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EQUIVALENCE OF LIPSCHITZ STRUCTURES

BY

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1. **Introduction.** In 1962, V. Ju. Sandberg gave a definition of a Lipschitz structure more general than that of a metric space. In [1], the author gave alternate definitions of a Lipschitz structure, one in terms of a family of pseudometrics and the other in terms of sequences of entourages. These definitions were proved equivalent to each other. In 1965, Sandberg gave conditions under which his Lipschitz structure was generated by a metric.

In this note, we modify Sandberg's conditions and show that the modified structures correspond in a natural way to the author's Lipschitz structures. (The modifications give an equivalent structure, but are possibly clearer than the original formulation.)

2. **Notation and definitions.** Given a space X, an entourage U is any subset of $X\times X$ which contains Δ(=the diagonal). We define a partial order for a sequences of entourages by $\{u_n\}_{n=0}^{\infty}\leq\{v_n\}_{n=0}^{\infty}$ if $u_n\subseteq v_n$ for each $n=0, 1, 2, \ldots$. We omit the indices when no confusion will occur. An entourage can be considered as a relation, so that u^k for a positive integer k and u^{-1} have the obvious meanings.

2.1 Definition. Let X be a space. An E-Lipschitz structure for X is a filter U of sequences of entourages (partially ordered by $\leq$) with a basis U' satisfying

$$\text{(E1) If } \{v_n\}\in U', \quad \text{then } v_{n+1}^2\subseteq v_n=v_n^{-1} \quad \text{for each } n,$$

and

$$\text{(E2) If } \{v_n\}\in U', \quad \text{then } \{v_{n+1}\}\in U'.$$

The pair (X, U) will be called on E-Lipschitz space. If $x, y\in X$ with $x\neq y$ implies that for some $\{u_n\}\in U$ $(x, y)\notin u_0$, the structure is called Hausdorff, or separated.

For a set X, let $\mathfrak{X}=\{\{a, b\}\mid a, b\in X\}$. Note that $\{a, a\}=\{a\}\in\mathfrak{X}$ for each $a\in X$. We denote by $\mathfrak{F}$ the family of functions from $\mathfrak{X}$ to N, the nonnegative integers, which are zero except at a finite number of points. For brevity, we denote $f(\{x, y\})$ by $f\{x, y\}$. The function with values $f\{a_i, b_i\}=n_i$ for $i=1, 2, \ldots k$ and $f\{a, b\}=0$ if $\{a, b\}\neq\{a_i, b_i\}$ will be denoted by $\sum_{i=1}^{k} n_i\{a_i, b_i\}$. (We assume of course that the $\{a_i, b_i\}$ are distinct.) Addition and multiplication by integers is pointwise. If F, $G\subseteq\mathfrak{F}$, then $F+G=\{f+g\mid f\in F, g\in G\}$. We write $f\leq g$ whenever $f\{a, b\}\leq g\{a, b\}$ for all $\{a, b\}\in\mathfrak{X}$.

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2.2 Definition. A nonempty subset F of $\mathfrak{F}$ will be called admissible if it satisfies

(A1) If $f \in F$ and $g \leq f$, then $g \in F$.
(A2) If $f+\{a, b\}+\{b, c\} \in F$, then $f+\{a, c\} \in F$.
(A3) If $f \in F$, then $f+\{a, a\} \in F$.

We define a natural partial order on the collection of subsets of $\mathfrak{F}$ by $F \underline{\alpha} G$ iff for all $f \in F$, there exists $g \in G$ such that $f \leq g$.

2.3 Definition. An S-Lipschitz structure on X is a filter Φ of subsets of $\mathfrak{F}$ satisfying

(S1) Φ has a basis of admissible sets
(S2) If $F \in \Phi$, there exists $G \in \Phi$ such that $G+G \underline{\alpha} F$.

If for all $x \neq y$ in X, there exists $F \in \Phi$ such that $\{x, y\} \notin F$, we say that Φ is separated.

2.4 Definition. A subset F of $\mathfrak{F}$ is called convex if $f+g \in F$ whenever $2f$ and $2g \in F$. It is called monotone of $f+g \in F$ implies that $2f$ or $2g \in F$.

Definition 2.3 appears superficially to be different from Sandberg's original definition, but is easily seen to be equivalent. The definition 2.4 appears in [4].

For each pseudometric d on a space X and each positive real number r, set

$$F(d, r) = \left\{ f \in \mathfrak{F} \,\middle|\, \sum_{\{x,y\} \in X} f\{x, y\}\, d(x, y) < r \right\}.$$

For $F \in \mathfrak{F}$, $\frac{1}{2}F = \{\{x, y\} \in X \mid 2\{x, y\} \in F\}$.

3. **Equivalence of the Lipschitz structures.** Sandberg in [4] proved that if there existed an admissible, convex, monotone set F such that $\{\frac{1}{2}^n F\}_{n=0}^{\infty}$ is a basis for the separated Lipschitz structure Φ, then there was a metric d on X such that $\{F(d, r) \mid r > 0\}$ formed a basis for Φ.

We prove the following extension.

3.1 Theorem. *Each E-Lipschitz structure on a space X induces in a natural way an S-Lipschitz structure on X. The S-Lipschitz structure has for a basis a family of monotone, convex, admissible sets. Conversely, each S-Lipschitz structure on X with a basis of monotone, convex, admissible sets induces in a natural way on E-Lipschitz structure on X. Moreover the process is reflexive.*

The theorem will follow from a series of lemmas. We present a complete proof (not utilizing Sandberg's results), since Sandberg's proof seems excessively complicated, and conceals the motivation. Since each metric determines a set $F(d, \frac{1}{2})$ in $\mathfrak{F}$, one might hope to recover this metric from the space in a constructive manner. This is in fact what we do.

It should be observed that an E-Lipschitz structure is separated iff the corresponding monotone, convex S-Lipschitz structure is separated.

Let F be a convex, admissible set. We generate an entourage sequence from F by the formula $(x, y) \in u_n$ iff $2^n\{x, y\} \in F$.

3.2 PROPOSITION. *The entourage sequence* $\{u_n\}_{n=0}^{\infty}$ *generated by the convex, admissible set F satisfies* $(E1)$.

Proof. It is clear that $u_n = \overset{-1}{u_n}$ for each integer n. Let (a, b) and $(b, c) \in u_{n+1}$. Then $2^{n+1}\{a, b\}$ and $2^{n+1}\{b, c\} \in F$. By convexity, $2^n\{a, b\}+2^n\{b, c\} \in F$. Finite induction on (A2) shows that $2^n\{a, c\} \in F$. Thus $(a, c) \in u_n$.

On the other hand, suppose we are given an entourage sequence $\{u_n\}$ satisfying $(E1)$. We state a theorem due to Gaal [2], in a slightly modified form.

3.3 THEOREM. [2, p. 164]. *For a space X, let* $\{u_n\}$ *be a sequence of entourages satisfying* $(E1)$. *Then there exists a pseudometric d for X with the property that*

$$u_{n+1} \subseteq \{(x, y) \mid d(x, y) \leq \tfrac{1}{2}^{n+1}\} \subseteq u_n, \quad \text{with } d(x, y) = 1 \quad \text{if} \quad (x, y) \notin u_0.$$

Using the pseudometric d generated by the formula given in Gaal [2, p. 165] we say that F is the subset of $\mathfrak{F}$ generated by the entourage sequence $\{u_n\}$ if $F = F(d, 1)$. It is then clear that

3.4. PROPOSITION. *The subset F generated by an entourage sequence satisfying* $(E1)$ *is a convex monotone admissible set.*

3.5. PROPOSITION. *Let F be a convex admissible subset of* $\mathfrak{F}$ *and G a monotone admissible subset of* $\mathfrak{F}$ *such that* $G+G+G+G \underline{\propto} F$. *Let* $\{u_n\}$ *be generated by F and H generated by* $\{u_n\}$. *Then* $G \underline{\propto} H \underline{\propto} F$.

Proof. Let d be the metric constructed from (3.3). If $d(a, b) = r$, then for each $s > r$, there is a chain $a = x_0, x_1, \ldots, x_n = b$ such that $(x_{i-1}, x_i) \in u_{n_i}$ and $\sum 2^{-n_i} = t < s$. We say that the chain has length t. Furthermore, if $2^{n_i}\{a_i, b_i\} \in F$ for $i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} 2^{-n_i} < 1$, then $\sum_{i=1}^{k} \{a_i, b_i\} \in F$ by convexity.

To show $H \underline{\propto} F$, choose $f \in H$. Then for some finite set $A = \{\{a_i, b_i\} \mid i = 1, 2, \ldots, m\}$ of distinct elements of F, $f = 0$ only on $\mathfrak{F} \backslash A$ and $\sum f\{a_i, b_i\} d(a_i, b_i) = r < 1$. Each pair (a_i, b_i) has a chain (described above) of length $(d(a_i, b_i)+\varepsilon/k)(f\{a_i, b_i\})^{-1}$, which we denote by $\{x_{ij} \mid j = 0, 1, \ldots, n_i\}$. If we set

$$f' = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \{x_{i,j-1}, x_{ij}\},$$

then $f' \in H$ for $0 < \varepsilon < 1-r$. But by the first remark above, $f' \in F$. Application of (A2) and finite induction proves that $f \in F$.

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We will prove that $G \underset{-}{\propto} H$ by contradiction. Suppose then that $g=\sum_{i=1}^{k}\{a_i, b_i\} \in G$, but that $g \notin H$ (i.e., $\sum_{i=1}^{k} d(a_i, b_i) \geq 1$). For each i, let $(a_i, b_i) \in U_{n_i-1} \backslash U_{n_i}$. (There is no loss of generality in assuming $d(a_i, b_i) \neq 0$ for all i.) Our purpose is now to find a pair $\{a_i, b_i\}$ such that $2^{m-1}\{a_i, b_i\} \in G$ and $2^{-m} \leq 2^{-n_i}$.

Since $(a_i, b_i) \in u_{n_i-1}$ we have $d(a_i, b_i) \leq 2^{-n_i+1}$. Summing over i, we obtain $\sum_{i=1}^{k} d(a_i, b_i) \leq \sum_{i=1}^{k} 2^{-n_i+1}$. Thus $1 \leq \sum_{i=1}^{k} 2^{-n_i+1}$ or $2^{-1} \leq \sum_{i=1}^{k} 2^{-n_i}$.

If $2^{-n_i} \geq 2^{-1}$ for some i, we are done. If not, relabelling if necessary, we can write $2^{-2} \leq \sum_{i-1}^{j} 2^{-n_i}$ and $2^{-2} \leq \sum_{i=j+1}^{k} 2^{-n_i}$. Since G is monotone, either $2\sum_{i=1}^{j}\{a_i, b_i\} \in G$ or $2\sum_{i=j+1}^{k}\{a_i, b_i\} \in G$. We assume it is the former (by relabelling if necessary). If $2^{-n_i} \geq 2^{-2}$ for some i, we are done. If not, by the same reasoning, we can assume that $2^{-3} \leq \sum_{i=1}^{\beta} 2^{-n_i}$, $2^{-3} \leq \sum_{i=\beta+1}^{j} 2^{-n_i}$ and $4\sum_{i=1}^{\beta}\{a_i, b_i\} \in G$. We proceed in this manner until single term satisfies $2^{m-1}\{a_i, b_i\} \in G$ and $2^{-m} \leq 2^{-n_i}$. Then $2^{m+1}\{a_i, b_i\} \in F$. That is,

$$d(a_i, b_i) \leq 2^{-m-1} \leq 2^{-n_i-1} < 2^{-n_i} \leq d(a_i, b_i),$$

the last inequality holding since $(a_i, b_i) \notin u_{n_i}$. The contradiction arose from the assumption that $g \in G \backslash H$.

3.6. Proposition. *Let $\{u_n\}$ be an entourage sequence satisfying (E1). If F is generated by $\{u_n\}$ and $\{v_n\}$ is generated by F, then $\{u_{n+1}\} \underset{-}{\propto} \{v_n\} \underset{-}{\propto} \{u_n\}$.*

Proof. Let $(x, y) \notin v_n$. Then $2^n\{x, y\} \notin F$. Thus $2^n d(x, y) \geq 1$ and $d(x, y) \geq 2^{-n} > 2^{-n-1}$. Thus $(x, y) \notin u_{n+1}$ and $u_{n+1} \subseteq v_n$ for all n. On the other hand, if $(x, y) \in v_n$, then $2^n\{x, y\} \in F$ and $2^n d(x, y) < 1$. Thus $d(x, y) < 2^{-n}$ so $(x, y) \in u_n$.

3.7. Proposition. *Let Φ be a convex, monotone S-Lipschitz structure. Set $U'=\{\{u_n\}_{n=0}^{\infty}|$. For some $F \in \Phi$, F generates $\{u_n\}_0^{\infty}.\}$. Then U' is a basis for an E-Lipschitz structure satisfying (E1) and (E2).*

Proof. Since (3.2) showed that (*E*1) is satisfied, we need only show that U' satisfies (*E*2) and is a filter base.

To prove (*E*2), we will show that if F, G generates $\{u_n\}$, $\{v_n\}$ respectively and $F+F \underset{-}{\propto} G$, then $\{u_n\} \underset{-}{\propto} \{v_{n+1}\}$.

So let $(x, y) \in u_n$. Then $2^n\{x, y\} \in F$ and $2^{n+1}\{x, y\} \in G$. But this implies that $(x, y) \in v_{n+1}$.

Now let $\{u_n\}$, $\{v_n\} \in U'$ be generated by F, G respectively. Then there exists $H \in \Phi$ such that H is convex, monotone and admissible and $H \underset{-}{\propto} F \cap G$. Letting $\{w_n\}$ be the entourage sequence generated by H, it is trivial to observe that $\{w_n\} \underset{-}{\propto} \{u_n \cap v_n\}$.

3.8. Definition. Let Φ and U' be as in (3.7). The *E*-Lipschitz structure U having U' for a base will be called the *E*-Lipschitz structure generated by Φ.

3.9. Proposition. *Let (X, U) be an E-Lipschitz structure. Let $\Phi'=\{F\in\mathfrak{F}|$. For some $\{u_n\}\in U$ satisfying $(E1)$, F is generated by $\{u_n\}\}$. Then Φ' forms a basis for a convex, monotone S-Lipschitz structure Φ.*

Proof. Let F and G belong to Φ'. Then there exist $\{u_n\}$ and $\{v_n\}$ in U satisfying $(E1)$ generating F, and G respectively. Since the entourage sequence $\{u_n \cap v_n\}$ satisfies $(E1)$, it generates some H in Φ'. It is easy to see that $H \underline{\alpha} F \cap G$.

3.10. Definition. Let (X, U) and Φ be as in (3.9). Then Φ is the S-Lipschitz structure generated by U.

Now it is easy to see that (3.1) follows from (3.5) and (3.6), since (3.7) and (3.9) show that the methods of transferral transfer filter bases. In fact, we have proved the stronger.

3.11. Theorem. *If (X, Φ) is an S-Lipschitz structure with a basic of convex admissible sets and a basis of monotone admissible sets, then (X, Φ) generates a E-Lipschitz structure U such that the S-Lipschitz structure Φ' generated by U is precisely Φ.*

The above result extends Sandberg's Theorem 1′ in [4].

4. **The metric case.** Let (X, Φ) be an S-Lipschitz structure. For $F\in\Phi$, define $\frac{1}{2}^k F=\{f\in F \mid 2^k f\in F\}$. Sandberg proves in [4] that (X, Φ) is generated by a single pseudometric iff Φ is convex, monotone and contains an admissible element F such that $\{\frac{1}{2}^k F\}_{k=0}^{\infty}$ is a filter base for Φ. Correspondingly, we have

4.1. Proposition. *Let (X, Φ) be a convex monotone S-Lipschitz structure. Then Φ is generated by a single pseudometric iff there exists $F\in\Phi$ which generates an entourage sequence $\{u_n\}$ such that $\{\{u_{n+k}\}_{n=0}^{\infty} \mid k=0, 1, \ldots\}$ is a basis for the E-Lipschitz structure generated by Φ.*

Proof. An E-Lipschitz structure is pseudometrizable iff there exists an entourage sequence $\{u_n\}\in U$ whose translates form a basis by (5.2) of [1]. Thus the theorem follows.

It is easy to see how this proposition relates to Sandberg's, since $(x, y)\in u_n$ iff $2^n\{x, y\}\in F$ iff $\{x, y\}\in\frac{1}{2}^n F$.

4.2. Corollary. *(X, Φ) is generated by a metric iff (X, Φ) is a separated S-Lipschitz structure satisfying the conditions of* (4.1).

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