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PROPER EMBEDDABILITY OF INVERSE SEMIGROUPS

A. SHEHADAH

Let S be an inverse semigroup. We prove that there is a ring with a proper involution $*$ in which S is $*$-embeddable. The ring will be a natural one, $R[S]$, the semigroup ring of S over any formally complex ring R; for example $\mathbb{R}$, $\mathbb{C}$.

1. Introduction

In [5] we gave a negative answer to a question in [3]: Given a semigroup S with a proper involution, does there exist a ring R with a proper involution in which S is $*$-embeddable. In this paper we answer the same question in the affirmative if S happens to be an inverse semigroup with an involution equal to its inverse map.

2. Preliminaries

DEFINITIONS: Let S be a semigroup. An involution on S is a map $*: S \to S$ such that, for $a, b \in S$, $a^{**} = a$, $(ab)^* = b^*a^*$. The involution $*$ is proper if, for $a, b \in S$, $aa^* = ab^* = bb^*$ implies $a = b$. A proper $*$-semigroup $(S,*)$ is a $*$-semigroup $(S,*)$ in which $*$ is proper.

Let R be a unital ring. An involution on R is a map $*: R \to R$ such that, for $A, B \in R$, $A^{**} = A$, $(AB)^* = B^*A^*$, and $(A + B)^* = A^* + B^*$.

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The involution $*$ is proper if, for $A \in R$, $AA^* = 0$ implies $A = 0$. A $*$-ring $(R,*)$ is a ring R with an involution $*$. A proper $*$-ring is a $*$-ring $(R,*)$ in which $*$ is proper.

Let $(S,*)$ be a $*$-semigroup and $(R,*)$ be a $*$-ring. A $*$-embedding of $(S,*)$ into $(R,*)$ is a 1-1 map $f:S \to R$, such that, for $a, b \in S$, $f(ab) = f(a)f(b)$, and $f(a^*) = (f(a))^*$. If such a map exists then we say that $(S,*)$ is $*$-embeddable in $(R,*)$.

Let $(R,*)$ be a $*$-ring. If for every $A_1, \ldots, A_n \in R$, $\sum_{i=1}^{n} A_i A_i^* = 0$ implies all $A_i = 0$, then $(R,*)$ will be called a formally complex ring. A formally complex ring is a proper $*$-ring: We take $n = 1$ in the definition.

Given a $*$-semigroup $(S,*)$ and a $*$-ring $(R,*)$, the $*$-semigroup ring $(R[S],*)$ with involution $*$ defined by $(\Sigma r_i s_i)^* = \Sigma r_i^* s_i^*$ is a $*$-ring in which $(S,*)$ is $*$-embeddable. If $(S,*)$ is a proper $*$-semigroup, there may not exist a proper $*$-ring $(R,*)$ in which $(S,*)$ is $*$-embeddable [5].

3. The results

LEMMA. *Let S be an inverse semigroup and let a, b, c be three elements of S such that $aa^{-1} = bc^{-1}$. Then $a^{-1}b = a^{-1}c$.*

Proof. We have $aa^{-1} = bc^{-1} = cb^{-1}$. Therefore $bb^{-1}.aa^{-1} = bb^{-1}.bc^{-1} = bb^{-1}b.c^{-1} = bc^{-1} = aa^{-1}$. Thus $bb^{-1}aa^{-1}a = aa^{-1}a$; therefore, $bb^{-1}a = a$. Now we have $a^{-1}b(a^{-1}b)^{-1} = a^{-1}b.b^{-1}a = a^{-1}.bb^{-1}a = a^{-1}a$, and $a^{-1}b(a^{-1}c)^{-1} = a^{-1}b.c^{-1}a = a^{-1}.bc^{-1}.a = a^{-1}.aa^{-1}.a = a^{-1}a$. Thus $a^{-1}b \le a^{-1}c$ in the Vagner - Preston partial ordering. Similarly $a^{-1}c \le a^{-1}b$ and so $a^{-1}b = a^{-1}c$.

THEOREM. *Let S be an inverse semigroup and $(R,*)$ be any formally complex ring. Then the involution $*$ induced on the semigroup ring $R[S]$ is proper.*

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Proof. We have to show, for every m, for every finite subset $\{s_1,\ldots,s_m\} \subseteq S$ and for every $A = \sum_{i=1}^{m} a_i a_i \in R[S]$, that $AA^* = 0$ only if $a_i = 0$ $(i = 1,\ldots,m)$. We prove this by complete induction on m. The case $m = 1$ is trivial, since here $aa.a^*s^{-1} = aa^*.ss^{-1}$, and so $(as).(as)^* = 0$ only if $aa^* = 0$ so that $a = 0$ (since R is a proper *-ring).

Given any positive integer n, assume the result is true for all $m \leq n$. Choose any subset $\{s_1,\ldots,s_n\} \subseteq S$, and let $a_1,\ldots,a_n \in R$ be such that $A = \sum_{i=1}^{n} a_i s_i$ satisfies $AA^* = 0$. Pick from the set $\{s_1,\ldots,s_n\}$ any s_j maximal with respect to the Vagner - Preston order, without loss of generality assume it is s_n, that is, $s_n \not\leq s_i$ $(i = 1,\ldots,n\text{-}1)$. We distinguish the following two cases, which exhaust all other possibilities:

Case 1. $s_n s_n^{-1} = s_u s_v^{-1}$ implies $u = v (\forall\, u,v = 1,\ldots,n)$.

Case 2. $s_n s_n^{-1} = s_u s_v^{-1}$ for some pair (u,v) such that $u \neq v$.

We treat each case separately.

In Case 1 the only $s_u s_v^{-1}$ which are equal to $s_n s_n^{-1}$ are of the form $s_i s_i^{-1}$. Without loss of generality, let these be $s_k s_k^{-1},\ldots,s_n s_n^{-1}$ for some k in the range $1 \leq k \leq n$. Then by collecting the coefficients of $s_n s_n^{-1}$ in AA^*, we have that $a_k a_k^* + \ldots + a_n a_n^* = 0$. Hence, since R is formally complex, $a_k = \ldots = a_n = 0$. Thus $A = \sum_{i=1}^{k-1} a_i s_i$.

In Case 2, apart from $s_n s_n^{-1}$ itself, there can be no $s_n s_v^{-1}$ or $s_v s_n^{-1}$ equal to $s_n s_n^{-1}$ since either would imply $s_n \leq s_v$ contrary to our choice of s_n. Also $s_n^{-1} s_n \neq s_n^{-1} s_v$ for otherwise $s_n = s_v$.

Now $s_n^{-1} A (s_n^{-1} A)^* = s_n^{-1} AA^* s_n = 0$ where $s_n^{-1} A = \sum_{i=1}^{n} a_i s_n^{-1} s_i$.

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Also $s_n s_n^{-1} = s_u s_v^{-1}$ for at least one pair (u,v) with u,v,n all different, and for all such (u,v) we have $s_n^{-1}s_u = s_n^{-1}s_v$, by the Lemma. Thus, in the formal sum $s_n^{-1}A = \sum_{i=1}^{n} a_i s_n^{-1} s_i$, there is at least one "collapsing" $s_n^{-1}s_u = s_n^{-1}s_v \neq s_n^{-1}s_n$ with u,v,n all different. In other words $s_n^{-1}A$ can be written in the form $s_n^{-1}A = b_n s_n^{-1}s_n + b_2 s_2' + \ldots + b_i s_i'$ for some $s_2', \ldots, s_i' \in S$, where $b_n = a_n$, $b_2, \ldots, b_i \in R$ and $i < n$. Since $s_n^{-1}A(s_n^{-1}A)^* = 0$, by the induction hypothesis, $a_n = b_n = 0$. Thus $A = \sum_{i=1}^{n-1} a_i s_i$. We have shown that, in both Case 1 and Case 2, A is a formal sum with fewer than n terms. By the induction hypothesis it follows that all $a_i = 0$.

PROPOSITION. *Let S be any inverse semigroup and let R be any formally complex *-ring. Then the semigroup ring $R[S]$ contains no nonzero nil ideal. (Equivalently, $R[S]$ has a zero nil radical).*

PROOF. The map $A = \Sigma r_i s_i \to A^* = \Sigma r_i^* s_i^{-1}$ defines an involution on $R[S]$. From the Theorem, $R[S]$ is a proper *-ring. Let I be any nil ideal in $R[S]$ and let $A \in I$. Now $AA^* \in I$ and hence, for some $n \geq 1$, $(AA^*)^n = 0$. By *-cancellation [2] $A = 0$. Thus $I = 0$.

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Department of Mathematics
Yarmouk University
Irbid
Jordan

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