mathrm{N} \to \mathrm{N}$, there exists $i \in \mathrm{N}$ such that $A \in \mathcal{S}_\alpha^f$ whenever $A \in \mathcal{S}_\beta^f$ and $\min A \ge i$.*

Proposition 22 *Let $f : \mathrm{N} \to \mathrm{N}$ be an increasing function. Given any $k \ge 2$, and $1 \le \beta < \alpha < \omega_1$, there exist $\mathsf{m} \in \mathcal{M}_\alpha$, $\varphi(\mathsf{m}) \ge k$, $x \in \mathcal{X}_{\mathsf{m}}$, $\min(\operatorname{supp} x) \ge k$, and $y \in c_{00}$ such that $\min(\operatorname{supp} y) \ge k$, $\langle y, x\rangle \ge k$, $\|y\|_\gamma \le 5\langle y, x\rangle$, and $\|y\chi_A\|_\gamma \le 10$ for $1 \le \gamma < \omega_1$ and all $A \in \mathcal{S}_\beta^f$.*

Proof The proof is by induction on α. Consider $\alpha = 2$ and $\beta = 1$. Pick $m_1, \ldots, m_k$ such that $m_1 \ge k$ and $m_{i+1} \ge \max\{2m_i, f(k + m_1 + m_2 + \cdots + m_i)\}$ for $1 \le i < k$. Then $\mathsf{m} = (m_1, \ldots, m_k) \in \mathcal{M}_2$, and $\varphi(\mathsf{m}) \ge k$. Furthermore,

$$x = \Bigl(\overbrace{0,\ldots,0}^{k}, \overbrace{\frac{1}{\sqrt{m_1}},\ldots,\frac{1}{\sqrt{m_1}}}^{m_1},\ldots,\overbrace{\frac{1}{\sqrt{m_k}},\ldots,\frac{1}{\sqrt{m_k}}}^{m_k}\Bigr) \in \mathcal{X}_{\mathsf{m}}.$$

Now let $y = x$. Then $y \in c_{00}$ and $\min(\operatorname{supp} x) = \min(\operatorname{supp} y) \ge k$. Computing directly, we have $\langle y, x\rangle = k$. Applying Lemma 20,

$$\begin{aligned}\|y\|_\gamma &\le \Bigl\|\Bigl(\overbrace{\frac{1}{\sqrt{m_1}},\ldots,\frac{1}{\sqrt{m_1}}}^{m_1}\Bigr)\Bigr\|_\gamma + \cdots + \Bigl\|\Bigl(\overbrace{\frac{1}{\sqrt{m_k}},\ldots,\frac{1}{\sqrt{m_k}}}^{m_k}\Bigr)\Bigr\|_\gamma\\ &\le 5k = 5\langle y, x\rangle.\end{aligned}$$

If $A \in \mathcal{S}_1^f$, choose i such that $k + m_1 + \cdots + m_{i-1} < \min A \leq k + m_1 + \cdots + m_i$, then

$$|A| \leq f(\min A) \leq f(k + m_1 + \cdots + m_i) \leq m_{i+1}.$$

Hence

$$\begin{aligned}\|y\chi_A\|_\gamma &\leq \Big\| \Big(\overbrace{\frac{1}{\sqrt{m_i}}, \dots, \frac{1}{\sqrt{m_i}}}^{m_i} \Big) \Big\|_\gamma + \Big\| \Big(\overbrace{\frac{1}{\sqrt{m_{i+1}}}, \dots, \frac{1}{\sqrt{m_{i+1}}}}^{|A|} \Big) \Big\|_\gamma \\ &\leq 5 + 5\sqrt{\frac{|A|}{m_{i+1}}} \leq 5 + 5 = 10.\end{aligned}$$

Suppose the proposition holds for all ordinals less than or equal to some $\alpha \geq 2$; let us prove it for $\alpha + 1$. Say $1 \leq \beta < \alpha + 1$. If $\beta < \alpha$, there is nothing to prove since $\mathcal{M}_\alpha \subseteq \mathcal{M}_{\alpha+1}$. So we may assume without loss of generality that $\beta = \alpha$. If α is a successor ordinal, apply the inductive hypothesis repeatedly to pick sequences $(\mathsf{m}_p)_{p=1}^k \subseteq \mathcal{M}_\alpha$, and $(x_p)_{p=1}^k$, $(y_p)_{p=1}^k \subseteq c_{00}$ such that

1. $\varphi(\mathsf{m}_1) \geq k$, and $\mathsf{m}_1 \ll \cdots \ll \mathsf{m}_k$,
2. $x_p \in \mathcal{X}_{\mathsf{m}_p}$, $1 \leq p \leq k$,
3. $\{k\} \leq \operatorname{supp} x_1 \cup \operatorname{supp} y_1 < \cdots < \operatorname{supp} x_k \cup \operatorname{supp} y_k$,
4. $\langle y_p, x_p \rangle \geq k$, and $\|y_p\|_\gamma \leq 5\langle y_p, x_p\rangle$, $1 \leq p \leq k$,
5. $\|y_p\chi_A\|_\gamma \leq 10$ for $1 \leq \gamma < \omega_1$ and all $A \in \mathcal{S}_{\alpha-1}^f$,
6.

$$10\, f\big(\max(\operatorname{supp} y_p)\big) \sum_{q=p+1}^{k} \frac{1}{\langle y_q, x_q\rangle} \leq 5 \quad \text{for } 1 \leq p < k.$$

Let $\mathsf{m} = \mathsf{m}_1 \smile \cdots \smile \mathsf{m}_k$. Because of condition (3), $\mathsf{m} \in \mathcal{M}_{\alpha+1}$, and $\varphi(\mathsf{m}) \geq k$. Also, $x = x_1 + \cdots + x_k \in \mathcal{X}_{\mathsf{m}}$, $\min(\operatorname{supp} x) \geq k$. Define

$$y = \frac{y_1}{\langle y_1, x_1\rangle} + \cdots + \frac{y_k}{\langle y_k, x_k\rangle}.$$

Then $y \in c_{00}$, $\min(\operatorname{supp} y) \geq k$, and $\langle y, x\rangle \geq k$. Furthermore,

$$\|y\|_\gamma \leq \frac{\|y_1\|_\gamma}{\langle y_1, x_1\rangle} + \cdots + \frac{\|y_k\|_\gamma}{\langle y_k, x_k\rangle} \leq 5k \leq 5\langle y, x\rangle$$

for $1 \leq \gamma < \omega_1$. Suppose $A \in \mathcal{S}_\alpha^f$. Then $A = A_1 \cup \cdots \cup A_l$, where $A_1 < \cdots < A_l$, $A_1, \dots, A_l \in \mathcal{S}_{\alpha-1}^f$ and $l \leq f(\min A)$. Choose i so that

$$\max(\operatorname{supp} y_{i-1}) < \min A \leq \max(\operatorname{supp} y_i).$$

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For $1 \le \gamma < \omega_1$,

$$\begin{aligned}
\|y\chi_A\|_\gamma &\le \frac{\|y_i\|_\gamma}{\langle y_i, x_i\rangle} + \sum_{q=i+1}^{k}\sum_{p=1}^{l} \frac{\|y_q\chi_{A_p}\|_\gamma}{\langle y_q, x_q\rangle} \\
&\le 5 + \sum_{q=i+1}^{k}\sum_{p=1}^{l} \frac{10}{\langle y_q, x_q\rangle} \quad \text{by conditions (3) and (3)} \\
&= 5 + 10l \sum_{q=i+1}^{k} \frac{1}{\langle y_q, x_q\rangle} \\
&\le 5 + 10\, f\big(\max(\operatorname{supp} y_i)\big) \sum_{q=i+1}^{k} \frac{1}{\langle y_q, x_q\rangle} \\
&\le 10 \quad \text{by condition (3).}
\end{aligned}$$

Let us turn to the case when α is a limit ordinal. Let (α_n) be the sequence used to define $\mathcal{S}_\alpha^f$ and $\mathcal{G}_\alpha$. Suppose $k \in \mathbb{N}$ is given. Pick sequences $(i_p)_{p=1}^k$, $(\mathsf{m}_p)_{p=1}^k$, $(x_p)_{p=1}^k$, and $(y_p)_{p=1}^k$ as follows. Let $i_1 = 2$. By the inductive hypothesis, there exist $\mathsf{m}_1 \in \mathcal{M}_{\alpha_2}$, $\varphi(\mathsf{m}_1) \ge k$, $x_1 \in \mathcal{X}_{\mathsf{m}_1}$, $\min(\operatorname{supp} x_1) \ge k$, and $y_1 \in c_{00}$ such that $\min(\operatorname{supp} y_1) \ge k$, $\langle y_1, x_1\rangle \ge k$, $\|y_1\|_\gamma \le 5\langle y_1, x_1\rangle$, and $\|y_1\chi_A\|_\gamma \le 10$ for $1 \le \gamma < \omega_1$ and all $A \in \mathcal{S}_{\alpha_1}^f$. Suppose all four sequences have been chosen up to p, where $1 \le p < k$. By Proposition 21, there exists $i_{p+1} > f\big(\max(\operatorname{supp} x_p)\big)$ such that $A \in \mathcal{S}^f_{\alpha_{f(\max(\operatorname{supp} x_p))}}$ whenever $A \in \mathcal{S}_{\alpha_j}^f$ for some $j \le f\big(\max(\operatorname{supp} x_p)\big)$ and $\min A \ge i_{p+1}$. By the inductive hypothesis (applied to the ordinals $\alpha_{f(\max(\operatorname{supp} x_p))} < \alpha_{i_{p+1}}$), pick

1. $\mathsf{m}_{p+1} \in \mathcal{M}_{\alpha_{i_{p+1}}}$, $\mathsf{m}_p \ll \mathsf{m}_{p+1}$, $\varphi(\mathsf{m}_{p+1}) \ge i_{p+1}$,
2. $x_{p+1} \in \mathcal{X}_{\mathsf{m}_{p+1}}$, $\min(\operatorname{supp} x_{p+1}) \ge i_{p+1}$, and
3. $y_{p+1} \in c_{00}$,

such that $\langle y_{p+1}, x_{p+1}\rangle \ge 2k$, $\|y_{p+1}\|_\gamma \le 5\langle y_{p+1}, x_{p+1}\rangle$, and $\|y_{p+1}\chi_A\|_\gamma \le 10$ for $1 \le \gamma < \omega_1$ and all $A \in \mathcal{S}^f_{\alpha_{f(\max(\operatorname{supp} x_p))}}$. Since $\mathsf{m}_p \in \mathcal{M}_{\alpha_{i_p}}$, and $\varphi(\mathsf{m}_p) \ge i_p$, $\mathsf{m}_p \in \mathcal{M}_\alpha$, $1 \le p \le k$. Now $\mathsf{m}_1 \ll \cdots \ll \mathsf{m}_k$, and $k \le \varphi(\mathsf{m}_1)$. Hence $\mathsf{m} = \mathsf{m}_1 \smile \cdots \smile \mathsf{m}_k \in \mathcal{M}_{\alpha+1}$. Define $x = x_1 + \cdots + x_k$. Then $x \in \mathcal{X}_{\mathsf{m}}$ and $\min(\operatorname{supp} x) \ge k$. Let

$$y = \frac{y_1\chi_{\operatorname{supp} x_1}}{\langle y_1, x_1\rangle} + \cdots + \frac{y_k\chi_{\operatorname{supp} x_k}}{\langle y_k, x_k\rangle}.$$

Then $y \in c_{00}$, $\min(\operatorname{supp} y) \ge k$, and $\langle y, x\rangle = k$. Furthermore,

$$\|y\|_\gamma \le \frac{\|y_1\|_\gamma}{\langle y_1, x_1\rangle} + \cdots + \frac{\|y_k\|_\gamma}{\langle y_k, x_k\rangle} \le 5k = 5\langle y, x\rangle$$

for $1 \le \gamma < \omega_1$. Suppose $A \in \mathcal{S}_\alpha^f$. Then $A \in \mathcal{S}_{\alpha_r}^f$ for some $r \le f(\min A)$. Choose p such that

$$\max(\operatorname{supp} x_{p-1}) < \min A \le \max(\operatorname{supp} x_p).$$

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If $p < q \le k$, let $A_q = A \cap \operatorname{supp} x_q \cap \operatorname{supp} y_q$. Then $A_q \in \mathcal{S}^f_{\alpha_r}$. Note that

$$r \le f(\min A) \le f\big(\max(\operatorname{supp} x_p)\big) \le f\big(\max(\operatorname{supp} x_{q-1})\big)$$

and $\min A_q \ge \min(\operatorname{supp} x_q) \ge i_q$. By the choice of i_q, we see that $A_q \in \mathcal{S}^f_{\alpha_{f(\max(\operatorname{supp} x_{q-1}))}}$. Hence $\|y_q \chi_{A_q}\|_\gamma \le 10$ for $1 \le \gamma < \omega_1$. Therefore,

$$\|y\chi_A\|_\gamma \le \frac{\|y_p\|_\gamma}{\langle y_p, x_p\rangle} + \sum_{q=p+1}^{k} \frac{\|y_q\chi_{A_q}\|_\gamma}{\langle y_q, x_q\rangle} \le 5 + \sum_{q=p+1}^{k} \frac{10}{\langle y_q, x_q\rangle} \le 10$$

since $\langle y_q, x_q\rangle \ge 2k$ for $1 < q \le k$.

Finally, suppose $\alpha_0 < \omega_1$ is a limit ordinal and the proposition holds for all $\alpha < \alpha_0$. Let (α_n) be the sequence used to define $\mathcal{M}_{\alpha_0}$. Let $k \in \mathbb{N}$, and $1 \le \beta < \alpha_0 < \omega_1$ be given. Choose n_0 such that $\beta < \alpha_{n_0}$. There exist $\mathsf{m} \in \mathcal{M}_{\alpha_{n_0}}$, $\varphi(\mathsf{m}) \ge k$, $x \in \mathcal{X}_{\mathsf{m}}$, $\min(\operatorname{supp} x) \ge \max\{k, n_0\}$, and $y \in c_{00}$ with $\min(\operatorname{supp} y) \ge k$ such that $\langle y, x\rangle \ge k$, $\|y\|_\gamma \le 5\langle y, x\rangle$, $\|y\chi_A\|_\gamma \le 10$ for $1 \le \gamma < \omega_1$ and all $A \in \mathcal{S}^f_\beta$. Since $n_0 \le \varphi(\mathsf{m})$, $\mathsf{m} \in \mathcal{M}_{\alpha_0}$. ■

Theorem 23 *For $1 \le \alpha < \omega_1$, let E_α be the completion of c_{00} with respect to the norm $\|\cdot\|_\alpha$. Then E_α has a 1-symmetric basis, E_α embeds into $C(\omega^{\omega^\alpha})$, but E_α does not embed into $C(\omega^{\omega^\beta})$ for any $\beta < \alpha$.*

Proof By Lemma 19, the coordinate unit vectors form a 1-symmetric basis of E_α. Note that $\mathcal{G}_\alpha \subseteq U_{E'_\alpha}$ is a norming subset of E'_α. By Lemma 18, $\mathcal{G}_\alpha \subseteq \mathcal{C}^g_\alpha$, where $g(n) = \sqrt{n}$. Therefore, E_α embeds into $C(\omega^{\omega^\alpha})$ by Theorem 16. Suppose $\beta < \alpha$ and E_α embeds into $C(\omega^{\omega^\beta})$. By Proposition 3, there exist an increasing function $f\colon \mathbb{N} \to \mathbb{N}$, and $K < \infty$ such that for all $y \in E_\alpha$,

$$\|y\|_\alpha \le K \sup\{\|y\chi_A\|_\alpha : A \in \mathcal{S}^f_\beta\}. \tag{1}$$

By Proposition 22, there exist $\mathsf{m} \in \mathcal{M}_\alpha$, $x \in \mathcal{X}_{\mathsf{m}}$, and $y \in c_{00}$ such that $\langle y, x\rangle > 10K$, and $\|y\chi_A\|_\alpha \le 10$ for all $A \in \mathcal{S}^f_\beta$. Since $x \in \mathcal{G}_\alpha$,

$$\|y\|_\alpha \ge \langle y, x\rangle > 10K \ge K \sup\{\|y\chi_A\|_\alpha : A \in \mathcal{S}^f_\beta\},$$

contrary to (1). ■

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