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APPROXIMATION BY Λ-SPLINES ON THE CIRCLE

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1. Introduction. Let $\Lambda = \{\lambda_0, \dots, \lambda_n\}$ denote a set of distinct integers and let $\Pi(\Lambda)$ denote the set of all generalized polynomials of the form

$$\sum_0^n a_j z^{\lambda_j}, \quad a_j \in \mathbf{C}.$$

For any given ζ on the unit circle U with

$$0 \leqq |\arg \zeta| \leqq \frac{2\pi}{k},$$

we consider the set Z_k of points $1, \zeta, \zeta^2, \dots, \zeta^{k-1}$ where

$$k > \max_{i,j} |\lambda_i - \lambda_j|.$$

We shall denote by $\mathscr{S}(\Lambda, Z_k)$ or $\mathscr{S}$ the class of Λ-splines $S(z)$ which satisfy the following conditions:

(i) $S(z) \in C^{n-1}(U)$

(ii) $S(z)|_{A_\nu} \in \Pi(\Lambda)$ where

$$A_\nu = \text{arc}(\zeta^\nu, \zeta^{\nu+1}), (\nu = 0, 1, \dots, k-2) \quad \text{and}$$

$$A_{k-1} = \text{arc}(\zeta^{k-1}, 1).$$

Λ-splines were introduced in [**8**] where their interpolation properties were studied. Although in [**8**], Λ is comprised of non-negative integers, there are no difficulties in allowing Λ to contain any integers. When $\Lambda = \{0, 1, \dots, n\}$, Λ-splines reduce to polynomial splines on the circle studied in [**1**], [**11**].

Our object here is to study approximation theoretic properties of Λ-splines and to obtain their trigonometric analogues. As in [**11**] and [**8**], a basic tool to this end will be the B-spline $M_\Lambda(z) \in \mathscr{S}$ which for $k \geqq n + 2$ has support on the arc$(1, \zeta^{n+1})$, (in fact the minimal support possible). We shall be concerned mainly with the case $\zeta^k = 1$ when the B-splines $M_\Lambda(z\zeta^{-\nu})$, $\nu = 0, 1, \dots, k-1$ will form a basis for $\mathscr{S}$.

In Section 2 we introduce the preliminaries and some definitions and in Section 3 we study the properties of the B-splines in $\mathscr{S}$ and the analogue of

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Marsden's identity [7]. We then examine approximation operators of the form

$$(1.1)\quad (\mathscr{L}g)(z) = \sum_{\nu=0}^{k-1} T_\nu(g)M_\Lambda(z\zeta^{-\nu}).$$

In Section 4, we take $T_\nu(g)$ to be a linear combination of $g^{(r)}(\tau_\nu)$, $r = 0, 1, \ldots, n$ for some prescribed points τ_ν on the arc$(\zeta^\nu, \zeta^{\nu+n+1})$. It is shown that there is a unique such operator which reproduces $\mathscr{S}$. This is the analogue of the quasi-interpolant (see [2]), a special case of which is due to Chen [3] when $\Lambda = \{0, 1, \ldots, n\}$. The order of approximation by this operator is the subject of Section 5 and generalizes the work in [3].

In Section 6 we consider (1.1) when $T_\nu(g)$ is a constant multiple of $g(\sigma_\nu)$ for some σ_ν. We show that there is a unique such operator which reproduces z^{λ_0} and z^{λ_1}. This is the analogue of the Bernstein-Schoenberg operator (B-S operator) (see [9]). Similar results for the case of generalized real polynomials are due to Hirschman and Widder [5]. Section 6 also deals with the order of approximation of this operator and in Section 7 we obtain an asymptotic formula which is reminiscent of a result of Voronovskaja [6] for Bernstein polynomials, thus generalizing the results in [4] for the case $\Lambda = \{0, 1, \ldots, n\}$.

Results of Sections 6 and 7 are analogous to the work of Marsden [7] for the B-S operator. However, unlike Marsden we keep n fixed $\leqq k - 2$, but our results as $k \to \infty$ are somewhat stronger in so far as we get convergence for all derivatives up to order $n - 1$ at all points.

By taking the λ_j's in Λ to be symmetric about 0, we can get corresponding results for trigonometric Λ-splines which is the subject of Section 8.

2. Preliminaries. For given distinct integers $\lambda_0, \lambda_1, \ldots, \lambda_n$ we denote by $\Lambda_{p,q}$ the set $\{\lambda_p, \ldots, \lambda_q\}$, but for simplicity we shall use Λ_p instead of $\Lambda_{0,p}$. In order to study the Λ-splines, it will be useful to consider the function $\phi_{\Lambda_n}(z) \in \Pi(\Lambda_n)$ satisfying the conditions

$$(2.1)\quad \phi_{\Lambda_n}^{(\nu)}(1) = \begin{cases} 0, & \nu = 0, 1, \ldots, n-1 \\ 1, & \nu = n. \end{cases}$$

It is easy to see that $\phi_{\Lambda_n}(z)$ is uniquely given by

$$(2.2)\quad \phi_{\Lambda_n}(z) = (-1)^n \begin{vmatrix} z^{\lambda_0} & z^{\lambda_1} & \ldots & z^{\lambda_n} \\ 1 & 1 & \ldots & 1 \\ \lambda_0 & \lambda_1 & \ldots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_0^{n-1} & \lambda_1^{n-1} & \ldots & \lambda_n^{n-1} \end{vmatrix} \div V(\lambda_0, \ldots, \lambda_n)$$

where $V(\lambda_0, \ldots, \lambda_n)$ denotes the Vandermondian. It follows from (2.1) and (2.2) that

$$(2.3) \qquad (\lambda_n - \lambda_0)\phi_{\Lambda_n}(z) = \phi_{\Lambda_{1,n}}(z) - \phi_{\Lambda_{n-1}}(z)$$

since the coefficients of z^{λ_n} on both sides are equal and all the derivatives up to order $n - 1$ at 1 vanish on both sides.

For any $\beta \in U$ we introduce the analogue of the truncated power function by setting

$$(2.4) \qquad \phi_{\Lambda_n}(z, \beta) = \begin{cases} 0 & , z \in \operatorname{arc}[1, \beta) \\ \phi_{\Lambda_n}(z\beta^{-1}), & z \in \operatorname{arc}[\beta, 1). \end{cases}$$

We shall prove the following

PROPOSITION 1. *The dimension of the space* $\mathcal{S}(\Lambda_n, Z_k) = k$.

Proof. If $S(z) \in \mathcal{S}$, then $S(z)$ can be written in the form

$$(2.5) \qquad S(z) = P(z) + \sum_{j=0}^{k-1} a_j\phi_{\Lambda_n}(z, \zeta^j), \quad P(z) \in \Pi(\Lambda_n),$$

where $S(z)$ on the arc(ζ^{k-1}, 1) is given by $P(z)$. Then it is easy to see that

$$(2.6) \qquad \sum_{j=0}^{k-1} a_j\phi_{\Lambda_n}(z\zeta^{-j}) = 0.$$

Moreover any $S(z)$ satisfying (2.5) and (2.6) belongs to $\mathcal{S}$. Equating to zero the coefficients of z^{λ_j}, we see that (2.6) is equivalent to the system of $n + 1$ equations:

$$\sum_{j=0}^{k-1} a_j\zeta^{-j\lambda_\nu} = 0, \quad \nu = 0, 1, \ldots, n.$$

Since $k > \max|\lambda_\mu - \lambda_\nu|$ and $|\arg \zeta| \leqq 2\pi/k$, it follows that the rank of the matrix of this system is $n + 1$, so that from (2.5) the dimension of $\mathcal{S}$ is $k - (n + 1) + (n + 1) = k$.

We shall now derive an analogue of Taylor's formula. To this end we set

$$D_jf(z) = z^{\lambda_j+1}\frac{d}{dz}(z^{-\lambda_j}f), \quad j = 0, 1, \ldots, n.$$

Observe that if $g(z) = f(az)$, then

$$D_jg(z) = (D_jf)(az),$$

for any constant a. Since

$$D_j\phi_{\Lambda_j} \in \Pi(\Lambda_{j-1})$$

and since it is easily seen from (2.1) that

$$\frac{d^\nu}{dz^\nu}(D_j\phi_{\Lambda_j})\Big]_{z=1} = \begin{cases} 0, \ \nu = 0, 1, \dots, j-2 \\ 1, \ \nu = j-1, \end{cases}$$

it follows that

$$(2.7) \quad D_j\phi_{\Lambda_j}(z) = \phi_{\Lambda_{j-1}}(z).$$

For $j = 1, 2, \dots, n + 1$, we define the differential operators L_j by

$$(2.8) \quad L_j = D_{j-1}D_{j-2} \dots D_0, \quad L_0 = I.$$

This enables us to get the following analogue of the Taylor's formula where $f \in C^{n+1}(U)$:

$$(2.9) \quad \begin{cases} f(z) = f(a)\phi_{\Lambda_0}(za^{-1}) + (L_1f)(a)\phi_{\Lambda_1}(za^{-1}) + \dots \\ \qquad\qquad + (L_nf)(a)\phi_{\Lambda_n}(za^{-1}) + R_n, \\ R_n = \int_a^z \phi_{\Lambda_n}(zv^{-1})v^{-1}(L_{n+1}f)(v)dv, \quad a, z \in U. \end{cases}$$

Formula (2.9) can be easily verified by integrating by parts and is perhaps known.

It is of interest to introduce the operators $\tilde{L}_j$ by

$$(2.10) \quad \tilde{L}_j = D_{n-j+1}D_{n-j+2} \dots D_n, \quad 1 \leqq j \leqq n + 1; \ \tilde{L}_0 = I.$$

In this case, we get an analogue of (2.9). Indeed we have

$$(2.11) \quad \begin{cases} f(z) = f(a)\phi_{\Lambda_{n,n}}(za^{-1}) + (\tilde{L}_1f)(a)\phi_{\Lambda_{n-1,n}}(za^{-1}) + \dots \\ \qquad\qquad + (\tilde{L}_nf)(a)\phi_{\Lambda_n}(za^{-1}) + \tilde{R}_n. \\ \tilde{R} = \int_a^z \phi_{\Lambda_n}(zv^{-1})v^{-1}(\tilde{L}_{n+1}f)(v)dv. \end{cases}$$

In order to define B-splines in $\mathcal{S}$, we introduce the Λ-divided difference of a function f on a subset of Z_k by the symbol $[1, \zeta, \dots, \zeta^{n+1}]_{\Lambda_n}f$ defined by the expression

$$(2.12) \quad \begin{vmatrix} 1 & 1 & \dots & 1 & f(1) \\ \zeta^{\lambda_0} & \zeta^{\lambda_1} & \dots & \zeta^{\lambda_n} & f(\zeta) \\ \cdot & \dots & \dots & \dots & \dots \\ \zeta^{(n+1)\lambda_0} & \zeta^{(n+1)\lambda_1} & \dots & \zeta^{(n+1)\lambda_n} & f(\zeta^{n+1}) \end{vmatrix} \div V(\zeta^{\lambda_0}, \dots, \xi^{\lambda_n}),$$

where $V(\zeta^{\lambda_0}, \dots, \zeta^{\lambda_n})$ is a Vandermondian. More generally, we set

$$(2.13) \quad [\zeta^\nu, \zeta^{\nu+1}, \dots, \zeta^{\nu+n+1}]_{\Lambda_n}f(z) = [1, \zeta, \dots, \zeta^{n+1}]_{\Lambda_n}f(z\zeta^\nu).$$

From (2.12) we can see that

$$(2.14) \quad [1, \zeta, \dots, \zeta^{n+1}]_{\Lambda_n}f = \sum_{\nu=0}^{n+1} (-1)^{n+1-\nu}S_{n+1-\nu}(\Lambda_n)f(\zeta^\nu)$$

where $S_\nu(\Lambda_n)$ is the ν-th elementary symmetric function of the numbers $\zeta^{\lambda_0}, \zeta^{\lambda_1}, \ldots, \zeta^{\lambda_n}$. From (2.14), it follows that

$$(2.15) \quad [1, \zeta, \ldots, \zeta^{n+1}]_{\Lambda_n} f = [\zeta, \zeta^2, \ldots, \zeta^{n+1}]_{\Lambda_{n-1}} f - \zeta^{\lambda_n} [1, \zeta, \ldots, \zeta^n]_{\Lambda_{n-1}} f = [\zeta, \zeta^2, \ldots, \zeta^{n+1}]_{\Lambda_{1,n}} f - \zeta^{\lambda_0} [1, \zeta, \ldots, \zeta^n]_{\Lambda_{1,n}} f.$$

Remark. If $\Lambda_n = \{0, 1, \ldots, n\}$, then our Λ-divided difference differs from the usual divided difference on the same points by a constant factor. More precisely, in this case

$$[1, \zeta, \ldots, \zeta^{n+1}]_{\Lambda_n} f = \prod_{j=0}^{n} (\zeta^{n+1} - \zeta^j)[1, \zeta, \ldots, \zeta^{n+1}] f$$

where the right hand divided difference is the usual one.

3. *B*-splines and their properties. Here and in the sequel we shall assume that $k \geqq n + 2$. We now define the *B*-spline $M_{\Lambda_n}(z)$ to be an element of $\mathscr{S}$ given by

$$(3.1) \quad M_{\Lambda_n}(z) = [1, \zeta^{-1}, \ldots, \zeta^{-n-1}]_{\Lambda_n} \phi_{\Lambda_n}(z, y^{-1}).$$

For $z \in \operatorname{arc}(\zeta^{n+1}, 1)$,

$$M_{\Lambda_n}(z) = [1, \zeta^{-1}, \ldots, \zeta^{-n-1}]_{\Lambda_n} \phi_{\Lambda_n}(zy)$$

because of (2.4) and so vanishes since

$$\phi_{\Lambda_n}(zy) \in \Pi(\Lambda_n).$$

We shall show that $M_{\Lambda_n}(z)$ is the spline of minimal support in $\mathscr{S}$.

PROPOSITION 2. *If $S(z) \in \mathscr{S}$ has support strictly contained in the arc $(1, \zeta^{n+1})$, then $S(z) \equiv 0$.*

Proof. Suppose the support of $S(z)$ lies in $(1, \zeta^n)$. Then $S(z)$ lies in the space of all Λ-splines with knots $1, \zeta, \ldots, \zeta^n$ which by Proposition 1 has dimension $n + 1$ and thus equals $\Pi(\Lambda_n)$. So $S(z) \in \Pi(\Lambda_n)$ and since $S(z)$ vanishes on an arc, $S(z) \equiv 0$.

We shall now prove

LEMMA 1. *The B-splines satisfy the following recurrence relations*:

$$(3.2) \quad \begin{cases} (\lambda_n - \lambda_0) M_{\Lambda_n}(z) = M_{\Lambda_{1,n}}(z\zeta^{-1}) - \zeta^{-\lambda_0} M_{\Lambda_{1,n}}(z) \\ \qquad\qquad - M_{\Lambda_{n-1}}(z\zeta^{-1}) + \zeta^{-\lambda_n} M_{\Lambda_{n-1}}(z), \end{cases}$$

and

$$(3.3) \quad D_n M_{\Lambda_n}(z) = M_{\Lambda_{n-1}}(z\zeta^{-1}) - \zeta^{-\lambda_n} M_{\Lambda_{n-1}}(z).$$

Proof. Using (2.3) we see from (3.1) that

$$(\lambda_n - \lambda_0)M_{\Lambda_n}(z) = [1, \zeta^{-1}, \ldots, \zeta^{-n-1}]_{\Lambda_n}(\phi_{\Lambda_{1,n}}(z, y^{-1}) - \phi_{\Lambda_{n-1}}(z, y^{-1})).$$

Next we use (2.15) which yields

$$\begin{aligned}(\lambda_n - \lambda_0)M_{\Lambda_n}(z) &= [\zeta^{-1}, \zeta^{-2}, \ldots, \zeta^{-n-1}]_{\Lambda_{1,n}}\phi_{\Lambda_{1,n}}(z, y^{-1}) \\ &\quad - \zeta^{-\lambda_0}[1, \zeta^{-1}, \ldots, \zeta^{-n}]_{\Lambda_{1,n}}\phi_{\Lambda_{1,n}}(z, y^{-1}) \\ &\quad - [\zeta^{-1}, \zeta^{-2}, \ldots, \zeta^{-n-1}]_{\Lambda_{n-1}}\phi_{\Lambda_{n-1}}(z, y^{-1}) \\ &\quad + \zeta^{-\lambda_n}[1, \zeta^{-1}, \ldots, \zeta^{-n}]_{\Lambda_{n-1}}\phi_{\Lambda_{n-1}}(z, y^{-1}).\end{aligned}$$

We now get (3.2) from (2.13) and (3.1).

Formula (3.3) follows on applying D_j to (3.1) and on using (2.7), (2.13) and (2.15).

As a simple application of the B-splines, we show that

$$(3.4) \qquad [1, \zeta^{-1}, \ldots, \zeta^{-n-1}]_{\Lambda_n} f = \int_U M_{\Lambda_n}(v^{-1})v^{-1}(L_{n+1}f)(v)dv.$$

In order to see this we use (2.4) and observe that for $a = 1$ in (2.9) we have

$$R_n = \int_U \phi_{\Lambda_n}(v^{-1}, z^{-1})v^{-1}(L_{n+1}f)(v)dv.$$

We now apply the difference operator $[1, \zeta^{-1}, \ldots, \zeta^{-n-1}]_{\Lambda_n}$ to both sides of (2.9) and using (3.1), we get (3.4).

In the sequel we shall suppose that ζ is a primitive k^{th} root of unity, i.e.,

$$\zeta = e^{2\pi i/k}, \ k \geqq n + 2.$$

We then have

PROPOSITION 3. *If ζ is a primitive k^{th} root of unity, then the B-splines $M_{\Lambda_n}(z\zeta^{-\nu})$, $\nu = 0, 1, \ldots, k - 1$ form a basis for the space $\mathcal{S}$ of Λ-splines.*

Proof. Since the dimension of $\mathcal{S}$ is k (Proposition 1), it is enough to show that $\{M_{\Lambda_n}(z\zeta^{-\nu})\}_0^{k-1}$ are linearly independent. We shall show that if there exists a relation

$$S(z) := \sum_{\nu=0}^{k-1} c_\nu M_{\Lambda_n}(z\zeta^{-\nu}) \equiv 0$$

then all the c_ν's are zero.

Consider the function $T(z)$ given by

$$T(z) = \begin{cases} 0 & z \in \text{arc}(1, \zeta^{k-n-1}) \\ \sum_{\nu=k-n-1}^{k-1} c_\nu M_{\Lambda_n}(z\zeta^{-\nu}), & z \in \text{arc}(\zeta^{k-n-1}, 1). \end{cases}$$

Since $M_{\Lambda_n}(z)$ vanishes outside the arc$(1, \zeta^{n+1})$ it follows that

$$T(z) = S(z) = 0, \quad \text{for } z \in \text{arc}(\zeta^{k-1}, 1).$$

Thus $T(z) \in \mathscr{S}$ and has support in the arc$(\zeta^{k-n-1}, \zeta^{k-1})$ and hence by Proposition 2, vanishes identically.

Observe that for $z \in \text{arc}(\zeta^{k-n-1}, \zeta^{k-n})$,

$$T(z) = c_{k-n-1}M_{\Lambda_n}(z\zeta^{-k+n+1}) = 0$$

which implies $c_{k-n-1} = 0$. Proceeding in this manner we see that

$$c_\nu = 0, \quad \nu = k - n - 1, \dots, k - 1.$$

Hence

$$\sum_{\nu=0}^{k-n-2} c_\nu M_{\Lambda_n}(z\zeta^{-\nu}) \equiv 0$$

and by the same argument as above, we see that c_ν's are all zero.

We shall now prove the analogue of Marsden's identity.

THEOREM 1. *If ζ is a primitive k-th root of unity and if $\psi_{\Lambda_n}(y) \in \Pi(\Lambda_n)$ satisfies the conditions*

$$(3.5) \qquad \begin{cases} \psi_{\Lambda_n}(\zeta^{-j}) = 0, & j = 1, 2, \dots, n \\ \psi_{\Lambda_n}(\zeta^{-n-1}) = -1, \end{cases}$$

then we have the identity

$$(3.6) \qquad \phi_{\Lambda_n}(zy) = \sum_{j=0}^{k-1} \psi_{\Lambda_n}(\zeta^j y)M_{\Lambda_n}(z\zeta^{-j}).$$

Proof. We prove the identity by induction on n. For $n = 0$, we have

$$\phi_{\Lambda_0}(z) = z^{\lambda_0}$$

while $M_{\Lambda_0}(z) = -z^{\lambda_0}\zeta^{-\lambda_0}$ on arc$(1, \zeta)$ and is 0 elsewhere. Also by (3.5),

$$\psi_{\Lambda_0}(y) = -\zeta^{\lambda_0}y^{\lambda_0}.$$

From this we can easily see that for $z \in \text{arc}(\zeta^\nu, \zeta^{\nu+1})$, we have

$$\sum_{j=0}^{k-1} \psi_{\Lambda_0}(\zeta^j y)M_{\Lambda_0}(z\zeta^{-j}) = \phi_{\Lambda_0}(\zeta^\nu y)M_{\Lambda_0}(z\zeta^{-\nu}) = (zy)^{\lambda_0} = \phi_{\Lambda_0}(zy).$$

We shall now suppose that (3.6) is true for any Λ-set containing n elements. Then using (2.3) and the inductive hypothesis, we obtain

$$(\lambda_n - \lambda_0)\psi_{\Lambda_n}(zy) = \phi_{\Lambda_{1,n}}(zy) - \phi_{\Lambda_{n-1}}(zy)$$

$$(3.7) \quad = \sum_{j=0}^{k-1} \psi_{\Lambda_{1,n}}(\zeta^j y)M_{\Lambda_{1,n}}(z\zeta^{-j}) - \sum_{j=0}^{k-1} \psi_{\Lambda_{n-1}}(\zeta^j y)M_{\Lambda_{n-1}}(z\zeta^{-j}).$$

We recall formula (3.2) Lemma 1 to obtain

$$(\lambda_n - \lambda_0)\sum_{j=0}^{k-1} \psi_{\Lambda_n}(\zeta^j y)M_{\Lambda_n}(z\zeta^{-j})$$

$$= \sum_{j=0}^{k-1} \psi_{\Lambda_n}(\zeta^j y)[M_{\Lambda_{1,n}}(z\zeta^{-j-1}) - \zeta^{-\lambda_0}M_{\Lambda_{1,n}}(z\xi^{-j})$$

$$- M_{\Lambda_{n-1}}(z\zeta^{-j-1}) + \zeta^{-\lambda_n}M_{\Lambda_{n-1}}(z\zeta^{-j})],$$

which after elementary manipulation gives

$$(3.8) \quad \sum_{j=0}^{k-1} \psi_{\Lambda_n}(\zeta^{j-1}y)\{M_{\Lambda_{1,n}}(z\zeta^{-j}) - M_{\Lambda_{n-1}}(z\zeta^{-j})\}$$

$$+ \sum_{j=0}^{k-1} \psi_{\Lambda_n}(\zeta^j y)\{-\zeta^{-\lambda_0}M_{\Lambda_{1,n}}(z\zeta^{-j}) + \zeta^{-\lambda_n}M_{\Lambda_{n-1}}(z\zeta^{-j})\}.$$

In order to prove (3.6) it is sufficient to show that the right side of (3.7) is equal to (3.8). This will be so if the following relations hold:

$$\left.\begin{aligned}\psi_{\Lambda_{1,n}}(\zeta^j y) &= \psi_{\Lambda_n}(\zeta^{j-1}y) - \zeta^{-\lambda_0}\psi_{\Lambda_n}(\zeta^j y)\\ \psi_{\Lambda_{n-1}}(\zeta^j y) &= \psi_{\Lambda_n}(\zeta^{j-1}y) - \zeta^{-\lambda_n}\psi_{\Lambda_n}(\zeta^j y)\end{aligned}\right\}, j = 0, 1, \dots, k-1$$

or equivalently,

$$(3.9) \quad \psi_{\Lambda_{1,n}}(y) = \phi_{\Lambda_n}(\zeta^{-1}y) - \zeta^{-\lambda_0}\psi_{\Lambda_n}(y)$$

$$(3.10) \quad \psi_{\Lambda_{n-1}}(y) = \psi_{\Lambda_n}(\zeta^{-1}y) - \zeta^{-\lambda_n}\psi_{\Lambda_n}(y).$$

Obviously both sides of (3.9) belong to the class $\Pi(\Lambda_{1,n})$ and by (3.5) they agree for $y = \zeta^{-j}(j = 1, 2, \dots, n)$. This shows that (3.9) is valid. In a similar way, we can show that (3.10) is true.

Remark. From (3.5) we can get an explicit representation for $\psi_{\Lambda_n}(y)$. Thus

$$(3.11) \quad \phi_{\Lambda_n}(y) = \frac{(-1)^{n-1}\zeta^{\lambda_0+\lambda_1+\dots+\lambda_n}}{V(\zeta^{-\lambda_0}, \dots, \zeta^{-\lambda_n})}\begin{vmatrix} y^{\lambda_0} & y^{\lambda_1} & \dots & y^{\lambda_n}\\ \zeta^{-\lambda_0} & \zeta^{-\lambda_1} & \dots & \zeta^{-\lambda_n}\\ \cdot & \cdot & \dots & \cdot\\ \zeta^{-n\lambda_0} & \zeta^{-n\lambda_1} & \dots & \zeta^{-n\lambda_n}\end{vmatrix},$$

whence we see that

$$(3.12)\quad \psi_{\Lambda_n}(1) = (-1)^{n-1}\zeta^{\lambda_0+\ldots+\lambda_n}.$$

4. The quasi-interpolant. It is known that for polynomial splines the quasi-interpolant plays a very useful role. An analogue of the quasi-interpolant for polynomial splines on the circle has recently been given in **[3]**.

In order to obtain the quasi-interpolant for Λ-splines, we choose points $\tau_\nu \in \mathrm{arc}(\zeta^\nu, \zeta^{\nu+n+1})$, $\nu = 0, 1, \ldots, k-1$ where ζ is a primitive kth root of unity. Consider an operator $\mathcal{L}: C^{(n)}(U) \to \mathcal{S}$ of the following form:

$$(4.1)\quad (\mathcal{L}g)(z) = \sum_{\nu=0}^{k-1} T_\nu(g) M_{\Lambda_n}(z\zeta^{-\nu})$$

where

$$(4.2)\quad T_\nu(g) = \sum_{r=0}^{n} a_{\nu,r}(L_r g)(\tau_\nu)$$

and $a_{\nu,r}$'s are constants depending on τ_ν, but independent of g. We shall show that there is a unique operator $\mathcal{L}$ of the form (4.1), (4.2) which reproduces splines in $\mathcal{S}$. We shall call such an operator the quasi-interpolant. We can now prove.

THEOREM 2. *For an operator $\mathcal{L}$ of the form* (4.1), (4.2) *we have*

$$(4.3)\quad (\mathcal{L}S)(z) = S(z) \quad \textit{for all } S(z) \in \mathcal{S},$$

if and only if

$$(4.4)\quad a_{\nu,r} = (\tilde{L}_{n-r}\psi_{\Lambda_n})(\tau_\nu^{-1}\zeta^\nu),\ \nu = 0, 1, \ldots, k-1;\ r = 0, 1, \ldots, n.$$

Proof. We shall first show that (4.3) implies (4.4). Note that (4.3) is equivalent to

$$(4.5)\quad T_\nu(M_{\Lambda_n}(z\zeta^{-j})) = \delta_{\nu j},\quad j, \nu = 0, 1, \ldots, k-1.$$

Applying the operator $\tilde{L}_j$ to the identity (3.6) with respect to the variable y and using (2.7) successively, we obtain

$$(4.6)\quad \phi_{\Lambda_{n-j}}(zy) = \sum_{l=0}^{k-1} (\tilde{L}_j\psi_{\Lambda_n})(\zeta^l y) M_{\Lambda_n}(z\zeta^{-l}).$$

Now applying the operator T_ν to both sides of (4.6) with respect to z and recalling (4.5), we have

$$(4.7)\quad \sum_{r=0}^{n} a_{\nu,r}(L_r\phi_{\Lambda_{n-j}})(\tau_\nu y) = (\tilde{L}_j\psi_{\Lambda_n})(\zeta^j y).$$

From (2.7) we see that

$$L_r\phi_{\Lambda_{n-j}} = D_{r-1}D_{r-2}\dots D_0\phi_{\Lambda_{n-j}} = \phi_{\Lambda_{r,n-j}},$$

and from (2.1) we note that

$$\phi_{\Lambda_{r,n-j}}(1) = \delta_{r,n-j}$$

so that putting $y = \tau_\nu^{-1}$ in (4.7) we have

$$a_{\nu,n-j} = (\widetilde{L}_j\psi_{\Lambda_n})(\tau_\nu^{-1}\zeta^\nu), \quad j = 0, 1, \dots, n$$

which is equivalent to (4.4).

We shall now show that (4.4) implies (4.3), which is equivalent to (4.5). Applying the operator T_ν to both sides of (3.1) after replacing z by $z\zeta^{-j}$ we have

$$(4.8) \quad T_\nu(M_{\Lambda_n}(z\zeta^{-j})) = [1, \zeta^{-1}, \dots, \zeta^{-n-1}]_{\Lambda_n}T_\nu(\phi_{\Lambda_n}(z\zeta^{-j}, y^{-1})).$$

In order to simplify the right side above we observe that (4.2) yields

$$(4.9) \quad T_\nu(\phi_{\Lambda_n}(zy)) = \sum_{r=0}^{n} a_{\nu,r}(L_r\phi_{\Lambda_n})\phi_{\Lambda_{r,n}}(\tau_\nu y) = \sum_{r=0}^{n} a_{\nu,r}\phi_{\Lambda_{r,n}}(\tau_\nu y).$$

We claim that

$$(4.10) \quad T_\nu(\phi_{\Lambda_n}(zy)) = \sum_{r=0}^{n} a_{\nu,r}\phi_{\Lambda_{r,n}}(\tau_\nu y) = \psi_{\Lambda_n}(y\zeta^\nu).$$

Since both sides belong to $\Pi(\Lambda_n)$, it is sufficient to show that

$$(4.11) \quad \left[\widetilde{L}_j\left(\sum_{r=0}^{n} a_{\nu,r}\phi_{\Lambda_{r,n}}(\tau_\nu y)\right)\right]_{\substack{y=\tau_\nu^{-1} \\ (j=0,1,\dots,n)}} = (\widetilde{L}_j\psi_{\Lambda_n})(\tau_\nu^{-1}\zeta^\nu),$$

To see this, we observe that on using (2.1) and (2.7) the left side in (4.11) becomes

$$\sum_{r=0}^{n} a_{\nu,r}\phi_{\Lambda_{r,n-j}}(1) = a_{\nu,n-j}$$

which by (4.4) equals the right side of (4.11). This proves the assertion (4.10).

In order to find $T_\nu(M_{\Lambda_n}(z\zeta^{-j}))$, we examine $T_\nu(\phi_{\Lambda_n}(z\zeta^{-j}, y^{-1}))$ in the light of (4.8). We observe that from (4.10),

$$(4.12) \quad T_\nu(\phi_{\Lambda_n}(z\zeta^{-j}, y^{-1})) = \begin{cases} 0 & , \tau_\nu \in \operatorname{arc}[\zeta^j, \zeta^j y^{-1}) \\ \psi_{\Lambda_n}(y\zeta^{\nu-j}), & \text{otherwise.} \end{cases}$$

Thus from (4.8) and (4.12) on using (2.13), we obtain

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(4.13) $T_\nu(M_{\Lambda_n}(z\zeta^{-j})) = [\zeta^{-j}, \zeta^{-j-1}, \dots, \zeta^{-j-n-1}]_{\Lambda_n}\Psi_\nu(y),$

where we set

$$\Psi_\nu(y) = \begin{cases} 0 & , \tau_\nu^{-1} \in \operatorname{arc}(y, \zeta^{-j}) \\ \psi_{\Lambda_n}(y\zeta^\nu), & \text{otherwise.} \end{cases}$$

We now consider three cases:

(a) $j < \nu$. In this case, take any l, $0 \leqq l \leqq n + 1$. If

$$\zeta^{-j-l} \in \operatorname{arc}(\tau_\nu^{-1}, \zeta^{-j}),$$

then

$$\tau_\nu^{-1} \in \operatorname{arc}(\zeta^{-j-l}, \zeta^{-j})$$

and so

$$\Psi_\nu(\zeta^{-j-l}) = \psi_{\Lambda_n}(\zeta^{-j-l+\nu}).$$

On the other hand, if

$$\zeta^{-j-l} \in \operatorname{arc}[\zeta^{-\nu-n-1}, \tau_\nu^{-1}],$$

then $-n - 1 < \nu - j - l < 0$ and so by (3.5),

$$\psi_{\Lambda_n}(\zeta^{\nu-j-l}) = 0.$$

Hence

(4.14) $\Psi_\nu(\zeta^{-j-l}) = 0 = \psi_{\Lambda_n}(\zeta^{\nu-j-l}).$

Thus we have

(4.15) $[\zeta^{-j}, \zeta^{-j-1}, \dots, \zeta^{-j-n-1}]_{\Lambda_n}\Psi_\nu(y)$
$= [\zeta^{-j}, \zeta^{-j-1}, \dots, \zeta^{-j-n-1}]_{\Lambda_n}\psi_{\Lambda_n}(y\zeta^\nu) = 0.$

(b) $j > \nu$. Again, as in case (a) we take any l, $0 \leqq l \leqq n + 1$. If

$$\zeta^{-j-l} \in \operatorname{arc}[\tau_\nu^{-1}, \zeta^{-j}],$$

then $-n - 1 < \nu - j - l < 0$ so that by (3.5), we have (4.14). If

$$\zeta^{-j-l} \in \operatorname{arc}[\zeta^{-j-n-1}, \tau_\nu^{-1}),$$

then this implies that

$$\tau_\nu^{-1} \in \operatorname{arc}[\zeta^{-j-l}, \zeta^{-j})$$

and so

$$\Psi_\nu(\zeta^{-j-l}) = 0.$$

Hence

$$[\zeta^{-j}, \zeta^{-j-1}, \dots, \zeta^{-j-n-1}]_{\Lambda_n}\Psi_\nu(y) = 0.$$

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(c) $j = \nu$. In this case we observe that from (3.5).

$$\psi_{\Lambda_n}(\zeta^{-j-l+\nu}) = \psi_{\Lambda_n}(\zeta^{-l}) = 0, \quad l = 1, 2, \dots, n.$$

Hence

$$\Psi_\nu(\zeta^{-j-l}) = 0, \quad l = 1, 2, \dots, n.$$

Since

$$\tau_\nu^{-1} \in \operatorname{arc}(\zeta^{-\nu-n-1}, \zeta^{-\nu}),$$

it follows that

$$\Psi_\nu(\zeta^{-j-n-1}) = 0.$$

Moreover

$$\Psi_\nu(\zeta^{-j}) = \psi_{\Lambda_n}(1) = (-1)^{n-1}\zeta^{\lambda_0+\dots+\lambda_n}$$

by (3.12). Thus when $j = \nu$, we see from (2.13) and (2.12) that

$$[\zeta^{-j}, \zeta^{-j-1}, \dots, \zeta^{-j-n-1}]_{\Lambda_n}\Psi_\nu(y)$$
$$= [1, \zeta^{-1}, \dots, \zeta^{-n-1}]_{\Lambda_n}\Psi_\nu(y\zeta^{-j}) = 1.$$

Combining the results of (a), (b) and (c) above, we see from (4.13) that (4.5) holds, which completes the proof.

5. Approximation by quasi-interpolants. We shall now examine the quasi-interpolant $\mathscr{L}$ as a tool for approximating functions of class $C^n(U)$. In order to do so, we recall the definition of the modulus of continuity for a function $f \in C(U)$. We set

$$\omega(f; h) = \sup\{ |f(z_1) - f(z_2)| : z_1, z_2 \in U, |z_1 - z_2| \leqq h\}.$$

We are interested in the approximating property of $\mathscr{L}$, for fixed Λ_n as $k \to \infty$. We shall prove

THEOREM 3. *For any $f \in C^n(U)$ and $z \in U$, we have the following estimates*:

$$(5.1) \qquad |(\mathscr{L}f)^{(s)}(z) - f^{(s)}(z)| \leqq \frac{K}{k^{n-s}}\omega\left(g; \frac{1}{k}\right), \ s = 0, 1, \dots, n$$

where $g(y) = y^{-\lambda_n}L_nf(y)$ and K is independent of f and k.

It may be observed that $\omega\left(g; \dfrac{1}{k}\right)$ vanishes whenever $f(z) = z^{\lambda_j}, j = 0, 1, \dots, n$.

For the proof of Theorem 3, we shall need two lemmas. In what follows for any $f \in U$, we set

$$\|f\| = \sup_{z \in U}|f(z)|.$$

LEMMA 2. *For* $j = 0, 1, \ldots, n$, *we have*

$$(5.2) \quad \|L_j M_{\Lambda_n}\| = O(k^{j-n}).$$

Proof. Using (2.9) with $a = 1$, $n = 0$, and replacing Λ_0 by $\Lambda_{n,n}$, we see that

$$M_{\Lambda_n}(z) = \int_1^z \phi_{\Lambda_{n,n}}(zv^{-1})v^{-1}D_n M_{\Lambda_n}(v)dv.$$

Since $M_{\Lambda_n}(z)$ has support in the arc$(1, \zeta^{n+1})$, we get

$$\|M_{\Lambda_n}\| = O\left(\frac{1}{k}\right)\|D_n M_{\Lambda_n}\| = O\left(\frac{1}{k}\right)\|M_{\Lambda_{n-1}}\|$$

on using (3.3). Hence

$$(5.3) \quad \|M_{\Lambda_n}\| = O(k^{-n})\|M_{\Lambda_O}\| = O(k^{-n}),$$

since

$$M_{\Lambda_0}(z) = -z^{\lambda_0}\zeta^{-\lambda_0}.$$

Again applying (3.3) successively, we obtain

$$\|L_j M_{\Lambda_n}\| \leqq 2^j\|M_{\Lambda_{j,n}}\| = O(k^{j-n}), \quad j = 0, 1, \ldots, n$$

on observing that $\Lambda_{j,n} = \{\lambda_j, \ldots, \lambda_n]$ and on using (5.3).

LEMMA 3. *For* $j = 0, 1, \ldots, n$, *we have*

$$(5.4) \quad \sup\{\,|\psi_{\Lambda_n}^{(j)}(z)\,|:z \in \text{arc}(\zeta^{-n-1}, 1)\,\} = O(k^j).$$

Proof. From (3.11) it can be seen that

$$\psi_{\Lambda_n}(z) = \sum_{j=0}^{n} \frac{(-1)^{n-1+j}\zeta^{\lambda_j}z^{\lambda_j}}{\prod_{\substack{\nu=0\\ \nu\neq j}}^{n} (\zeta^{-\lambda_j} - \zeta^{-\lambda_\nu})}$$

whence we easily obtain

$$(5.5) \quad \|\psi_{\Lambda_n}^{(\nu)}\| = O(k^n), \quad \nu = 0, 1, \ldots, n.$$

Furthermore, it is known that for any $z \in U$, we have

$$(5.6) \quad [z, \zeta^{-1}, \ldots, \zeta^{-n}]\psi_{\Lambda_n}(y) = \frac{1}{n!}\int_U M(\omega|z, \zeta^{-1}, \ldots, \zeta^{-n})\psi_{\Lambda_n}^{(n)}(\omega)d\omega$$

where the divided difference on the left is the usual divided difference and the B-spline on the right in the integral is the usual B-spline on the circle. If $z \in \text{arc}(\zeta^{-n-1}, 1)$, then by (5.6) and (3.5), we see that

$$(5.7) \quad \psi_{\Lambda_n}(z) = \frac{1}{n!} \int_{\zeta^{-n-1}}^{1} F(z, \omega)\psi_{\Lambda_n}^{(n)}(\omega)d\omega$$

where

$$F(z, \omega) = \prod_{j=1}^{n} (z - \zeta^{-j})M(\omega|z, \zeta^{-1}, \ldots, \zeta^{-n}).$$

For any $\omega, \xi \in (\zeta^{-n-1}, 1)$, we define the truncated power function

$$(\xi - \omega)_+^{n-1} = \begin{cases} (\xi - \omega)^{n-1}, & \text{if } \omega \in \text{arc}(\zeta^{-n-1}, \xi) \\ 0 & , \text{if } \omega \in \text{arc}(\xi, 1). \end{cases}$$

Since the B-spline is the divided difference of the truncated power function, we see that

$$\frac{1}{n+1}F(z, \omega) = (z - \omega)_+^{n-1} - \sum_{l=1}^{n} (\zeta^{-l} - \omega)_+^{n-1} \times \sum_{\substack{r=1 \\ r\neq l}}^{n} \left(\frac{z - \zeta^{-r}}{\zeta^{-l} - \zeta^{-r}}\right).$$

From the above it is easy to see that

$$(5.8) \quad \left|\frac{\partial^j}{\partial z^j}F(z, \omega)\right| \leqq \frac{C}{k^{n-1-j}}, \quad j = 0, 1, \ldots, n - 1$$

for all $z, \omega \in \text{arc}(\zeta^{-n-1}, 1)$, where C is a constant independent of k. Differentiating (5.7) j times and using (5.5) and (5.8) we obtain (5.4).

LEMMA 4. *If $G(t) \in C^n(U)$ and if for some $z \in U$,*

$$G^{(\nu)}(z) = 0, \quad \nu = 0, 1, \ldots, n - 1,$$

then for $\omega \in U$, we have

$$(5.9) \quad |L_rG(\omega)| \leqq C_1|\omega - z|^{n-r} \sup_{t\in \text{arc}(\omega,z)} |L_nG(t)|$$

for $r = 0, 1, \ldots, n - 1$, where C_1 is independent of G, ω and z.

Proof. Using (2.9) with f replaced by L_rG, we get for $r = 0, 1, \ldots, n - 1$,

$$(5.10) \quad L_rG(\omega) = \int_z^\omega \phi_{\Lambda_{r,n-1}}(\omega\nu^{-1})\nu^{-1}L_nG(\nu)d\nu.$$

Now from (2.1), we know that

$$\phi_{\Lambda_{r,n-1}}^{(\nu)}(1) = 0 \quad \text{for } \nu = 0, 1, \ldots, n - 2 - r, \text{ and}$$
$$\phi_{\Lambda_{r,n-1}}^{(n-1-r)}(1) = 1.$$

So by the classical Taylor's formula with remainder,

$$\phi_{\Lambda_{r,n-1}}(t) = \frac{1}{(n-2-r)!}\int_1^t (t-v)^{n-2-r}\phi_{\Lambda_{r,n-1}}^{(n-1-r)}(v)dv.$$

Hence we get

$$(5.11)\quad |\phi_{\Lambda_{r,n-1}}(t)| \leqq C_1|t-1|^{n-1-r}$$

which combined with (5.10), yields (5.9).

Proof of Theorem 3. In order to prove (5.1) it is enough to show that

$$(5.12)\quad |L_s(\mathscr{L}f)(z) - L_sf(z)| \leqq \frac{K}{k^{n-s}}\omega\left(g; \frac{1}{k}\right), \quad s = 0, 1, \dots, n.$$

Set

$$G(t) = f(t) - P_z(t)$$

where

$$P_z(t) \in \Pi(\Lambda_n) \quad \text{and}$$
$$(f^{(\nu)} - P_z^{(\nu)})(z) = 0, \quad (\nu = 0, 1, \dots, n).$$

Then

$$(5.13)\quad \begin{aligned} L_s(\mathscr{L}f)(z) - L_sf(z) &= L_s(\mathscr{L}f)(z) - L_sP_z(z) \\ &= L_s(\mathscr{L}f)(z) - L_s(\mathscr{L}P_z)(z), \quad \text{by (4.3)} \\ &= L_s(\mathscr{L}G)(z). \end{aligned}$$

From (4.1), we see that

$$(5.14)\quad L_s(\mathscr{L}G)(z) = \sum_{\nu=0}^{k-1} T_\nu(G)L_sM_{\Lambda_n}(z\zeta^{-\nu})$$

where from (4.2), we have

$$(5.15)\quad T_\nu(G) = \sum_{r=0}^{n} a_{\nu,r}(L_rG)(\tau_\nu).$$

By Lemma 4, we can see that for $r = 0, 1, \dots, n-1$

$$(5.16)\quad |(L_rG)(\tau_\nu)| \leqq C_1|\tau_\nu - z|^{n-r} \sup_{t\in \text{arc}(\tau_\nu,z)} |L_nG(t)|.$$

From the definition of $P_z(t)$ it follows that

$$L_nP_z(v) = C_2v^{\lambda_n}$$

(C_2 a constant) and

$$L_nP_z(z) = L_nf(z),$$

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so that

$$
\begin{aligned}
L_nG(v) &= L_nf(v) - L_nP_z(v) \\
&= L_nf(v) - (vz^{-1})^{\lambda_n}L_nf(z) \qquad (5.17)\\
&= v^{\lambda_n}(g(v) - g(z)).
\end{aligned}
$$

where

$$g(v) = v^{-\lambda_n}L_nf(v).$$

From (5.16) and (5.17), we obtain

$$|L_rG(\tau_\nu)| \leqq C_1|\tau_\nu - z|^{n-r}\omega(g;|\tau_\nu - z|), \quad r = 0, 1, \dots, n$$

which from (5.15) yields

$$(5.18) \quad |T_\nu(G)| \leqq C_1 \sum_{r=0}^{n} |a_{\nu,r}|\,|\tau_\nu - z|^{n-r}\omega(g;|\tau_\nu - z|).$$

Since $\tau_\nu \in \operatorname{arc}(\zeta^\nu, \zeta^{\nu+n+1})$, i.e., $\tau_\nu^{-1}\zeta^\nu \in \operatorname{arc}(\zeta^{-n-1}, 1)$, it follows from (4.4) and (5.4) that

$$(5.19) \quad |a_{\nu,r}| = O(k^{n-r}), \quad r = 0, 1, \dots, n.$$

Observe that $M_{\Lambda_n}(z\zeta^{-\nu})$ is non-zero only if $z \in \operatorname{arc}(\zeta^\nu, \zeta^{\nu+n+1})$ and since τ_ν also lies in this arc, we have

$$|\tau_\nu - z| = O(k^{-1}),$$

so that (5.18) and (5.19) give

$$|T_\nu(G)| \leqq C_2\omega\left(g; \frac{1}{k}\right).$$

Hence from (5.14), we obtain

$$|L_s(\mathcal{L}G)(z)| \leqq \frac{C_3}{k^{n-s}}\omega\left(g; \frac{1}{k}\right),$$

which is equivalent to (5.12) because of (5.13).

6. Bernstein-Schoenberg type operator. While the quasi-interpolant requires information about the value of the function and its derivative up to order n at k points, the B-S operator needs only function-values at k points. In view of this, it is of some interest to define the B-S type operator for Λ-splines.

Using (2.2) and (3.11) and comparing coefficients of y^{λ_j} on both sides in (3.6), we obtain

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$$(6.1)\qquad z^{\lambda_j} = C_j(\Lambda_n)\sum_{\nu=0}^{k-1}\zeta^{\nu\lambda_j}M_{\Lambda_n}(z\zeta^{-\nu}),\quad j = 0, 1, \ldots, n$$

where

$$(6.2)\qquad C_j(\Lambda_n) = (-1)\zeta^{\lambda_j}\prod_{\substack{l=0\\ l\neq j}}^{n}\frac{\lambda_j-\lambda_l}{\zeta^{-\lambda_j}-\zeta^{-\lambda_l}}.$$

We shall show that there is a unique linear operator

$$(6.3)\qquad (Sf)(z) = \sum_{\nu=0}^{k-1} b_\nu f(\sigma_\nu)M_{\Lambda_n}(z\zeta^{-\nu})$$

which reproduces z^{λ_0} and z^{λ_1}. This requirement gives, in view of (6.1)

$$b_\nu\sigma_\nu^{\lambda_0} = C_0(\Lambda_n)\zeta^{\nu\lambda_0},\quad b_\nu\sigma_\nu^{\lambda_1} = C_1(\Lambda_n)\zeta^{\nu\lambda_1}.$$

It is easy to see that

$$b_\nu = \{C_0(\Lambda_n)\}^{\lambda_1/(\lambda_1-\lambda_0)}\{C_1(\Lambda_n)\}^{\lambda_0/(\lambda_0-\lambda_1)} =: b(\Lambda_n)$$

and

$$\sigma_\nu := \sigma_\nu(\Lambda_n) = \left\{\frac{C_1(\Lambda_n)}{C_0(\Lambda_n)}\right\}^{(\lambda_1-\lambda_0)^{-1}}\zeta^\nu.$$

From (6.2) it follows by elementary computation that

$$\sigma_\nu = R\zeta^{1/2(n+1)+\nu}$$

where

$$(6.4)\qquad R^{\lambda_1-\lambda_0} = \prod_{l=2}^{n}\left(\frac{\sin\frac{\lambda_l-\lambda_0}{k}\pi}{\frac{\lambda_l-\lambda_0}{k}\pi}\right)\left(\frac{\sin\frac{\lambda_l-\lambda_1}{k}\pi}{\frac{\lambda_l-\lambda_1}{k}\pi}\right).$$

We now renormalize our B-splines $M_{\Lambda_n}(z)$ and set

$$(6.5)\qquad N_{\Lambda_n}(z) = b(\Lambda_n)M_{\Lambda_n}(z).$$

From (6.2) and Lemma 2, we get

$$(6.6)\qquad N_{\Lambda_n}(z) = O(1).$$

Our operator $(Sf)(z)$ now takes the form

$$(6.7)\qquad (Sf)(z) = \sum_{\nu=0}^{k-1} f(\sigma_\nu)N_{\Lambda_n}(z\zeta^{-\nu}),\quad \sigma_\nu = R\zeta^{1/2(n+1)+\nu}.$$

When $\lambda_0 = 0$, we note that (6.7) shows that the normalized B-splines $N_{\Lambda_n}(z\zeta^{-\nu})$, $(\nu = 0, 1, \ldots, k - 1)$ form a partition of unity.

For a study of the convergence of this operator, we shall prove

LEMMA 5. *For $r = 0, 1, \ldots, n - 1$ we have the identity*

$$(6.8) \quad \widetilde{L}_r(Sf)(z) = b(\Lambda_n) \sum_{\nu=0}^{k-1} [1, \zeta^{-1}, \ldots, \zeta^{-r}]_{\Lambda_{n-r+1,n}} \times f(\sigma_\nu y) M_{\Lambda_{n-r}}(z\zeta^{-\nu}).$$

Proof. We shall prove (6.8) by induction on r. For $r = 0$, (6.8) reduces to (6.7). We assume then that (6.8) is true for some $r < n - 1$. Then

$$\widetilde{L}_{r+1}(Sf)(z) = D_{n-r}\widetilde{L}_r(Sf)(z).$$

Applying our inductive hypothesis and observing that by (3.3),

$$D_{n-r}M_{\Lambda_{n-r}}(z\zeta^{-\nu}) = M_{\Lambda_{n-r-1}}(z\zeta^{-\nu-1}) - \zeta^{-\lambda_{n-r}}M_{\Lambda_{n-r-1}}(z\zeta^{-\nu})$$

we have after elementary rearrangement

$$\widetilde{L}_{r+1}(Sf)(z) = b(\Lambda_n)\sum_{\nu=0}^{k-1} [1, \zeta^{-1}, \ldots, \zeta^{-r}]_{\Lambda_{n-r+1,n}} F_\nu(y) M_{\Lambda_{n-r-1}}(z\zeta^{-\nu})$$

where

$$F_\nu(y) = f(\sigma_{\nu-1}y) - \zeta^{-\lambda_{n-r}}f(\sigma_\nu y).$$

We note that $\sigma_{\nu-1} = \sigma_\nu\zeta^{-1}$ and apply (2.13) and (2.15) to derive (6.8) with r replaced by $r + 1$ which completes the proof.

We shall now prove

THEOREM 4. *Let $f(z)$ be defined on some annulus $\{z:\rho_1 \leqq |z| \leqq \rho_2\}$ for some $\rho_1 < 1 < \rho_2$. Suppose that for any η, $\rho_1 \leqq \eta \leqq \rho_2$, the function $f(\eta z)$ lies in $C^r(U)$, $z \in U$ for some r, $0 \leqq r \leqq n-1$. Moreover let*

$$H_r(\eta z) := (\eta z)^{-\lambda_0}\widetilde{L}_r f(\eta z)$$

be continuous for $z \in U$, $\rho_1 \leqq \eta \leqq \rho_2$. Then

$$(6.9) \quad |\widetilde{L}_r(Sf)(z) - \widetilde{L}_r f(z)| \leqq C\omega\left(H_r; \frac{1}{k}\right)$$

where C is independent of f and k.

Proof. Since the operator S reproduces z^{λ_0}, it follows from (6.7) that

$$(6.10) \quad z^{\lambda_0} = \sum_{\nu=0}^{k-1} (\sigma_\nu(\Lambda_{n-r}))^{\lambda_0} N_{\Lambda_{n-r}}(z\zeta^{-\nu}).$$

Then from (6.8) we obtain

$$(6.11)\quad \tilde{L}_r(Sf)(z) - \tilde{L}_r f(z) = \frac{b(\Lambda_n)}{b(\Lambda_{n-r})} \sum_{\nu=0}^{k-1} (\Delta_\nu f) N_{\Lambda_{n-r}}(z\zeta^{-\nu})$$

where

$$(6.12)\quad (\Delta_\nu f) = [1, \zeta^{-1}, \ldots, \zeta^{-r}]_{\Lambda_{n-r+1,n}} f(\sigma_\nu(\Lambda_n)y) - \frac{b(\Lambda_{n-r})}{b(\Lambda_n)} \tilde{L}_r f(z) z^{-\lambda_0} (\sigma_\nu(\Lambda_{n-r}))^{\lambda_0}.$$

From (3.4) it follows that

$$(6.13)\quad [1, \zeta^{-1}, \ldots, \zeta^{-r}]_{\Lambda_{n-r+1,n}} f = \int_U M_{\Lambda_{n-r+1,n}}(v^{-1}) v^{-1} (\tilde{L}_r f)(v) dv.$$

In particular for $f(z) = z^{\lambda_0}$, this yields (from (6.2)),

$$(6.14)\quad \int_U M_{\Lambda_{n-r+1,n}}(v^{-1}) v^{\lambda_0 - 1} dv = \prod_{j=n-r+1}^{n} \frac{\zeta^{\lambda_0} - \zeta^{\lambda_j}}{\lambda_0 - \lambda_j} = \frac{C_0(\Lambda_{n-r})}{C_0(\Lambda_n)}.$$

Hence from (6.12), (6.13) and (6.14) after some simplification, we get

$$(6.15)\quad \Delta_\nu f = \int_U M_{\Lambda_{n-r+1,n}}(v^{-1}) v^{\lambda_0 - 1} (\sigma_\nu(\Lambda_\nu))^{\lambda_0} \{H_r(\sigma_\nu(\Lambda_n)v) - H_r(z)\} dv.$$

For a fixed $z \in U$, we shall estimate $\Delta_\nu f$ in (6.11) for those values of ν for which

$$N_{\Lambda_{n-r}}(z\zeta^{-\nu}) \neq 0,$$

i.e., for $z \in \operatorname{arc}(\zeta^\nu, \zeta^{\nu+n-r+1})$. Moreover, the integrand in (6.15) is non-zero only for values of v in the $\operatorname{arc}(\zeta^{-r-1}, 1)$. Recalling that

$$\sigma_\nu(\Lambda_n) = R\zeta^{1/2(n+1)+\nu} \quad \text{and} \quad 1 - R = O(k^{-2}),$$

we see that

$$|v\sigma_\nu(\Lambda_n) - z| = O(k^{-1})$$

so that using (5.2) of Lemma 2 in (6.15) we obtain

$$|\Delta_\nu f| = O(k^{-r}) \omega\left(H_r; \frac{1}{k}\right).$$

Since

$$|b(\Lambda_n)/b(\Lambda_{n-r})| = O(k^r) \quad \text{and} \quad |N_{\Lambda_{n-r}}(z)| = O(1),$$

the result follows from (6.11).

Remark. The B-S operator (6.7) is defined only for functions f which are defined on some annulus $\{z: \rho_1 \leqq |z| \leqq \rho_2\}$, $\rho_1 < 1 < \rho_2$. However, any function $f \in C(U)$ can be extended to $\tilde{f}$ which is continuous on an

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annulus in a number of ways. Perhaps the simplest way is to set

$$\tilde{f}(\eta z) = f(z), \quad z \in U, \eta > 0.$$

Using this extension we can easily derive from Theorem 4, the following

COROLLARY. *For $f \in C(U)$, set*

$$(6.16) \quad (\tilde{S}f)(z) = \sum_{\nu=0}^{k-1} f(\zeta^{1/2(n+1)+\nu}) N_{\Lambda_n}(z\zeta^{-\nu}).$$

If $f \in C^r(U)$ for some r, $0 \leqq r \leqq n - 1$, then for $z \in U$,

$$|\tilde{L}_r(\tilde{S}f)(z) - \tilde{L}_r f(z)| \leqq C_1 \left\{ \frac{1}{k} \|\tilde{L}_r f\| + \omega\left(\tilde{L}_r f; \frac{1}{k}\right) \right\}$$

where C_1 is independent of f and k.

In particular

$$(\tilde{S}f)^{(\nu)}(z) \rightarrow f^{(\nu)}(z), \quad (\nu = 0, 1, \ldots, r)$$

uniformly on U as $k \rightarrow \infty$.

7. An asymptotic formula. If we suppose the function $f(z)$ to be analytic in a neighbourhood of U, then it is possible to get a more precise result for the error of approximation to f by the B-S type operator. We shall indeed prove

THEOREM 5. *If f is holomorphic in a neighbourhood $\mathcal{D}$ of U, then we have*

$$(7.1) \quad \lim_{k \rightarrow \infty} k^2 \{ (Sf)(z) - f(z) \} = -\frac{1}{6}(n + 1)\pi^2 L_2 f(z).$$

The proof of Theorem 5 will be based on

LEMMA 6. *If $E_{2,k}(z)$ is given by*

$$(7.2) \quad E_{2,k}(z) = \sum_{\nu=0}^{k-1} \phi_{\Lambda_2}(\sigma_\nu z^{-1}) N_{\Lambda_n}(z\zeta^{-\nu})$$

then

$$(7.3) \quad E_{2,k}(z) = -\frac{(n + 1)\pi^2}{6k^2} + O\left(\frac{1}{k^4}\right).$$

Proof. From (2.2) we see that

$$(7.4) \quad V(\lambda_0, \lambda_1, \lambda_2) E_{2,k}(z)$$

$$= \sum_{\nu=0}^{k-1} \begin{vmatrix} (\sigma_\nu z^{-1})^{\lambda_0} & (\sigma_\nu z^{-1})^{\lambda_1} & (\sigma_\nu z^{-1})^{\lambda_2} \\ 1 & 1 & 1 \\ \lambda_0 & \lambda_1 & \lambda_2 \end{vmatrix} N_{\Lambda_2}(z\zeta^{-\nu}).$$

From (6.1) and (6.5) we have

$$\sum_{\nu=0}^{k-1} (\sigma_\nu z^{-1})^{\lambda_j} N_{\Lambda_n}(z\zeta^{-\nu}) = \begin{matrix} 1, j = 0, 1, \\ K, j = 2, \end{matrix}$$

where

$$K = \frac{(C_0(\Lambda_n))^{(\lambda_1-\lambda_2)/(\lambda_1-\lambda_0)}(C_1(\Lambda_n))^{(\lambda_0-\lambda_2)/(\lambda_0-\lambda_1)}}{C_2(\Lambda_n)}.$$

Using (6.2), elementary calculation shows that

$$\text{(7.5)} \quad K = 1 - \frac{(n+1)\pi^2 V(\lambda_0, \lambda_1, \lambda_2)}{6k^2(\lambda_1 - \lambda_0)} + O\left(\frac{1}{k^4}\right).$$

The result now follows from (7.4) and (7.5).

Proof of Theorem 5. Since f is holomorphic in a domain $\mathscr{D}$, formula (2.9) is valid for any points z, a in $\mathscr{D}$. Thus for $z \in U$, $w \in \mathscr{D}$, we have

$$\text{(7.6)} \quad f(\omega) = f(z)\phi_{\Lambda_0}(\omega z^{-1}) + (L_1 f)(z)\phi_{\Lambda_1}(\omega z^{-1}) + (L_2 f)(z)\phi_{\Lambda_2}(\omega z^{-1}) + O(|\omega - z|^3).$$

Using (7.6) with $\omega = \sigma_\nu$, $\nu = 0, 1, \ldots, k-1$ we have

$$\text{(7.7)} \quad (Sf)(z) = \sum_{\nu=0}^{k-1} f(\sigma_\nu) N_{\Lambda_n}(z\zeta^{-\nu})$$
$$= f(z)E_{0,k}(z) + (L_1 f)(z)E_{1,k}(z) + (L_2 f)(z)E_{2,k}(z) + O(|\omega - z|^3).$$

where

$$E_{j,k}(z) = \sum_{\nu=0}^{k-1} \phi_{\Lambda_j}(\sigma_\nu z^{-1}) N_{\Lambda_\nu}(z\zeta^{-\nu}), \quad j = 0, 1, 2.$$

From (2.2),

$$\phi_{\Lambda_0}(\sigma_\nu z^{-1}) = (\sigma_\nu z^{-1})^{\lambda_0} \quad \text{and}$$
$$\phi_{\Lambda_1}(\sigma_\nu z^{-1}) = [(\sigma_\nu z^{-1})^{\lambda_1} - (\sigma_\nu z^{-1})^{\lambda_0}]/(\lambda_1 - \lambda_0),$$

so that using the reproducing property of the B-S operator we have

(7.8) $\quad E_{0,k}(z) = 1 \quad \text{and} \quad E_{1,k}(z) = 0.$

The result then follows from (7.7), (7.8) and (7.3).

Remark. We observe that

$$L_2 f(z) = z^2 f''(z) + (1 - \lambda_0 - \lambda_1) z f'(z) + \lambda_0 \lambda_1 f(z),$$

which shows that the asymptotic formula depends upon λ_0 and λ_1 and not on $\lambda_2, \ldots, \lambda_n$.

8. Trigonometric Λ-splines. We shall consider the special case when the numbers λ_j in Λ are symmetric about the origin, or equivalently, when

(8.1) $$\Lambda_n = \begin{cases} \{\pm\mu_1, \ldots, \pm\mu_m\}, & n = 2m - 1 \\ \{0, \pm\mu_1, \ldots, \pm\mu_m\}, & n = 2m. \end{cases}$$

In this case $\Pi(\Lambda_n)$ is related to the class of trigonometric polynomials $T(\Lambda_n)$ spanned by

$$\{\cos \mu_j\theta, \sin \mu_j\theta\}_1^m \quad \text{when } n = 2m - 1$$

or by

$$\{1, \cos \mu_j\theta, \sin \mu_j\theta\}_1^m \quad \text{when } n = 2m.$$

Indeed, $p(z) \in \Pi(\Lambda_n)$ if and only if $p(e^{i\theta}) \in T(\Lambda_n)$ when Λ_n is given by (8.1).

For a positive integer $k > 2 \max|\mu_j|$ we shall denote by $\mathscr{T}_k(\Lambda_n)$ the class of trigonometric splines $t(\theta)$ which satisfy

i) $t(\theta + 2\pi) = t(\theta)$, $t(\theta) \in C^{n-1}(R)$,
ii) $t(\theta)|_{(jh,jh+h)} \in T(\Lambda_n)$, for all integers j, where $h = 2\pi/k$.

It follows that taking

$$Z_k = \{1, e^{ih}, \ldots, e^{i(k-1)h}\},$$

$S(z) \in \mathscr{S}(\Lambda_n, Z_k)$ if and only if

$$S(e^{i\theta}) \in \mathscr{T}_k(\Lambda_n).$$

From Proposition 1, we see that

$$\dim \mathscr{T}_k(\Lambda_n) = k.$$

Let $q_{\Lambda_n}(\theta) \in T(\Lambda_n)$ be such that

$$q_{\Lambda_n}^{(\nu)}(0) = \begin{cases} 0, & \nu = 0, 1, \ldots, n - 1 \\ 1, & \nu = n. \end{cases}$$

It is easy to see from (2.1) that

(8.2) $$q_{\Lambda_n}(\theta) = i^{-n} \phi_{\Lambda_n}(e^{i\theta}).$$

It is now possible to define the trigonometric B-splines $Q_{\Lambda_n}(\theta)$ as a trigonometric Λ-divided difference of $q_{\Lambda_n}(\theta - y)(\theta - y)_+^0$. However for the sake of brevity, we set

(8.3) $\quad Q_{\Lambda_n}(\theta) = -i^n M_{\Lambda_n}(e^{i\theta}).$

It follows immediately from Proposition 3 that $\{Q_{\Lambda_n}(\theta - \nu h)\}_0^{k-1}$ form a basis for the space $\mathscr{T}_k(\Lambda_n)$.

We shall use the symbol Λ_n^p to denote the set $\Lambda_n \backslash \{\pm\mu_p\}$. Using (8.3) and Lemma 1, we shall prove

LEMMA 6. *The B-splines $Q_{\Lambda_n}(\theta)$ satisfying the following recurrence relations*:

$$(8.4) \quad (\mu_m^2 - \mu_1^2)Q_{\Lambda_n}(\theta) = Q_{\Lambda_n^m}(\theta - 2h) - 2\cos\mu_m h Q_{\Lambda_n^m}(\theta - h) + Q_{\Lambda_n^m}(\theta) - \{Q_{\Lambda_n^1}(\theta - 2h) - 2\cos\mu_1 h Q_{\Lambda_n^1}(\theta - h) + Q_{\Lambda_n^1}(\theta)\}, \quad n \geqq 3$$

and for n even

$$(8.5) \quad Q'_{\Lambda_n}(\theta) = Q_{\Lambda_{n-1}}(\theta) - Q_{\Lambda_{n-1}}(\theta - h).$$

Proof. In order to prove (8.4), we use (3.2) with $\zeta = e^{ih}$, $\lambda_0 = \mu_1$, $\lambda_n = \mu_m$, and obtain

$$(\mu_m - \mu_1)M_{\Lambda_n}(z) = M_A(ze^{-ih}) - e^{-\mu_1 ih}M_A(z) - M_B(ze^{-ih}) + e^{-i\mu_m h}M_B(z).$$

where $A = \Lambda_n \backslash \{\mu_1\}$ and $B = \Lambda_n \backslash \{\mu_m\}$.

We again apply (3.2) to $M_A(ze^{-ih})$ and $M_A(z)$ with $\lambda_0 = -\mu_1$ and $\lambda_n = \mu_m$. Also we use (3.2) for $M_B(ze^{-ih})$ and $M_B(z)$ with $\lambda_0 = \mu_1$ and $\lambda_n = -\mu_m$. After simplification, we get

$$(\mu_m^2 - \mu_1^2)M_{\Lambda_n}(z) = M_{\Lambda^1_n}(ze^{-2ih}) - 2\cos\mu_1 h M_{\Lambda_n^1}(ze^{-ih}) + M_{\Lambda_n^1}(z) - \{M_{\Lambda_n^m}(ze^{-2ih}) - 2\cos\mu_m h M_{\Lambda_n^m}(ze^{-ih}) + M_{\Lambda_n^m}(z)\}.$$

Formula (8.4) follows now on using (8.3).

In order to prove (8.5) we use (8.3) and (3.3).

Remark. As an application of (8.4) and (8.5) we show that $Q_{\Lambda_n}(\theta)$ is real. When $n = 1$,

$$Q_{\Lambda_1}(\theta) = \frac{\sin\mu_1\theta}{\mu_1}, \quad \text{for } 0 < \theta < h \quad \text{and}$$

$$Q_{\Lambda_1}(\theta) = \frac{\sin \mu_1(2h - \theta)}{\mu} \quad \text{for } h < \theta < 2h.$$

It follows from (8.4) that $Q_{\Lambda_n}(\theta)$ is real for all odd n. From this and from (8.5) we see that $Q'_{\Lambda_n}(\theta)$ is real for n even. But from (3.1) and (8.3) we observe that for $0 < \theta < h$,

$$Q_{\Lambda_n}(\theta) = q_{\Lambda_n}(\theta),$$

whence it follows that for n even, $Q_{\Lambda_n}(\theta)$ is real.

Putting $z = e^{i\theta}$, $y = e^{-i\alpha}$ in (3.6) we can deduce from Theorem 1 an analogue of Marsden's identity. We state without proof

THEOREM 6. *If* $V_{\Lambda_n}(\theta) \in T(\Lambda_n)$ *and satisfies the conditions*

$$(8.6) \qquad \begin{cases} V_{\Lambda_n}(0) = 1 \\ V_{\Lambda_n}(jh) = 0, \quad j = 1, 2, \dots, n \end{cases}$$

then we have the identity

$$(8.7) \qquad q_{\Lambda_n}(\theta - \alpha) = \sum_{j=0}^{k-1} V_{\Lambda_n}(\alpha - jh) Q_{\Lambda_n}(\theta - jh).$$

We note from (3.11) and (3.12) that

$$V_{\Lambda_n}(\theta) = (-1)^{n-1} \psi_{\Lambda_n}(e^{-i\theta}).$$

In order to define the quasi-interpolant for trigonometric Λ-splines, we need to introduce some differential operators. We shall denote in the sequel $d/d\theta$ by D. If $n = 2m - 1$ and $\Lambda_n = \{\pm\mu_1, \dots, \pm\mu_m\}$, we set

$$(8.8) \qquad \Theta_0 = I, \ \Theta_{2r} = \prod_{j=1}^{r} (D^2 + \mu_j^2), \ \Theta_{2r+1} = D\Theta_{2r}.$$

Similarly if $n = 2m$ and $\Lambda_n = \{0, \pm\mu_1, \dots, \pm\mu_m\}$, we set

$$(8.9) \qquad \Theta_0 = I, \ \Theta_{2r-1} = D \prod_{j=1}^{r-1} (D^2 + \mu_j^2), \ \Theta_{2r} = D\Theta_{2r-1}.$$

For n even (or odd) we set

$$\tilde{\Theta}_0 = 1, \ \tilde{\Theta}_{2r} = \prod_{j=m-r+1}^{m} (D^2 + \mu_j^2), \ \tilde{\Theta}_{2r+1} = D\tilde{\Theta}_{2r}.$$

We now choose points $\tau_\nu (\nu = 0, 1, \dots, k - 1)$ with $\tau_\nu \in (\nu h, (\nu + n + 1)h)$ and consider a linear operator

$$\mathscr{L}^*: C_2^n(R) \to \mathscr{T}_k(\Lambda_n)$$

of the following form:

$$(8.11)\quad (\mathscr{L}^*f)(\theta) = \sum_{\nu=0}^{k-1} T_\nu^*(f) Q_{\Lambda_n}(\theta - \nu h)$$

where

$$(8.12)\quad T_\nu^*(f) = \sum_{r=0}^{n} b_{\nu,r}(\Theta_r f)(\tau_\nu)$$

and $b_{\nu,r}$ are constants depending on τ_ν but not on f.

We can then prove

THEOREM 7. *An operator* $\mathscr{L}^*$ *of the form given by* (8.11) *and* (8.12) *satisfies*

$$(8.13)\quad (\mathscr{L}^*S)(\theta) = S(\theta), \quad \textit{for all } S(\theta) \in \mathscr{T}_k(\Lambda_n)$$

if and only if

$$(8.14)\quad b_{\nu,r} = (-1)^{n-r}(\widetilde{\Theta}_{n-r}V_{\Lambda_n})(\tau_\nu - \nu h), \quad \nu = 0, 1, \dots, k-1.$$

where V_{Λ_n} *is given by* (8.6).

For $\mathscr{L}^*$ of the form (8.11) and (8.12), define an operator

$$\mathscr{L}: C^n(\nu) \to \mathscr{S}$$

by

$$\mathscr{L}g(e^{i\theta}) = \mathscr{L}^*f(\theta), \quad \text{when } g(e^{i\theta}) = f(\theta).$$

It is easily seen that $\mathscr{L}$ is of the form (4.1) and (4.2). Moreover $\mathscr{L}^*$ satisfies (8.13) if and only if $\mathscr{L}$ satisfies (4.3); also $\mathscr{L}^*$ satisfies (8.14), if and only if $\mathscr{L}$ satisfies (4.4). Theorem 7 then follows from Theorem 2.

From Theorem 3 we can deduce

THEOREM 8. *If* $f(\theta) \in C_{2\pi}^n(R)$, *then the following estimate holds*:

$$(8.15)\quad |(\mathscr{L}^*f)^{(s)}(\theta) - f^{(s)}(\theta)| \leqq Kh^{n-s}\{\omega(g_1; h) + \omega(g_2; h)\} \quad (s = 0, 1, \dots, n)$$

where

$$(8.16)\quad g_1(\theta) + ig_2(\theta) = e^{2i\mu_m\theta}D(e^{-i\mu_m\theta})\Theta_{n-1}f.$$

If $n = 2m$, *the right hand side of* (8.15) *can be replaced by*

$$Kh^{n-s}\omega(\widetilde{\Theta}_n f; h).$$

It may be observed that $\omega(g_1; h)$ and $\omega(g_2; h)$ both vanish when $f \in T(\Lambda_n)$.

We now consider the B-S operator (6.7) where Λ_n is given by (8.1) and

$\lambda_0 = \mu_1$, $\lambda_1 = -\mu_1$. It is easily seen from (6.4) that in this case $R = 1$ so that

$$(8.17)\quad (Sg)(z) = \sum_{\nu=0}^{k-1} g(\zeta^{1/2(n+1)+\nu})N_{\Lambda_n}(z\zeta^{-\nu}).$$

Thus in this case S coincides with $\widetilde{S}$ given by (6.16). We now define an operator

$$S^*: C_{2\pi}(R) \rightarrow \mathcal{T}_k(\Lambda_n)$$

by

$$(8.18)\quad (S^*f)(\theta) = (Sg)(e^{i\theta}),\quad g(e^{i\theta}) = f(\theta).$$

It follows from (8.17) and (8.16) that S^* reproduces $\cos \mu_1\theta$ and $\sin \mu_1\theta$. An explicit formula for S^*f can be derived from (8.18), (8.17), (6.5) and (8.3). Indeed we have

$$(8.19)\quad (S^*f)(\theta) = \sum_{\nu=0}^{k-1} f\left(\frac{1}{2}(n+1)h + \nu h\right)A_1(\Lambda_n)Q_{\Lambda_n}(\theta - \nu h)$$

where

$$A_1(\Lambda_n) = \begin{cases} \dfrac{\mu_1}{\sin \mu_1 h} \displaystyle\prod_{j=2}^{m} \dfrac{\left(\frac{1}{2}\mu_1\right)^2 - \left(\frac{1}{2}\mu_j\right)^2}{\sin^2 \frac{1}{2}\mu_1 h - \sin^2 \frac{1}{2}\mu_j h}, & n = 2m - 1, \\[2ex] \dfrac{1}{\cos \frac{1}{2}\mu_1 h} \, \dfrac{\left(\frac{1}{2}\mu_1\right)^2}{\sin^2 \frac{1}{2}\mu_1 h} \displaystyle\prod_{j=2}^{m} \dfrac{\left(\frac{1}{2}\mu_1\right)^2 - \left(\frac{1}{2}\mu_j\right)^2}{\sin^2 \frac{1}{2}\mu_1 h - \sin^2 \frac{1}{2}\mu_j h}, & n = 2m. \end{cases}$$

From Corollary to Theorem 4, we can deduce

THEOREM 9. *If $f(\theta) \in C^r_{2\pi}(R)$ for some r, $0 \leqq r \leqq n - 1$, then*

$$(8.20)\quad |(S^*f)^{(r)}(\theta) - f^{(r)}(\theta)| \leqq C\left\{h\sum_{\nu=0}^{r} \|f^{(\nu)}\| + \omega(f^{(r)}; h)\right\},$$

where C is independent of f and h.

Finally we consider an analogue of the asymptotic formula (7.1), which was proved under the assumption that f is holomorphic in a neighbourhood of U. However if the number R occurring in the definition of Sf is 1, then we can prove (7.1) even for $f \in C^3(U)$, because then we require formula (7.6) only when $\omega, z \in U$. Thus from Theorem 5 we can deduce

THEOREM 10. *If* $f \in C^3_{2\pi}(R)$, *then*

$$\lim_{h \to 0} h^{-2}\{ (S^*f)(\theta) - f(\theta) \} = \frac{1}{24}(n + 1)(f''(\theta) + \mu_1^2 f(\theta)).$$

For $\Lambda = \{0, 1, \ldots, n\}$ it is shown in [**4**] that the B-S operator S^* is variation-diminishing, i.e., the number of times which S^*f changes sign in $[0, 2\pi]$ is no greater than the number of times which f changes sign in $[0, 2\pi]$. It would seem plausible that S^* is also variation-diminishing for more general Λ, possibly under a restriction on the size of h.

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