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Enumeration of three-quadrant walks via invariants: some diagonally symmetric models

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Abstract. In the past 20 years, the enumeration of plane lattice walks confined to a convex cone—normalized into the first quadrant—has received a lot of attention, stimulated the development of several original approaches, and led to a rich collection of results. Most of these results deal with the nature of the associated generating function: for which *models* is it algebraic, D-finite, D-algebraic? By *model*, what we mean is a finite collection of allowed steps.

More recently, similar questions have been raised for nonconvex cones, typically the three-quadrant cone $\mathcal{C} = \{(i, j) : i \geq 0 \text{ or } j \geq 0\}$. They turn out to be more difficult than their quadrant counterparts. In this paper, we investigate a collection of eight models in \mathcal{C} , which can be seen as the first level of difficulty beyond quadrant problems. This collection consists of diagonally symmetric models in $\{-1, 0, 1\}^2 \setminus \{(-1, 1), (1, -1)\}$. Three of them are known not to be D-algebraic. We show that the remaining five can be solved in a uniform fashion using Tutte's notion of *invariants*, which has already proved useful for some quadrant models. Three models are found to be algebraic, one is (only) D-finite, and the last one is (only) D-algebraic. We also solve in the same fashion the diagonal model $\{\nearrow, \nwarrow, \swarrow, \searrow\}$, which is D-finite. The three algebraic models are those of the Kreweras trilogy, $\mathcal{S} = \{\nearrow, \leftarrow, \downarrow\}$, $\mathcal{S}^* = \{\rightarrow, \uparrow, \swarrow\}$, and $\mathcal{S} \cup \mathcal{S}^*$.

Our solutions take similar forms for all six models. Roughly speaking, the square of the generating function of three-quadrant walks with steps in \mathcal{S} is an explicit rational function in the *quadrant* generating function with steps in $\mathcal{S} := \{(j-i, j) : (i, j) \in \mathcal{S}\}$. We derive various exact or asymptotic corollaries, including an explicit algebraic description of a positive harmonic function in \mathcal{C} for the (reverses of the) five models that are at least D-finite.

1 Introduction

1.1 Walks in a quadrant

Over the last two decades, the enumeration of walks in the nonnegative quadrant

$$\mathcal{Q} := \{(i, j) : i \geq 0 \text{ and } j \geq 0\}$$

has attracted a lot of attention and established its own scientific community with close to 100 research papers (see, e.g., [10] and citing papers). Most of the attention has

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focused on walks *with small steps*, that is, taking their steps in a fixed subset \mathcal{S} of $\{-1, 0, 1\}^2 \setminus \{(0, 0)\}$. We often call \mathcal{S} -walk, a walk taking its steps in \mathcal{S} . For each step set \mathcal{S} (often called a *model*, henceforth), one considers a three-variate generating function $Q(x, y; t)$ defined by

$$(1.1) \quad Q(x, y; t) = \sum_{n \geq 0} \sum_{i, j \in \Omega} q_{i, j}(n) x^i y^j t^n,$$

where $q_{i, j}(n)$ is the number of quadrant \mathcal{S} -walks starting from $(0, 0)$, ending at (i, j) , and having in total n steps. For each small step model \mathcal{S} , one now knows whether and where the series $Q(x, y; t)$ fits in the following classical hierarchy of series:

$$(1.2) \quad \text{rational} \subset \text{algebraic} \subset \text{D-finite} \subset \text{D-algebraic}.$$

Recall that a series (say $Q(x, y; t)$ in our case) is *rational* if it is the ratio of two polynomials, *algebraic* if it satisfies a polynomial equation (with coefficients that are polynomials in the variables), *D-finite* if it satisfies three *linear* differential equations (one in each variable), again with polynomial coefficients, and finally *D-algebraic* if it satisfies three *polynomial* differential equations. One central result in the classification of quadrant problems is that, for the 79 (nontrivial and essentially distinct) models with small steps, the series $Q(x, y; t)$ is D-finite if and only if a certain group, which is easy to construct from the step set \mathcal{S} , is finite [5, 6, 10, 24, 25, 27].

1.2 Walks in a three-quadrant cone

In 2016, the author turned her attention to nonconvex cones [9] and initiated the enumeration of lattice paths confined to the three-quadrant cone

$$\mathcal{C} := \{(i, j) : i \geq 0 \text{ or } j \geq 0\}$$

(see Figure 1). We can say that such walks *avoid the negative quadrant*. These problems turn out to be significantly harder than their quadrant counterparts. In [9], the two most natural models were solved: *simple walks* with steps in $\{\rightarrow, \uparrow, \leftarrow, \downarrow\}$ and *diagonal walks* with steps in $\{\nearrow, \nwarrow, \swarrow, \searrow\}$. The associated series $C(x, y; t)$, defined analogously to (1.1),

$$C(x, y; t) = \sum_{n \geq 0} \sum_{(i, j) \in \mathcal{C}} c_{i, j}(n) x^i y^j t^n,$$

was proved to be D-finite for both models. It then became natural to explore more three-quadrant problems, in particular to understand whether the D-finiteness of $C(x, y; t)$ was again related to the finiteness of the associated group—or even whether the series $C(x, y; t)$ and $Q(x, y; t)$ would always lie *in the same class* of the hierarchy (1.2) (with the exception of five “singular” models that are nontrivial for the quadrant problem but become trivial, with a rational series, for the three-quadrant cone). This conjecture of Dreyfus and Trotignon [17] holds so far for all solved cases, and will only be reinforced by this paper.¹

¹In a very recent preprint [20], Elvey Price proves this conjecture for the dependence in x .

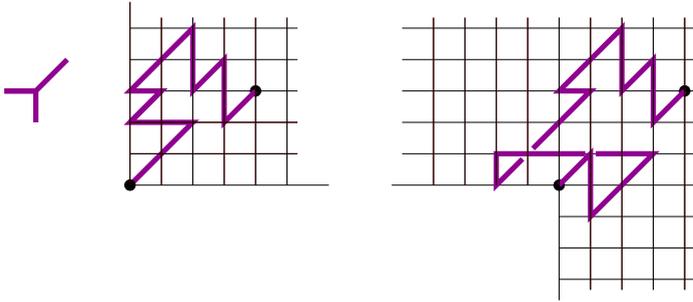


Figure 1: Two walks with Kreweras steps $\nearrow, \leftarrow, \downarrow$, one in the first quadrant \mathcal{Q} (left), and one in the three-quadrant cone \mathcal{C} (right). The associated generating functions are algebraic.

Using an asymptotic argument, Mustapha [28] quickly proved that the 51 three-quadrant problems associated with an infinite group have, as their quadrant counterparts, a non-D-finite solution. Regarding exact solutions, Raschel and Trotignon obtained in [31] sophisticated integral expressions for $C(x, y; t)$ for the step sets \mathcal{S} of Table 1 (apart from the diagonal model). The first four have a finite group, and the expressions of [31] imply that $C(x, y; t)$ is indeed D-finite for these models (at least in x and y). The other four have an infinite group, and $C(x, y; t)$ is non-D-finite by [28]. These four non-D-finite models, labeled from #6 to #9 in Table 1, were further studied by Dreyfus and Trotignon [17]: they proved that $C(x, y; t)$ is D-algebraic for the first one (#6), but not for the other three. More recently, the method used in the original paper [9] was adapted to solve the so-called king model where all eight steps are allowed [11, 12], again with a D-finite solution (and a finite group). Finally, remarkable results of Budd [13] and Elvey Price [19] on the winding number of various families of plane walks provide explicit D-finite expressions for several series counting walks in \mathcal{C} with prescribed endpoints.

1.3 Invariants

One of the many approaches that have been applied to quadrant walks relies on the notion of *invariants*, introduced by William Tutte in the seventies and eighties in his study of colored planar triangulations [33]. These invariants come by pairs and consist of two series $I(x; t)$ and $J(y; t)$ satisfying some properties (precise definitions will be given later). It follows from [2, 4, 15, 16, 22] that invariants play a crucial role in the classification of quadrant models. In particular, it was first proved in [2] that:

- among the 79 nontrivial quadrant models, exactly 13 admit invariants involving the series $Q(x, y; t)$: four of these models have a finite group and nine an infinite group;
- one can use these invariants to prove, in a uniform manner, that $Q(x, y; t)$ is *algebraic* for the four models with a finite group (and invariants);
- one can use these invariants to prove, in a uniform manner, that $Q(x, y; t)$ is *D-algebraic* for the nine models with an infinite group (and invariants).

Moreover, it was already known at that time that the other 19 models with a finite group are *transcendental* (i.e., not algebraic) [4]. Hence, for finite group models, algebraicity

is equivalent to the existence of invariants involving Q . Similarly, the D-algebraicity results for the nine infinite group models were complemented using differential Galois theory [15, 16], proving that for infinite group models, D-algebraicity is equivalent to the existence of invariants involving Q . This equivalence has recently been generalized to quadrant walks with weighted steps (in the infinite group case) [22].

1.4 Main results

The aim of this paper is to explore the applicability of invariants in the solution of three-quadrant models. We focus on the nine models of Table 1, because the series $C(x, y; t)$ can then be described by an equation that is reminiscent of a quadrant equation, although more complex. These models (apart from the diagonal one) are

	Kreweras	Reverse Kreweras	Double Kreweras	Simple	Diagonal
\mathcal{S}					
\mathcal{I}					
				Gessel	
$C_{\mathcal{S}}(x, y; t)$	Alg. Sec. 4	Alg. Sec. 5	Alg. Sec. 6	DF Sec. 8.1 [9, 31]	DF Sec. 8.2 [9, 31]

	#6	#7	#8	#9
\mathcal{S}				
\mathcal{I}				
$C_{\mathcal{S}}(x, y; t)$	DA Sec. 7 [17]	Non-DA [17]	Non-DA [17]	Non-DA [17]

Table 1: The nine models \mathcal{S} considered in this paper. One is the diagonal model (shaded column). The others are the eight models with x/y -symmetry and no step \nearrow nor \searrow . Each model is shown with its companion model \mathcal{I} , with step polynomial $S(1/x, xy)$ (or $S(1/\sqrt{x}, \sqrt{xy})$ for the diagonal model). The first five models have a finite group, and the other four an infinite group.

already those considered in [31]. Our first contribution is, for this (admittedly small) set of models, a collection of results that are perfect analogues of the above quadrant results:

- Exactly four of these nine models admit invariants involving the series $C(x, y; t)$: three of them (those of the *Kreweras trilogy* on the left of the top table) have a finite group, and one (#6) has an infinite group. Moreover, these four models are also those that admit invariants involving the series $Q(x, y; t)$ (Proposition 3.1).
- We use these invariants to prove, in a uniform manner, that the series $C(x, y; t)$ is *algebraic* for the three models with a finite group (and invariants); this series was already known to be D-finite in x and y [31], and conjectured to be algebraic [9].
- We use these invariants to (re)prove that the series $C(x, y; t)$ is *D-algebraic* for the (unique) model with an infinite group and invariants. This series was already known to be D-algebraic [17], but the expression that we obtain is new.

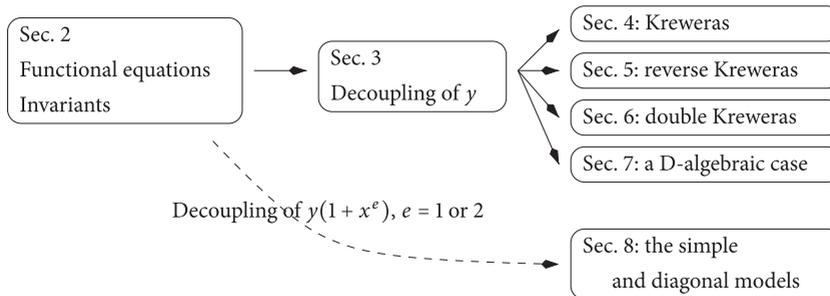
As in the quadrant case, these “positive” results are complemented by “negative” ones from the existing literature, which imply a tight connection between invariants and (D)-algebraicity: the series $C(x, y; t)$ is transcendental for the fourth and fifth models with a finite group (simple walks and diagonal walks) [9], and non-D-algebraic for the other three models with an infinite group [17]. These properties are summarized in Table 1.

Our second contribution is a new solution of two (transcendental) D-finite models via invariants: simple walks with steps $\rightarrow, \uparrow, \leftarrow, \downarrow$ and diagonal walks with steps $\nearrow, \nwarrow, \swarrow, \searrow$ (fourth and fifth in the table). This may seem surprising, since, for finite group models, invariants have so far been used to prove *algebraicity* results. However, as shown in [9] for both models, it is natural to introduce a series $A(x, y; t)$ that differs from $C(x, y; t)$ by a simple explicit D-finite series (expressed in terms of $Q(x, y; t)$), and then the heart of the solution of [9] is to prove that $A(x, y; t)$ is algebraic. What we show here is how to reprove this algebraicity via invariants.

In the six cases that we solve via invariants (the first six in Table 1), what we actually establish is an algebraic relation between the three-quadrant series $C(x, y; t)$ (or $A(x, y; t)$ in the simple and diagonal cases) for the model \mathcal{S} , and the quadrant series $\mathcal{Q}(x, y; t)$ for a companion model \mathcal{S}' also shown in Table 1. The (D)-algebraicity of $C(x, y; t)$ (or $A(x, y; t)$) then follows from the (D)-algebraicity of $\mathcal{Q}(x, y; t)$. (Note that $\mathcal{Q}(x, y; t)$ is also D-algebraic for the three models that we do *not* solve, but the lack of invariants involving $C(x, y; t)$ prevents us from relating $C(x, y; t)$ and $\mathcal{Q}(x, y; t)$.) We thus obtain either explicit algebraic expressions of $C(x, y; t)$ or $A(x, y; t)$, or, for the model that is only D-algebraic, an expression in terms of the *weak invariant* of [3].

Finally, for the five models \mathcal{S} that are (at least) D-finite, we derive from our results explicit algebraic expressions for the series $\sum_{i,j} H_{i,j} x^i y^j$, where $(H_{i,j})_{(i,j) \in \mathcal{C}}$ is a discrete positive harmonic function in \mathcal{C} associated with the reverse set of steps, $\mathcal{S}^* := \{(-i, -j) : (i, j) \in \mathcal{S}\}$. It is widely believed that there exists a unique such function, up to a multiplicative constant. As can now be expected, the above series is related to the series describing the/a harmonic function associated with the step set \mathcal{S}^* in the quadrant. We prove that this relation still holds, under natural assumptions, for the sixth model that we solve.

1.5 Outline of the paper



The paper is organized as follows. In Section 2, we introduce general tools, like basic functional equations for walks in \mathbb{C} , and also the key notion of *pairs of invariants*. For each of the models \mathcal{S} of Table 1, we give one or two pairs of \mathcal{S} -invariants, borrowed from [2]: one pair involves the generating function $\mathcal{Q}(x, y; t)$ of quadrant walks with steps in \mathcal{S} , and the other one, which only exists for finite group models, is rational. In Section 3, we construct, for models of the Kreweras trilogy and for the 6th model of the table, a new pair of \mathcal{S} -invariants involving the generating function of walks in \mathbb{C} with steps in \mathcal{S} (note the change in the step set). This construction is based on a certain *decoupling property* in these models, which we relate to the decoupling property used in the solution of quadrant walks with steps in \mathcal{S} by invariants (Proposition 3.1). A basic, but useful, *invariant lemma* (Lemma 2.6) allows us to relate these new invariants to the known ones. This connection is made explicit in Sections 4–7 for Kreweras steps, reverse Kreweras steps, double Kreweras steps, and for the 6th model of Table 1, respectively. We thus relate the generating function $C(x, y; t)$ for three-quadrant \mathcal{S} -walks to the generating function $\mathcal{Q}(x, y; t)$ for quadrant \mathcal{S} -walks. In Section 8, we apply the same approach to solve the simple and diagonal models, proving that the series $A(x, y; t)$ is algebraic in both cases. We finish in Section 9 with comments and perspectives.

The paper is accompanied by six Maple sessions (one per step set) available on the author’s web page (<http://www.labri.fr/perso/bousquet/publis.html>).

1.6 Kreweras’ model

To illustrate our results, we now present those that deal with Kreweras’ model, $\mathcal{S} = \{\nearrow, \leftarrow, \downarrow\}$. Note that we often omit the dependency in t of our series, denoting, for instance, $C(x, y)$ for $C(x, y; t)$. We also use the notation $\bar{x} = 1/x$ and $\bar{y} = 1/y$.

Theorem 1.1 *The generating function $C(x, y)$ of walks with steps in $\{\nearrow, \leftarrow, \downarrow\}$ starting from $(0, 0)$ and avoiding the negative quadrant is algebraic of degree 96. It is given by the following identity:*

$$(1 - t(\bar{x} + \bar{y} + xy))C(x, y) = 1 - t\bar{y}C_-(\bar{x}) - t\bar{x}C_-(\bar{y}),$$

where $C_-(x)$ is a series in t with polynomial coefficients in x , algebraic of degree 24. This series can be expressed explicitly in terms of the unique formal power series $V \equiv V(t)$ satisfying

$$(1.3) \quad V = t(2 + V^3).$$

Indeed,

$$2 \left(t\bar{x}C_-(x) + x - \frac{1}{2t} \right)^2 = \frac{(1 - V^3)^{3/2}}{V^2} + (1 - xV)^2 \left(\frac{1}{V^2} - \frac{1}{x} \right) + \left(\bar{x} + V - \frac{2x}{V} \right) \sqrt{1 - V \frac{4 + V^3}{4} x + \frac{V^2}{4} x^2}.$$

One recognizes, as announced, some ingredients of the quadrant generating function for the companion model $\mathcal{S} = \{\rightarrow, \uparrow, \swarrow\}$, which satisfies [10, 26]

$$(1.4) \quad \mathcal{Q}(x, 0) = \frac{V(4 - V^3)}{16t} - \frac{t - x^2 + tx^3}{2xt^2} + \left(\bar{x} + V - \frac{2x}{V} \right) \sqrt{1 - V \frac{4 + V^3}{4} x + \frac{V^2}{4} x^2}.$$

Note, however, that $\mathcal{Q}(x, 0)$ has degree 6 only.

We can complement Theorem 1.1 with the following one, which counts walks in \mathcal{C} ending on the diagonal: as before, the variable t records the length, and the variable x the abscissa of the endpoint. Note that the expressions in the two theorems only differ by a sign in front of the square root.

Theorem 1.2 *The generating function $D(x)$ of walks with steps in $\{\nearrow, \leftarrow, \downarrow\}$ starting from $(0, 0)$, avoiding the negative quadrant, and ending on the first diagonal is algebraic of degree 24. It can be written explicitly in terms of the series V defined by (1.3):*

$$\frac{\Delta(x)}{2} \left(xD(x) + \frac{1}{t} \right)^2 = \frac{(1 - V^3)^{3/2}}{V^2} + (1 - xV)^2 \left(\frac{1}{V^2} - \frac{1}{x} \right) - \left(\bar{x} + V - \frac{2x}{V} \right) \sqrt{1 - V \frac{4 + V^3}{4} x + \frac{V^2}{4} x^2},$$

where $\Delta(x) = (1 - tx)^2 - 4t^2\bar{x}$.

The generating function of *excursions* (that is, walks ending at $(0, 0)$) was conjectured in [9] to be algebraic of degree 6. More generally, it is natural to ask about the generating function $C_{i,j}$ of walks ending at a specific point (i, j) . In order to describe these series, we introduce an extension of degree 4 of $\mathbb{Q}(V)$ (hence of degree 12 over $\mathbb{Q}(t)$). First, due to the periodicity of the model (walks ending at position (i, j) have length $-i - j$ modulo 3), it makes sense to consider t^3 as the natural variable for this problem. Note that

$$V^3 = t^3(2 + V^3)^3,$$

Series	Defining equation	Expression	Degree over $\mathbb{Q}(t)$
V	$V = t(2 + V^3)$	—	3
W	$4W(1 - W) = V^3, \quad W(0) = 0$	$\frac{1 - \sqrt{1 - V^3}}{2}$	6
Z	$2Z = W(1 + Z^2)$	$\frac{1 - \sqrt{1 - W^2}}{W}$	12

Table 2: Relevant extensions of $\mathbb{Q}(t)$ for Kreweras (and reverse Kreweras) steps.

so that all series in t^3 can be written as series in V^3 . We then define the series W and Z as described in Table 2. Both are formal power series in t^3 , and we have the following tower of extensions:

$$\mathbb{Q}(t^3) \xrightarrow{3} \mathbb{Q}(V^3) \xrightarrow{2} \mathbb{Q}(W) \xrightarrow{2} \mathbb{Q}(Z).$$

Corollary 1.3 *Let us define $V, W,$ and Z as above. For any $(i, j) \in \mathbb{C}$, the generating function $C_{i,j}$ of walks avoiding the negative quadrant and ending at (i, j) is algebraic of degree (at most) 12, and belongs to $\mathbb{Q}(t, Z)$. More precisely, $t^{i+j}C_{i,j}$ belongs to $\mathbb{Q}(Z)$. If $i = j$ or $i = j \pm 1$ with $i, j \geq 0$, then $t^{i+j}C_{i,j}$ has degree 6 at most and belongs to $\mathbb{Q}(W)$.*

For instance, we have

$$(1.5) \quad C_{0,0} = \frac{(1 + 2W - 2W^2)(3W^3 - 2W^2 - 4W + 4)}{4(1 - W)},$$

$$tC_{0,1} = \frac{C_{0,0} - 1}{2} = \frac{W(-6W^4 + 10W^3 + 7W^2 - 18W + 8)}{8(1 - W)},$$

$$t^2C_{1,1} = \frac{W(35W^6 - 100W^5 + 20W^4 + 144W^3 - 96W^2 - 32W + 32)}{64(1 - W)(1 + 2W - 2W^2)},$$

$$(1.6) \quad t^2C_{-1,0} = \frac{Z(Z^3 + 3Z^2 - 3Z + 1)}{Z^4 + 4Z^3 - 6Z^2 + 4Z + 1}.$$

We can derive from such results asymptotic estimates for the number of walks ending at (i, j) via singularity analysis [21, Chapter VII.7]. For instance,

$$c_{0,0}(3m) \sim -\frac{27\sqrt{3}}{4\Gamma(-3/4)} 3^{3m} (3m)^{-7/4}.$$

Note that this asymptotic behavior was already known, but only up to a constant [28]. More generally, we obtain an explicit description of a discrete positive harmonic function associated with S^* -walks in the three-quadrant cone \mathbb{C} , where we recall that $S^* = \{(-i, -j), (i, j) \in \mathcal{S}\}$.

Corollary 1.4 *For $(i, j) \in \mathbb{C}$, there exists a positive constant $H_{i,j}$ such that, as $n \rightarrow \infty$ with $n + i + j \equiv 0 \pmod{3}$,*

$$(1.7) \quad c_{i,j}(n) \sim -\frac{H_{i,j}}{\Gamma(-3/4)} 3^n n^{-7/4}.$$

The function H is discrete harmonic for the step set $\mathcal{S}^* = \{\swarrow, \rightarrow, \uparrow\}$. That is,

$$(1.8) \quad H_{i,j} = \frac{1}{3} (H_{i-1,j-1} + H_{i+1,j} + H_{i,j+1}),$$

where $H_{i,j} = 0$ if $(i, j) \notin \mathcal{C}$. By symmetry, $H_{i,j} = H_{j,i}$. The generating function

$$(1.9) \quad \mathcal{H}(x, y) := \sum_{j \geq 0, i \leq j} H_{i,j} x^{j-i} y^j,$$

which is a formal power series in x and y , is algebraic of degree 16, given by

$$(1.10) \quad (1 + xy^2 + x^2y - 3xy) \mathcal{H}(x, y) = \mathcal{H}_-(x) + \frac{1}{2} (2 + xy^2 - 3xy) \mathcal{H}_d(y),$$

where

$$(1.11) \quad \mathcal{H}_-(x) := \sum_{i > 0} H_{-i,0} x^i = \frac{9x}{2} \sqrt{\frac{1+2x}{1-x} \sqrt{\frac{4-x}{1-x}}} + 2$$

and

$$(1.12) \quad \mathcal{H}_d(y) := \mathcal{H}(0, y) = \sum_{i \geq 0} H_{i,i} y^i = \frac{9}{(1-y)\sqrt{y(4-y)}} \sqrt{\frac{1+2y}{1-y} \sqrt{\frac{4-y}{1-y}}} - 2.$$

Remark 1.5 It may seem surprising to read that we derive, from the asymptotic enumeration of \mathcal{S} -walks in \mathcal{C} , the expression of a function H that is \mathcal{S}^* -harmonic, rather than \mathcal{S} -harmonic. This is simply due to the definition of discrete harmonic functions (e.g., [30]): a function $f_{i,j}$ is harmonic for the (uniform) random walk with steps in \mathcal{P} if

$$f_{i,j} = \frac{1}{|\mathcal{P}|} \sum_{(a,b) \in \mathcal{P}} f_{i+a,j+b}.$$

A \mathcal{P} -harmonic function typically occurs in the asymptotic enumeration of \mathcal{P} -walks starting (rather than ending) at (i, j) . In many cases, this number behaves like $f_{i,j} \mu^n n^\gamma$, for constants μ and γ , where the function f is \mathcal{P} -harmonic. As already mentioned, it is widely believed (and proved in some cases [7, 18, 23, 29]) that for many domains \mathcal{D} and step sets with zero drift, there exists (up to a multiplicative constant) a unique positive harmonic function in \mathcal{D} that vanishes at the boundary.

The first expression of a/the harmonic function, implying its algebraicity, was given by Trotignon [32]. It is, however, less explicit than in Corollary 1.4. The connection between walks in \mathcal{C} with steps in \mathcal{S} and walks in \mathcal{Q} with steps in the companion model \mathcal{S} , already observed at the level of generating functions (see Theorem 1.1 and (1.4)), is also visible at the level of harmonic functions. Indeed, the number of quadrant \mathcal{S} -walks of length n ending at $(i, j) \in \mathcal{Q}$ is

$$(1.13) \quad \tilde{q}_{i,j}(n) \sim \frac{h_{i,j}}{\Gamma(-3/2)} 3^n n^{-5/2}$$

(for $n \equiv i + j \pmod 3$) with

$$(1.14) \quad \sum_{i \geq 0} h_{i,0} x^i = \frac{9}{4} \left(\frac{1+2x}{1-x} \sqrt{\frac{4-x}{1-x}} + 2 \right).$$

This should be compared to (1.11) and (1.12). This expression can be derived from (1.4) by singularity analysis. See also [30] for a more general expression that applies to any quadrant model with small steps and zero drift. We explain in Remark 4.1 how one can predict, under a natural assumption, the relation between the \mathcal{S}^* -harmonic function $H_{i,j}$ in \mathbb{C} and the \mathcal{S} -harmonic function $h_{i,j}$ in \mathbb{Q} . Note that (1.13) also describes the asymptotic number of walks with steps in the original model \mathcal{S} confined to the nonpositive quadrant, joining $(0, 0)$ to $(-i, -j)$.

Let us finally consider the total number of n -step walks avoiding the negative quadrant, which we denote by c_n .

Corollary 1.6 *The generating function $C(1,1)$ counting all walks confined to the three-quadrant cone \mathcal{C} is algebraic of degree 24, and given by*

$$(1.15) \quad \begin{aligned} \frac{1}{2}(1-3t)^2 \left(C(1,1) + \frac{1}{t} \right)^2 &= 2 \left(tC_-(1) + 1 - \frac{1}{2t} \right)^2 \\ &= \frac{(1-V^3)^{3/2}}{V^2} + (1-V)^2 \left(\frac{1}{V^2} - 1 \right) + \left(1+V - \frac{2}{V} \right) \sqrt{1 - V \frac{4+V^3}{4} + \frac{V^2}{4}}, \end{aligned}$$

where V is the series defined by (1.3). The series $C(1,1)$ has radius of convergence $1/3$, with dominant singularities at $\zeta/3$, where ζ is any cubic root of unity. For the first-order asymptotics of the coefficients c_n , only the singularity at $1/3$ contributes, and

$$(1.16) \quad c_n \sim \frac{3^{3/4} \sqrt{2-\sqrt{2}}}{\Gamma(5/8)} 3^n n^{-3/8}.$$

2 General tools

We first introduce some basic notation. For a ring R , we denote by $R[t]$ (resp. $R[[t]]$) the ring of polynomials (resp. formal power series) in t with coefficients in R . If R is a field, then $R(t)$ stands for the field of rational functions in t , and $R((t))$ for the field of Laurent series in t , that is, series of the form $\sum_{n \geq n_0} a_n t^n$, with $n_0 \in \mathbb{Z}$ and $a_n \in R$. This notation is generalized to several variables. For instance, the generating function $C(x, y; t)$ of walks confined to \mathcal{C} belongs to $\mathbb{Q}[x, \bar{x}, y, \bar{y}][[t]]$, where we denote $\bar{x} = 1/x$ and $\bar{y} = 1/y$. We often omit in our notation the dependency in t of our series, writing, for instance, $C(x, y)$ instead of $C(x, y; t)$. For a series $F(x, y) \in \mathbb{Q}[x, \bar{x}, y, \bar{y}][[t]]$ and two integers i and j , we denote by $F_{i,j}$ the coefficient of $x^i y^j$ in $F(x, y)$. This is a series in $\mathbb{Q}[[t]]$. Similarly, if $F(x) \in \mathbb{Q}[x, \bar{x}][[t]]$, then F_i stands for the coefficient of x^i in $F(x)$.

We consider a set of “small” steps \mathcal{S} in \mathbb{Z}^2 , meaning that $\mathcal{S} \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$. We define the *step polynomial* $S(x, y)$ by

$$S(x, y) = \sum_{(i,j) \in \mathcal{S}} x^i y^j,$$

and six (Laurent) polynomials $H_0, H_-, H_+, V_0, V_-,$ and V_+ by

$$(2.1) \quad S(x, y) = \bar{y}H_-(x) + H_0(x) + yH_+(x) = \bar{x}V_-(y) + V_0(y) + xV_+(y).$$

The notation H (resp. V) recalls that these polynomials record horizontal (resp. vertical) moves: for instance, $H_-(x)$ describes the horizontal displacements of steps that move down. The *group* associated with \mathcal{S} can be easily defined at this stage, but we do not use it in this paper, and simply refer the interested reader to, e.g., [10].

2.1 Functional equations for walks avoiding a quadrant

In this subsection, we first restrict our attention to step sets \mathcal{S} satisfying the following two assumptions:

- (2.2) • \mathcal{S} is symmetric in the first diagonal.
- \mathcal{S} does not contain the steps $(1, -1)$ nor $(-1, 1)$.

This corresponds to the models shown in Table 1, apart from the diagonal model. As we shall explain, the latter model should still be considered in the same class, as the associated generating function satisfies the same kind of functional equation.

It is easy to write a functional equation defining the series $C(x, y)$, based on a step-by-step construction of the walks:

$$C(x, y) = 1 + tS(x, y)C(x, y) - t\bar{y}H_-(x)C_-(\bar{x}) - t\bar{x}H_-(y)C_-(\bar{y}) - t\bar{x}\bar{y}C_{0,0} \mathbb{1}_{(-1,-1) \in \mathcal{S}},$$

where the series $C_-(\bar{x})$ counts walks ending on the negative x -axis:

$$(2.3) \quad C_-(\bar{x}) = \sum_{i < 0, n \geq 0} c_{i,0}(n) x^i t^n.$$

In the above functional equation, each term with a minus sign corresponds to a forbidden move: moving down from the negative x -axis, moving left from the negative y -axis, or performing a South-West step from the point $(0, 0)$. Observe that we have exploited the x/y -symmetry of \mathcal{S} . The above equation rewrites as

$$(2.4) \quad K(x, y)C(x, y) = 1 - t\bar{y}H_-(x)C_-(\bar{x}) - t\bar{x}H_-(y)C_-(\bar{y}) - t\bar{x}\bar{y}C_{0,0} \mathbb{1}_{(-1,-1) \in \mathcal{S}},$$

where $K(x, y) := 1 - tS(x, y)$ is the *kernel* of the equation.

Following [31], we will now work in convex cones by splitting the three-quadrant cone in two symmetric halves. The rest of this subsection simply rephrases Section 2.2

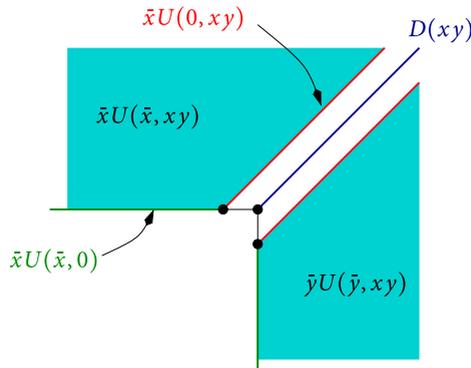


Figure 2: The series $D(xy)$ counts walks ending on the diagonal, and $\bar{x}U(\bar{x}, xy)$ those ending above the diagonal.

of [31], with a slightly different presentation. (Another strategy, namely splitting in three quadrants, is applied in [9, 11, 12, 20].) We write

$$(2.5) \quad C(x, y) = \bar{x}U(\bar{x}, xy) + D(xy) + \bar{y}U(\bar{y}, xy),$$

with $D(y) \in \mathbb{Q}[y][[t]]$ and $U(x, y) \in \mathbb{Q}[x, y][[t]]$. Observe that these properties, together with the above identity, define the series D and U uniquely: the series $\bar{x}U(\bar{x}, xy)$ (resp. $D(xy)$, $\bar{y}U(\bar{y}, xy)$) counts walks ending strictly above (resp. on, strictly below) the diagonal (see Figure 2). We will now write functional equations for U and D , based again on a step-by-step construction (we could also derive them algebraically from (2.4), but we prefer a combinatorial argument). This requires to classify steps of \mathcal{S} according to whether they lie above, on or below the main diagonal. We write accordingly

$$(2.6) \quad S(x, y) = \bar{x}\mathcal{V}_+(xy) + \mathcal{V}_0(xy) + x\mathcal{V}_-(xy).$$

Note that this is only possible because we have forbidden steps $(1, -1)$ and $(-1, 1)$, which would contribute terms of the form $x^2(\bar{x}\bar{y})$ and $\bar{x}^2(xy)$, respectively. Let us explain the notation \mathcal{V}_+ , \mathcal{V}_0 , \mathcal{V}_- , which is reminiscent of (2.1). We define the companion set of steps

$$(2.7) \quad \mathcal{S} := \{(j - i, j) : (i, j) \in \mathcal{S}\},$$

with step polynomial $\mathcal{S}(x, y) = S(\bar{x}, xy)$ (we hope that using the same notation \mathcal{S} for the set of steps and its generating polynomial will not cause any inconvenience). Then

$$\mathcal{S}(x, y) = S(\bar{x}, xy) = x\mathcal{V}_+(y) + \mathcal{V}_0(y) + \bar{x}\mathcal{V}_-(y),$$

which is indeed the decomposition of \mathcal{S} along horizontal moves. Note that the transformation $\mathcal{S} \mapsto \mathcal{S}$ maps Kreweras steps to reverse Kreweras steps, and vice versa, and leaves double Kreweras steps globally unchanged. The full correspondence between \mathcal{S} and \mathcal{S} is shown in Table 1.

We claim that the generating function of walks ending on the diagonal satisfies

$$D(xy) = 1 + t\mathcal{V}_0(xy)D(xy) - t\bar{x}\bar{y}D_0\mathbb{1}_{(-1,-1)\in\mathcal{S}} + 2t(x\mathcal{V}_-(xy))(\bar{x}U(0,xy)) - 2t\bar{x}\bar{y}U_{0,0}\mathbb{1}_{(0,-1)\in\mathcal{S}}.$$

The two terms involving D on the right-hand side count walks whose last step starts from the diagonal, and those involving U count walks whose last step starts from the lines $j = i \pm 1$. For instance, $\bar{x}U(0,xy)$ counts walks ending on the line $j = i + 1$, and the multiplication by $t(x\mathcal{V}_-(xy))$ corresponds to adding a subdiagonal step. The factor 2 accounts for the x/y -symmetry of the model. The final term prevents from moving from $(-1, 0)$ to $(-1, -1)$ (or symmetrically). For walks ending (strictly) above the diagonal, we obtain

$$\bar{x}U(\bar{x},xy) = t\bar{x}\mathcal{V}_+(xy)D(xy) + tS(x,y)\bar{x}U(\bar{x},xy) - t\bar{y}H_-(x)\bar{x}U(\bar{x},0) - t(x\mathcal{V}_-(xy))\bar{x}U(0,xy) + t\bar{x}\bar{y}U_{0,0}\mathbb{1}_{(0,-1)\in\mathcal{S}}.$$

Again, the terms involving D (resp. U) count walks whose last step starts on (resp. above) the diagonal. The term involving $H_-(x)$ corresponds to walks that would enter the negative quadrant through the negative x -axis, and the term involving $\mathcal{V}_-(xy)$ counts walks that would in fact end on the diagonal. The final term corresponds to walks ending at $(-1, 0)$, extended by a South step: they have been subtracted twice, once in each of the previous two terms.

We now perform the change of variables $(x, y) \mapsto (\bar{x}, xy)$. This gives

$$(1 - t\mathcal{V}_0(y))D(y) = 1 - t\bar{y}D_0\mathbb{1}_{(-1,-1)\in\mathcal{S}} + 2t\mathcal{V}_-(y)U(0,y) - 2t\bar{y}U_{0,0}\mathbb{1}_{(0,-1)\in\mathcal{S}}, \tag{2.8}$$

$$(1 - tS(\bar{x},xy))xU(x,y) = tx\mathcal{V}_+(y)D(y) - t\bar{y}H_-(\bar{x})U(x,0) - t\mathcal{V}_-(y)U(0,y) + t\bar{y}U_{0,0}\mathbb{1}_{(0,-1)\in\mathcal{S}}. \tag{2.9}$$

Using the first equation, we can eliminate $U(0, y)$ (and $U_{0,0}$) from the second. Multiplying finally by $2y$, we obtain the functional equation that we are going to focus on:

$$2xy\mathcal{K}(x,y)U(x,y) = y + y(t\mathcal{V}_0(y) + 2tx\mathcal{V}_+(y) - 1)D(y) - 2tH_-(\bar{x})U(x,0) - tD_0\mathbb{1}_{(-1,-1)\in\mathcal{S}},$$

with $\mathcal{K}(x, y) = 1 - tS(\bar{x}, xy)$. It will be convenient to write it in terms of \mathcal{S} only: we thus observe that

$$(-1, -1) \in \mathcal{S} \Leftrightarrow (0, -1) \in \mathcal{S} \quad \text{and} \quad H_-(\bar{x}) = x\mathcal{H}_-(x), \tag{2.10}$$

where we write

$$\mathcal{S}(x, y) = \bar{x}\mathcal{V}_-(y) + \mathcal{V}_0(y) + x\mathcal{V}_+(y) = \bar{y}\mathcal{H}_-(x) + \mathcal{H}_0 + y\mathcal{H}_+(x). \tag{2.11}$$

Lemma 2.1 *For a step set \mathcal{S} satisfying the assumptions (2.2), the series $U(x, y)$ and $D(y)$ defined by (2.5) are related by*

$$2xy\mathcal{K}(x,y)U(x,y) = y + y(t\mathcal{V}_0(y) + 2tx\mathcal{V}_+(y) - 1)D(y) - 2tx\mathcal{H}_-(x)U(x,0) - tD_0\mathbb{1}_{(0,-1)\in\mathcal{S}}, \tag{2.12}$$

with $\mathcal{K}(x, y) := K(\bar{x}, xy) = 1 - tS(\bar{x}, xy)$. Note that $\mathcal{K}(x, y)$ is the kernel associated with the step set \mathcal{S} defined by (2.7), which may not be $x|y$ -symmetric.

This equation holds for the diagonal model $S = \{ \nearrow, \nwarrow, \swarrow, \searrow \}$ as well, provided we define the series D and U by

$$(2.13) \quad C(x, y) = \bar{x}^2 U(\bar{x}^2, xy) + D(xy) + \bar{y}^2 U(\bar{y}^2, xy),$$

and take $\mathcal{S} = \{ \nearrow, \uparrow, \swarrow, \downarrow \}$, that is,

$$\mathcal{K}(x, y) = 1 - t\mathcal{S}(x, y) = 1 - tS(\sqrt{\bar{x}}, \sqrt{\bar{x}y}) = 1 - t(xy + y + \bar{x}\bar{y} + \bar{y}).$$

Proof We have already justified the equation under the assumptions (2.2). Now, consider the diagonal model. The only points (i, j) that can be visited by a walk starting from $(0, 0)$ are such that $i + j$ is even. This allows us to write $C(x, y)$ as (2.13), where $U(x, y)$ and $D(y)$ are in $\mathbb{Q}[x, y][[t]]$ and $\mathbb{Q}[y][[t]]$, respectively. One then writes basic step-by-step equations for $D(xy)$ and $\bar{x}^2 U(\bar{x}^2, xy)$, and then performs the change of variables $(x, y) \mapsto (\sqrt{\bar{x}}, \sqrt{\bar{x}y})$. This leads to the following counterparts of (2.8) and (2.9):

$$\begin{aligned} (1 - t(\bar{y} + y)) D(y) &= 1 - t\bar{y}D_0 + 2t\bar{y}U(0, y) - 2t\bar{y}U_{0,0}, \\ (1 - t(y + \bar{y} + \bar{x}\bar{y} + xy)) xU(x, y) &= txyD(y) - t\bar{y}(1 + x)U(x, 0) \\ &\quad - t\bar{y}U(0, y) + t\bar{y}U_{0,0}. \end{aligned}$$

The combination of these two equations gives

$$(2.14) \quad \begin{aligned} &2xy(1 - t(y + \bar{y} + xy + \bar{x}\bar{y})) U(x, y) = \\ &y + y(t(y + \bar{y}) + 2txy - 1)D(y) - 2t(1 + x)U(x, 0) - tD_0, \end{aligned}$$

which coincides with (2.12). ■

Remark 2.2 The (single) equation in Lemma 2.1 really defines the two series $U(x, y)$ and $D(y)$. Indeed, equation (2.8) relating $D(y)$ and $U(0, y)$ can be recovered by setting $x = 0$ in (2.12).

Our solution of three-quadrant models will involve the generating function of walks with steps in \mathcal{S} confined to the first (nonnegative) quadrant \mathcal{Q} . This series $\mathcal{Q}(x, y) \equiv \mathcal{Q}(x, y; t) \in \mathbb{Q}[x, y][[t]]$ satisfies a similar looking equation [10]:

$$(2.15) \quad xy\mathcal{K}(x, y)\mathcal{Q}(x, y) = xy - tx\mathcal{K}_-(x)\mathcal{Q}(x, 0) - ty\mathcal{V}_-(y)\mathcal{Q}(0, y) + t\mathcal{Q}_{0,0}\mathbb{1}_{(-1,-1) \in \mathcal{S}},$$

where we have used the notation (2.11). This equation is simpler than (2.12), because its right-hand side, apart from the simple term xy , is the sum of a function of x and a function of y . This is not the case in (2.12), because the factor $(t\mathcal{V}_0(y) + 2tx\mathcal{V}_+(y) - 1)$ involves x . However, the square of this factor is a function of y only—modulo $\mathcal{K}(x, y)$. This property is already exploited in [31].

Lemma 2.3 Let \mathcal{S} be a collection of small steps, and define \mathcal{V}_+ , \mathcal{V}_0 , and \mathcal{V}_- by (2.11). Then

$$(t\mathcal{V}_0(y) + 2tx\mathcal{V}_+(y) - 1)^2 = \Delta(y) - 4tx\mathcal{V}_+(y)\mathcal{K}(x, y),$$

where $\mathcal{K}(x, y) = 1 - t\mathcal{S}(x, y)$ and

$$(2.16) \quad \Delta(y) = (1 - t\mathcal{V}_0(y))^2 - 4t^2\mathcal{V}_-(y)\mathcal{V}_+(y)$$

is the discriminant (in x) of $x\mathcal{K}(x, y)$.

The proof is a simple calculation, which we leave to the reader.

2.2 Invariants

We now define the notion of *invariant* that we use in this paper. We refer to [1, 2] for a slightly different notion that applies to rational functions only. The definition that we adopt here allows us a uniform treatment of all the models that we solve, whereas in [2], quadrant walks with reverse Kreweras steps had to be handled by a different argument. We will only use our notion of invariants for small step models, but this subsection could actually be applied to any model in $\{-1, 0, 1, 2, 3, \dots\}^2$.

For reasons that are related to the form of the functional equation (2.12), we consider in this subsection a set of steps that we denote \mathcal{S} rather than \mathcal{S} . Before all, observe that the series

$$(2.17) \quad \frac{1}{\mathcal{K}(x, y)} = \frac{1}{1 - t\mathcal{S}(x, y)} = \sum_{n \geq 0} t^n \mathcal{S}(x, y)^n$$

counts *all* walks with steps in \mathcal{S} , starting from the origin, by the length (variable t) and the coordinates of the endpoint (variables x and y). The coefficient of t^n in this series is a Laurent polynomial in x and y . As soon as \mathcal{S} is not contained in a half-plane, the collection of these polynomials has unbounded degree and valuation.

Recall that $\mathbb{Q}(x, y)((t))$ is the field of Laurent series in t with rational coefficients in x and y . We denote by $\mathbb{Q}_{\text{mult}}(x, y)((t))$ the subring consisting of series in which for each n , the coefficient of t^n is of the form $p(x, y)/(d(x)d'(y))$, where $p(x, y) \in \mathbb{Q}[x, y]$, $d(x) \in \mathbb{Q}[x]$, and $d'(y) \in \mathbb{Q}[y]$. The reason why we focus on this subring is that:

- all the series that we handle are of this type;
- later (in the proof of Lemma 2.6) we will expand their coefficients as Laurent series in x and y , and with this condition the two expansions can be performed independently, without having to prescribe an order on x and y (as would be the case if we had to expand $1/(x - y)$, for instance).

Definition 2.1 A Laurent series $H(x, y)$ in $\mathbb{Q}_{\text{mult}}(x, y)((t))$ is said to have *poles of bounded order at 0* if the collection of its coefficients (in the t -expansion) has poles of bounded order at $x = 0$, and also at $y = 0$. In other words, there exists a monomial $x^i y^j$ such that $x^i y^j H(x, y)$ expands as a series in t whose coefficients have no pole at $x = 0$ nor at $y = 0$.

Clearly, the series $1/\mathcal{K}(x, y)$ shown in (2.17) lies in $\mathbb{Q}_{\text{mult}}(x, y)((t))$, but does not have poles of bounded order at 0 (unless \mathcal{S} is contained in the first quadrant).

Definition 2.2 (Divisibility) Fix a step set \mathcal{S} with kernel $\mathcal{K}(x, y) = 1 - t\mathcal{S}(x, y)$. A Laurent series $F(x, y)$ in $\mathbb{Q}_{\text{mult}}(x, y)((t))$ is said to be *divisible* by $\mathcal{K}(x, y)$ if the ratio $F(x, y)/\mathcal{K}(x, y)$ has poles of bounded order at 0.

Note that if $F(x, y)$ is in $\mathbb{Q}_{\text{mult}}(x, y)((t))$, the ratio $F(x, y)/\mathcal{K}(x, y)$ is always an element of $\mathbb{Q}_{\text{mult}}(x, y)((t))$ (by (2.17)). The following alternative characterization of divisibility will not be used, but it may clarify the notion.

Lemma 2.4 Assume that $\mathcal{K}(x, y)$ has valuation -1 and degree 1 in x and in y . The Laurent series $F(x, y)$ in $\mathbb{Q}_{\text{mult}}(x, y)((t))$ is divisible by $\mathcal{K}(x, y)$ if and only if:

- (1) $F(x, y)$ has poles of bounded order at 0;
- (2) if $\mathcal{X} \equiv \mathcal{X}(y)$ (resp. $\mathcal{Y} \equiv \mathcal{Y}(x)$) denotes the unique root of $\mathcal{K}(\cdot, y)$ (resp. $\mathcal{K}(x, \cdot)$) that is a formal power series in t (with coefficients in some algebraic closure of $\mathbb{Q}(y)$ or $\mathbb{Q}(x)$), then $F(\mathcal{X}, y) = F(x, \mathcal{Y}) = 0$.

Proof Assume that $F(x, y)$ is divisible by $\mathcal{K}(x, y)$, and write

$$(2.18) \quad F(x, y) = \mathcal{K}(x, y)H(x, y),$$

where $H(x, y)$ has poles of bounded order at 0. Since $\mathcal{K}(x, y)$ is a Laurent polynomial in x, y , and t , then $F(x, y)$ has poles of bounded order at 0 as well. Regarding Property (2), it suffices to prove it for \mathcal{X} , by symmetry. Observe that the equation $\mathcal{K}(x, y) = 0$ can be rewritten as

$$x = tx\mathcal{L}(x, y) = t(\mathcal{V}_-(y) + x\mathcal{V}_0(y) + x^2\mathcal{V}_+(y)).$$

This shows that exactly one of the roots of $\mathcal{K}(\cdot, y)$, denoted by \mathcal{X} , is a formal power series in t ; moreover, it has constant term 0 and coefficients in $\mathbb{Q}(y)$. Since $F(x, y)$ and $H(x, y)$ have poles of bounded order at 0, the series $F(\mathcal{X}, y)$ and $H(\mathcal{X}, y)$ are well defined and belong to $\mathbb{Q}(y)((t))$. We now specialize x to \mathcal{X} in (2.18), and this proves that $F(\mathcal{X}, y) = 0$.

Now, assume that the two conditions of the lemma hold, and let us prove that the series $H(x, y)$ defined by (2.18) has poles of bounded order at 0. By symmetry, it suffices to prove this for the variable x . The second root of $\mathcal{K}(\cdot, y)$ is

$$\mathcal{X}'(y) = \frac{\mathcal{V}_-(y)}{\mathcal{V}_+(y)} \frac{1}{\mathcal{X}(y)}.$$

It is a Laurent series in t of valuation -1 , with coefficients in $\mathbb{Q}(y)$. A partial fraction expansion of $1/\mathcal{K}(x, y)$ (in the variable x) gives

$$\frac{1}{\mathcal{K}(x, y)} = \frac{1}{\sqrt{\Delta(y)}} \left(\frac{1}{1 - \bar{x}\mathcal{X}(y)} + \frac{x}{1 - x/\mathcal{X}'(y)} \right),$$

where $\Delta(y)$ is given by (2.16). By assumption, $F(x, y)$ has poles of bounded order at $x = 0$. Since $\mathcal{X}(y) = \mathcal{O}(t)$ and $1/\mathcal{X}'(y) = \mathcal{O}(t)$, only the first part of the above series has poles at $x = 0$, and it suffices to prove that

$$(2.19) \quad \frac{F(x, y)}{1 - \bar{x}\mathcal{X}(y)} = F(x, y) \sum_{n \geq 0} \bar{x}^n \mathcal{X}(y)^n,$$

has poles of bounded order at $x = 0$. By assumption, there exists an integer i such that

$$x^i F(x, y) = \sum_{n \geq 0} \frac{p_n(x, y)}{d_n(x)d'_n(y)} t^n,$$

for polynomials $p_n, d_n,$ and d'_n such that $d_n(0) \neq 0$. Then, by the assumption $F(\mathcal{X}, y) = 0$, we have

$$\frac{x^i F(x, y)}{1 - \bar{x}\mathcal{X}} = \frac{x^i F(x, y) - \mathcal{X}^i F(\mathcal{X}, y)}{1 - \bar{x}\mathcal{X}} = \sum_{n \geq 0} \left(\frac{p_n(x, y)}{d_n(x)} - \frac{p_n(\mathcal{X}, y)}{d_n(\mathcal{X})} \right) \frac{x}{x - \mathcal{X}} \frac{t^n}{d'_n(y)}.$$

The n th summand can be written

$$\frac{xq_n(x, \mathcal{X}, y)}{d_n(x)d_n(\mathcal{X})} \cdot \frac{t^n}{d'_n(y)},$$

for some trivariate polynomial q_n . As a series in t , it belongs to $\mathbb{Q}_{\text{mult}}(x, y)((t))$, and its coefficients have no pole at $x = 0$ since $d(0) \neq 0$ (they are even multiples of x). We thus conclude that the series (2.19), as claimed, has poles of bounded order at $x = 0$. ■

Definition 2.3 (Invariants) Fix a step set \mathcal{S} with kernel $\mathcal{K}(x, y) = 1 - t\mathcal{S}(x, y)$. Let $I(x)$ and $J(y)$ be Laurent series in t with coefficients in $\mathbb{Q}(x)$ and $\mathbb{Q}(y)$, respectively. We say that $(I(x), J(y))$ is a pair of \mathcal{S} -invariants if $I(x) - J(y)$ is divisible by $\mathcal{K}(x, y)$, in the sense of Definition 2.2.

Note that $I(x) - J(y)$ is always an element of $\mathbb{Q}_{\text{mult}}(x, y)((t))$. Furthermore, if $(I(x), J(y))$ is a pair of invariants, then $I(x)$ and $J(y)$ have poles of bounded order at 0.

For any series $A \in \mathbb{Q}((t))$, the pair (A, A) is a (trivial) invariant. More interesting examples will be given in the next subsection (see, for instance, (2.20)).

The term *invariant* comes from the fact that, if both roots \mathcal{X} and \mathcal{X}' of $\mathcal{K}(\cdot, y)$ can be substituted for x in $I(x)$, then $I(\mathcal{X}) = I(\mathcal{X}')$. Of course, a similar observation holds for $J(y)$.

Lemma 2.5 The componentwise sum of two pairs of invariants $(I_1(x), J_1(y))$ and $(I_2(x), J_2(y))$ is still a pair of invariants. The same holds for their componentwise product.

Proof The first result is obvious by linearity. For the second, we observe that

$$\begin{aligned} I_1(x)I_2(x) - J_1(y)J_2(y) &= (I_1(x) - J_1(y))I_2(x) + J_1(y)(I_2(x) - J_2(y)) \\ &= \mathcal{K}(x, y)(H_1(x, y)I_2(x) + J_1(y)H_2(x, y)), \end{aligned}$$

where $H_k(x, y) = (I_k(x) - J_k(y))/\mathcal{K}(x, y)$, for $k = 1, 2$. The result follows because series with poles of bounded order at 0 form a ring. ■

The following lemma is a key tool when playing with invariants.

Lemma 2.6 (Invariant lemma) Let $I(x)$ and $J(y)$ be Laurent series in t with coefficients in $\mathbb{Q}(x)$ and $\mathbb{Q}(y)$, respectively. Assume that for all n , the coefficient of t^n in the ratio series

$$\frac{I(x) - J(y)}{\mathcal{K}(x, y)}$$

vanishes at $x = 0$ and at $y = 0$. Then $(I(x), J(y))$ is a pair of invariants by Definition 2.3, but in fact $I(x)$ and $J(y)$ depend only on t , and are equal. That is, the above ratio is zero.

Proof The argument is adapted from Lemma 4.13 in [2]. Let us assume, without loss of generality, that $I(x)$ and $J(y)$ contain no negative power of t . By assumption, we have

$$I(x) - J(y) = xy\mathcal{K}(x, y)G(x, y),$$

where for each n , the coefficient of t^n in $G(x, y)$ is of the form $p(x, y)/(d(x)d'(y))$ and has no pole at $x = 0$ nor $y = 0$. Hence, this coefficient expands as a formal power series in x and y , and we consider $G(x, y)$ itself as an element of $\mathbb{Q}[[x, y, t]]$. We define a total order on monomials $t^n x^i y^j$, with $(n, i, j) \in \mathbb{N}^3$, by the lexicographic order on the exponents (n, i, j) . Observe that $xy\mathcal{K}(x, y) \in \mathbb{Q}[t, x, y]$, and the smallest monomial that occurs in it is xy . Assume that $G(x, y) \neq 0$, and let M be the smallest monomial in $G(x, y)$. Then xyM is the smallest monomial in $xy\mathcal{K}(x, y)G(x, y)$, and hence in $I(x) - J(y)$ (once expanded as a series in t , whose coefficients are Laurent series in x or in y). However, the latter series cannot contain monomials involving both x and y . Hence, $G(x, y) = 0$ and the lemma is proved. ■

2.3 Some known invariants

For each model \mathcal{S} of Table 1, at least one nontrivial pair of invariants is already known from [2, Section 4], as we now review.

Consider, for instance, the set $\mathcal{S} = \{\nearrow, \leftarrow, \downarrow\}$ of Kreweras' steps, with $\mathcal{K}(x, y) = 1 - t(xy + \bar{x} + \bar{y})$. The first pair of invariants exhibited in [2] is rational, and symmetric in x and y :

$$I_0(x) = \bar{x}^2 - \bar{x}/t - x, \quad J_0(y) = I_0(y).$$

Indeed, it is elementary to check that

$$(2.20) \quad I_0(x) - J_0(y) = \mathcal{K}(x, y) \frac{x - y}{txy},$$

and the "series" $\frac{x-y}{txy}$ has poles of bounded order at 0. To describe the second pair, let $\mathcal{Q}(x, y)$ be the generating function of quadrant walks with steps in \mathcal{S} . Then $\mathcal{Q}(x, y)$ satisfies the functional equation

$$(2.21) \quad xy\mathcal{K}(x, y)\mathcal{Q}(x, y) = xy - tx\mathcal{Q}(x, 0) - ty\mathcal{Q}(0, y),$$

and the x/y -symmetry entails that $\mathcal{Q}(x, 0) = \mathcal{Q}(0, x)$. Observe that

$$(2.22) \quad xy = \frac{1}{t} - \bar{x} - \bar{y} - \frac{1}{t}\mathcal{K}(x, y).$$

Then the second pair of invariants $(I_1(x), J_1(y))$ is derived from (2.21) and (2.22):

$$I_1(x) = tx\mathcal{Q}(x, 0) + \bar{x} - \frac{1}{2t}, \quad J_1(y) = -I_1(y) = -ty\mathcal{Q}(0, y) - \bar{y} + \frac{1}{2t}.$$

\mathcal{S}			
\mathcal{S}			
$I_0(x)$	$\bar{x} + x/t - x^2$	$\bar{x}^2 - \bar{x}/t - x$	$\bar{x} - x - \frac{1+2t}{t(1+x)}$
$J_0(y)$	$\bar{y} + y/t - y^2$	$\bar{y}^2 - \bar{y}/t - y$	$\bar{y} - y - \frac{1+2t}{t(1+y)}$
$f(x)$	$\frac{x}{t} - x^2$	$\frac{1}{2t} - \bar{x}$	$\frac{x-t-tx^2}{t(1+x)}$
$g(y)$	$-\bar{y}$	$\frac{1}{2t} - \bar{y}$	$-\bar{y}$

\mathcal{S}		
\mathcal{S}		
$I_0(x)$	$x + \bar{x} - t(\bar{x} - x)^2$	$\frac{x}{t(1+x)^2} + \frac{t(1+x)^2}{x}$
$J_0(y)$	$\frac{y}{t(1+y)^2} + \frac{t(1+y)^2}{y}$	$y + \bar{y} - t(\bar{y} - y)^2$
$f(x)$	$-\bar{x}$	$\frac{x}{t(1+x)}$
$g(y)$	$\frac{y}{t(1+y)}$	$-\bar{y}$

Table 3: Rational \mathcal{S} -invariants (I_0, J_0), and \mathcal{S} -decoupling of xy by f and g for finite group models. The associated invariants $I_1(x), J_1(y)$ defined by (2.24) are algebraic.

\mathcal{S}				
\mathcal{S}				
$f(x)$	$-\bar{x}$	$-\bar{x}$	$-\bar{x} + \frac{1}{t}$	$-\bar{x}$
$g(y)$	$\frac{y(1-ty)}{t(1+y)}$	$-1 + \frac{y}{t} - y^2$	$-\bar{y} - y$	$\frac{y-t}{t(1+y)}$

Table 4: \mathcal{S} -decoupling of xy by f and g for infinite group models. The associated invariants $I_1(x), J_1(y)$ defined by (2.24) are D-algebraic.

Indeed,

$$I_1(x) - J_1(y) = -\mathcal{K}(x, y) \left(xy\mathcal{Q}(x, y) + \frac{1}{t} \right).$$

More generally, it is proved in [2] that among the nine models \mathcal{S} of Table 1, exactly the first five (those associated with a finite group) admit a pair of rational invariants $(I_0(x), J_0(y))$. We make them explicit in Table 3. On the other hand, the above construction of the \mathcal{Q} -related pair $(I_1(x), J_1(y))$ generalizes to all nine models, upon noticing that for each of them, the monomial xy decouples explicitly as the sum of a function of x and a function of y —modulo \mathcal{K} :

$$(2.23) \quad xy = f(x) + g(y) + h(x, y)\mathcal{K}(x, y),$$

for rational series $f(x), g(y)$, and $h(x, y)$ having poles of bounded order at 0. A possible choice of $f(x)$ and $g(y)$, borrowed from [2], is given in Tables 3 (for finite groups) and 4 (for infinite groups). The decoupling identity (2.23) allows us to rewrite the quadrant equation (2.15) as

$$\mathcal{K}(x, y) (h(x, y) - xy\mathcal{Q}(x, y)) = I_1(x) - J_1(y),$$

where

$$(2.24) \quad \begin{aligned} I_1(x) &= tx\mathcal{H}_-(x)\mathcal{Q}(x, 0) - f(x), \\ J_1(y) &= -ty\mathcal{V}_-(y)\mathcal{Q}(0, y) + t\mathcal{Q}_{0,0}\mathbb{1}_{(-1,-1)\in\mathcal{S}} + g(y). \end{aligned}$$

Since $h(x, y) - xy\mathcal{Q}(x, y)$ has poles of bounded order at 0, the pair $(I_1(x), J_1(y))$ is indeed a pair of invariants. This pair is used in [2] to prove that $\mathcal{Q}(x, y; t)$ is algebraic (for finite group models) or D-algebraic (for infinite group models), for all models \mathcal{S} in Table 1. The same approach applies to five other models \mathcal{S} that are not relevant for this paper [2].

3 Decoupling and invariants for three-quadrant walks

3.1 A new type of decoupling

For the nine models \mathcal{S} of Table 1, we have exhibited in the previous section invariants $(I_1(x), J_1(y))$ involving the generating function $\mathcal{Q}(x, y; t)$ for quadrant walks, by combining the functional equation (2.15) with the decoupling of xy modulo \mathcal{K} , given by (2.23).

Consider now the three-quadrant equation (2.12). One main difference with (2.15) is that the coefficient of $yD(y)$, being $(t\mathcal{V}_0(y) + 2tx\mathcal{V}_+(y) - 1)$, involves both x and y . Our aim will be to write the right-hand side of (2.12) as the sum of a function of x and a function of y multiplied by $(t\mathcal{V}_0(y) + 2tx\mathcal{V}_+(y) - 1)$, modulo the kernel $\mathcal{K}(x, y)$. Clearly, the only difficulty is to write the constant term y in this form. This turns out to be possible in exactly four of the nine cases, and this property is in fact related to “classical” decoupling of xy modulo the original model \mathcal{S} (not \mathcal{S}' !).

Proposition 3.1 *Let \mathcal{S} be one of the nine models of Table 1, and \mathcal{S}' its companion model, with associated kernels $K(x, y)$ and $\mathcal{K}(x, y)$, respectively. Define $\mathcal{V}_-, \mathcal{V}_0$, and \mathcal{V}_+ by (2.11). The following conditions are equivalent:*

\mathcal{S}				
\mathcal{I}				
$c(x, y)$	$ty + 2tx - 1$	$t\bar{y} + 2txy - 1$	$t(\bar{y} + y) + 2tx(1 + y) - 1$	$ty + 2xt(y + 1) - 1$
$F(x)$	$1/t - 2x$	$-\bar{x}^2 + \bar{x}/t - x$	$\frac{1+2t}{t(1+x)} - 1 - x - \bar{x}$	$\frac{1}{t} - 2x - 2\bar{x}$
$G(y)$	$1/t$	\bar{y}/t	$\frac{1}{t(1+y)}$	$\frac{1-y}{t(1+y)}$
$H(x, y)$	0	$\bar{x}\bar{y}(x - y)/t$	$\frac{x-y}{t(1+x)(1+y)}$	$-\frac{2y}{t(1+y)}$

Table 5: Decoupling of y in the form $y = c(x, y)G(y) + F(x) + \mathcal{K}(x, y)H(x, y)$, with $c(x, y) = t\mathcal{V}_0(y) + 2tx\mathcal{V}_+(y) - 1$, for four symmetric models in the three-quadrant cone.

- (a) There exist rational functions $F(x) \in \mathbb{Q}(x, t)$ and $G(y) \in \mathbb{Q}(y, t)$ such that the numerator of

$$(3.1) \quad y - (t\mathcal{V}_0(y) + 2tx\mathcal{V}_+(y) - 1)G(y) - F(x)$$

contains a factor $xy\mathcal{K}(x, y)$.

- (b) There exist rational functions $f(x) \in \mathbb{Q}(x, t)$ and $g(y) \in \mathbb{Q}(y, t)$ such that the numerator of

$$(3.2) \quad xy - f(x) - g(y)$$

contains a factor $xyK(x, y)$.

The only models of Table 1 for which these properties hold are the three models of the Kreweras trilogy, and the 6th model. A solution is then given by

$$(3.3) \quad F(x) = \bar{x}^2 + \frac{K(\bar{x}, \bar{x})}{tH_-(\bar{x})}, \quad tG(y) = \frac{1 + H_-(\bar{y})}{yH_+(\bar{y})} - 1,$$

and

$$(3.4) \quad f(x) = g(x) = \frac{1}{2} \left(x^2 + \frac{K(x, x)}{tH_-(x)} \right).$$

For this choice, stronger properties hold, as the rational functions (3.1) and (3.2) are then divisible by $\mathcal{K}(x, y)$ and $K(x, y)$, respectively, in the sense of Definition 2.2.

The explicit values of the rational functions $F(x)$ and $G(y)$ are given in Table 5 for further reference.

Remark 3.2 It follows from Lemma 2.4 that if a rational function is divisible by $K(x, y)$, its numerator contains a factor $xyK(x, y)$. However, the converse is wrong, as shown by $xyK(x, y)/(x - t)$. Hence, the term “stronger property” is used in the proposition.

Proof of Proposition 3.1 We first consider the eight models with no step \nearrow nor \searrow . Assume that the numerator of (3.1) contains a factor $xy\mathcal{K}(x, y)$. Since $K(x, y) = \mathcal{K}(\bar{x}, xy)$, it is equivalent to say that the numerator of

$$xy - (t\mathcal{V}_0(xy) + 2t\bar{x}\mathcal{V}_+(xy) - 1)G(xy) - F(\bar{x})$$

contains a factor $xyK(x, y)$. Since $K(x, y) = K(y, x)$, the same holds for the numerator of

$$xy - (t\mathcal{V}_0(xy) + 2t\bar{y}\mathcal{V}_+(xy) - 1)G(xy) - F(\bar{y}),$$

and, by summing the two previous rational functions, for the numerator of

$$2xy - 2(t\mathcal{V}_0(xy) + t(\bar{x} + \bar{y})\mathcal{V}_+(xy) - 1)G(xy) - F(\bar{x}) - F(\bar{y}).$$

Since $S(x, y) = S(y, x)$, it follows from the expression (2.6) of $S(x, y)$ that $\bar{y}\mathcal{V}_+(xy) = x\mathcal{V}_-(xy)$. Hence, the coefficient of $G(xy)$ in the above function is in fact $2K(x, y)$. Since the denominator of $G(xy)$ cannot contain a factor $xyK(x, y)$ (all steps of \mathcal{S} would be on the diagonal), we conclude that the numerator of $2xy - F(\bar{x}) - F(\bar{y})$ contains a factor $xyK(x, y)$, which means that Condition (b) holds with $f(x) = g(x) = F(\bar{x})/2$.

It is straightforward to adapt the above argument to the diagonal case, where now $S(x, y) = \mathcal{S}(\bar{x}^2, xy) = x^2\mathcal{V}_-(xy) + \mathcal{V}_0(xy) + \bar{x}^2\mathcal{V}_+(xy)$, with $\bar{y}^2\mathcal{V}_+(xy) = x^2\mathcal{V}_-(xy)$. One then finds a solution to Problem (b) with $f(x) = g(x) = F(\bar{x}^2)/2$.

Conversely, assume that Condition (b) holds. As proved in [2, Section 4.2], then \mathcal{S} is either one of three models of the Kreweras trilogy, or the 6th model of Table 1. Moreover, Condition (b) then holds for a pair (f, g) such that $f = g$: it suffices to take for f the half-sum of the two functions F and G of [2, Tables 4 and 5], and this yields (3.4). We leave it to the reader to check that the functions $F(x)$ and $G(y)$ defined by (3.3), and listed in Table 5, then satisfy Condition (a) of the lemma. Finally, since f, g, F , and G are Laurent polynomials in t , the same holds when dividing (3.1) by $\mathcal{K}(x, y)$ or (3.2) by $K(x, y)$, and the resulting ratios thus have poles of bounded order at zero. This establishes the claimed divisibility properties. ■

3.2 A new pair of invariants

We now restrict our attention to the four models of Table 5, for which the conditions of Proposition 3.1 hold. We return to the functional equation (2.12) that relates $U(x, y)$ and $D(y)$. We rewrite the first term y in the right-hand side using Proposition 3.1, and obtain

$$(3.5) \quad \mathcal{K}(x, y) (2xyU(x, y) - H(x, y)) = (t\mathcal{V}_0(y) + 2tx\mathcal{V}_+(y) - 1)S(y) - R(x),$$

with

$$H(x, y) = \frac{y - (t\mathcal{V}_0(y) + 2tx\mathcal{V}_+(y) - 1)G(y) - F(x)}{\mathcal{K}(x, y)}$$

and

$$(3.6) \quad S(y) = yD(y) + G(y), \quad R(x) = 2tx\mathcal{H}_-(x)U(x, 0) - F(x) + tD_0\mathbb{1}_{(0,-1)\in\mathcal{S}}.$$

The functions $F(x)$, $G(y)$, and $H(x, y)$ are those of Table 5. Upon multiplying (3.5) by

$$(t\mathcal{V}_0(y) + 2tx\mathcal{V}_+(y) - 1)S(y) + R(x),$$

and using Lemma 2.3, we exhibit a new pair of invariants.

Proposition 3.3 *With the above notation, the pair $(I(x), J(y)) := (R(x)^2, \Delta(y)S(y)^2)$ is a pair of invariants for the step set \mathcal{S} , in the sense of Definition 2.3. More precisely,*

$$(3.7) \quad \frac{I(x) - J(y)}{\mathcal{H}(x, y)} = -4tx\mathcal{V}_+(y)S(y)^2 + (H(x, y) - 2xyU(x, y)) \left((t\mathcal{V}_0(y) + 2tx\mathcal{V}_+(y) - 1)S(y) + R(x) \right),$$

where we recall that $\Delta(y) = (1 - t\mathcal{V}_0(y))^2 - 4t^2\mathcal{V}_-(y)\mathcal{V}_+(y)$.

Proof The identity is straightforward, and we only need to check that all series that occur on the right-hand side of (3.7) have poles of bounded order at 0. First, this holds for the series F , G , and H (see Proposition 3.1 or Table 5). Then $D(y)$ belongs to $\mathbb{Q}[y][[t]]$, $U(x, y)$ to $\mathbb{Q}[x, y][[t]]$, and $\mathcal{H}_-(x)$ is a Laurent polynomial in x . Thus, the series $R(x)$ and $S(y)$ defined by (3.6) have poles of bounded order at 0. Since $\mathcal{V}_0(y)$ and $\mathcal{V}_+(y)$ are Laurent polynomials in y , the conclusion follows. ■

Remark 3.4 There is a useful alternative expression of $R(x)$, and thus of $I(x) = R(x)^2$. Let us return to the basic functional equation (2.4) for $C(x, y)$, written at (\bar{x}, \bar{x}) , and observe that $C_-(x) = xU(x, 0)$ and $C_{0,0} = D_0$ (see (2.5)). This gives

$$\begin{aligned} K(\bar{x}, \bar{x})C(\bar{x}, \bar{x}) &= 1 - 2tx^2H_-(\bar{x})U(x, 0) - tx^2D_0\mathbb{1}_{(-1,-1)\in\mathcal{S}} \\ &= 1 - x^2(R(x) + F(x)) && \text{by (3.6) and (2.10)} \\ &= -x^2\left(R(x) + \frac{K(\bar{x}, \bar{x})}{tH_-(\bar{x})}\right) && \text{by (3.3).} \end{aligned}$$

We conclude that

$$R(x) = -K(\bar{x}, \bar{x})\left(\bar{x}^2C(\bar{x}, \bar{x}) + \frac{1}{tH_-(\bar{x})}\right).$$

In particular, for $x = 1$, we obtain

$$(3.8) \quad R(1) = -(1 - |S|t)\left(C(1, 1) + \frac{1}{tH_-(1)}\right).$$

Since we are going to provide the value of $I(x) = R(x)^2$ explicitly, we will obtain an expression for the square of the above series. This is reflected in the characterization of $C(1, 1)$ already given for Kreweras’ model (see (1.15)). Further illustrations are (5.5), (6.9), and (7.7).

3.3 General strategy

Let us now describe the strategy that we are going to apply to each of the four models of Table 5, in Sections 4–7. For each of these models, we have at hand two, or sometimes three, pairs of \mathcal{S} -invariants:

- the pair $(I(x), J(y))$ from Proposition 3.3; it involves the series $U(x, 0) = \bar{x}C_-(x)$ and $D(y)$, which are keys in the determination of the generating function $C(x, y)$ of walks avoiding the negative quadrant;
- a rational one, $(I_0(x), J_0(y))$, for the three models of the Kreweras trilogy (Table 3);
- for all four models, the pair $(I_1(x), J_1(y))$ defined by (2.24), where the functions f and g are given in Tables 3 and 4. This pair of invariants involves series counting quadrant walks with steps in \mathcal{S} , and these series are known. They are algebraic for the Kreweras trilogy [10], and D-algebraic for the last model [2].

Using Lemma 2.5, we are going to form polynomial combinations of these pairs to construct a new pair of invariants involving $U(x, 0)$ and $D(y)$ and satisfying the condition of Lemma 2.6—and thus trivial. This will give us expressions of $I(x)$ and $J(y)$, and hence $U(x, 0)$ and $D(y)$, in terms of some known quadrant series. For the Kreweras trilogy, an alternative would be to construct trivial invariants based on $(I(x), J(y))$ and $(I_0(x), J_0(y))$ only, as is done in [1, 2] to (re)derive quadrant generating functions. However, exploiting the three pairs that we have at hand yields more direct derivations.

4 Kreweras steps

In this section, we take $\mathcal{S} = \{\nearrow, \leftarrow, \downarrow\}$, so that $\mathcal{S}' = \{\rightarrow, \uparrow, \swarrow\}$. We prove the results that were stated at the end of the introduction (Section 1.6): first the exact results in Section 4.1, then the asymptotic ones in Section 4.2. We refer to the Maple session available on the author’s web page (<http://www.labri.fr/perso/bousquet/publis.html>) for the details of the calculations.

4.1 Generating functions

To begin with, we establish the expressions of $U(x, 0)$ and $D(y)$.

Proof of Theorems 1.1 and 1.2 The first equation in Theorem 1.1 is of course the basic functional equation (2.4). The \mathcal{S} -invariants of Proposition 3.3 are

$$(4.1) \quad I(x) = \left(2tU(x, 0) + 2x - \frac{1}{t}\right)^2, \quad J(y) = \Delta(y) \left(yD(y) + \frac{1}{t}\right)^2$$

with $\Delta(y) = (1 - ty)^2 - 4t^2\bar{y}$. The rational invariants (I_0, J_0) are given in Table 3 (but we will not use them, in fact). The invariants related to quadrant walks with steps in \mathcal{S} , defined by (2.24), are

$$(4.2) \quad I_1(x) = t\mathcal{Q}(x, 0) - x/t + x^2, \quad J_1(y) = -t\mathcal{Q}(0, y) - \bar{y} + t\mathcal{Q}_{0,0}.$$

They satisfy

$$(4.3) \quad I_1(x) - J_1(y) = -\frac{x}{t}\mathcal{K}(x, y)(1 + ty\mathcal{Q}(x, y)).$$

We want to construct a pair $(\tilde{I}(x), \tilde{J}(y))$ of invariants satisfying the condition of Lemma 2.6: the ratio $(\tilde{I}(x) - \tilde{J}(y))/\mathcal{K}(x, y)$ should be a multiple of xy (here we are abusing terminology, since the property that we actually require involves the coefficient of t^n for each n). The ratio $(I(x) - J(y))/\mathcal{K}(x, y)$ is the right-hand side of (3.7). It is easily seen to be a Laurent series in t with coefficients in $\mathbb{Q}[x, y]$, and in fact a multiple of x . It equals $-4x/t$ at $y = 0$. Similarly, the ratio $(I_1(x) - J_1(y))/\mathcal{K}(x, y)$, derived from (4.3), is a multiple of x and equals $-x/t$ at $y = 0$. This leads us to introduce the following pair of invariants:

$$(\tilde{I}(x), \tilde{J}(y)) := (I(x) - 4I_1(x), J(y) - 4J_1(y)),$$

which satisfies the condition of Lemma 2.6. This pair is thus of the form (A, A) , where A is a series in t . Returning to the explicit values (4.1) and (4.2) of our invariants, this gives

$$(4.4) \quad \begin{aligned} I(x) - 4I_1(x) &= \left(2tU(x, 0) + 2x - \frac{1}{t}\right)^2 - 4t\mathcal{Q}(x, 0) + 4x/t - 4x^2 = A, \\ J(y) - 4J_1(y) &= \Delta(y) \left(yD(y) + \frac{1}{t}\right)^2 + 4t\mathcal{Q}(0, y) + 4\tilde{y} - 4t\mathcal{Q}_{0,0} = A. \end{aligned}$$

To complete the solution, it suffices to:

- inject the known algebraic expressions of $\mathcal{Q}(x, 0) = \mathcal{Q}(0, x)$ and $\mathcal{Q}_{0,0}$, taken, for instance, from Proposition 14 in [10]. These expressions involve a series in t denoted by W in [10], which is the series V of Theorem 1.1. We refer to the Maple session for details;
- specialize the second identity obtained in this way, of the form

$$\Delta(y) \left(yD(y) + \frac{1}{t}\right)^2 - 4J_1(y) = A,$$

(where $J_1(y)$ is now explicit), at $y = V^2$, which is the (only) root of $\Delta(y)$ lying in $\mathbb{Q}[[t]]$. This gives

$$A = -4J_1(V^2) = 2 \frac{(1 - V^3)^{3/2}}{V^2} + \frac{V^6 + 12V^3 + 8}{4V^2}.$$

These two ingredients yield, after elementary manipulations involving the equation $V = t(2 + V^3)$, the expressions announced in Theorems 1.1 and 1.2, which are those of $I(x)/2$ and $J(x)/2$ (recall that $U(x, 0) = \tilde{x}C_-(x)$).

Since V has degree 3 over $\mathbb{Q}(t)$, it follows from these expressions that $C_-(x)$ and $D(x)$ have degree at most 24. This bound is proved to be tight by computing explicit minimal polynomials, at $x = 2$ for instance (this is lighter than keeping the indeterminate x). Now, the identities of Theorem 1.1 show that $C(x, y)$ will have degree $24 \times 4 = 96$ at most, and at any rate the same degree as $x C_-(\tilde{x}) + y C_-(\tilde{y}) = U(\tilde{x}, 0) + U(\tilde{y}, 0)$. So we only need to determine the degree of $U(x, 0) + U(y, 0)$, which we do at $x = 2$ and $y = 3$ (by successive eliminations as before). We find it to be 96 indeed, which concludes the proof of Theorems 1.1 and 1.2. ■

Now, we prove the results that deal with walks ending at a prescribed position (i, j) .

Proof of Corollary 1.3 We begin with walks ending on the diagonal, the generating function of which is given in Theorem 1.2. We consider the series

$$(4.5) \quad xD(x/V) + \frac{V}{t},$$

which, due to the periodicity of the model and the fact that V/t is a series in t^3 , is also a series in t^3 (or equivalently, in V^3) with coefficients in $\mathbb{Q}[x]$. Using Theorem 1.2 and the identity $t = V/(2 + V^3)$, we can write

$$\begin{aligned} \left(xD(x/V) + \frac{V}{t}\right)^2 &= V^2 \frac{J(x/V)}{\Delta(x/V)} \\ &= \frac{2(2 + V^3)^2}{(x - V^3)(4 - 4x - xV^3 + x^2)} \\ &\quad \times \left(x(1 - V^3)^{3/2} + (1 - x)^2(x - V^3) - (V^3 + xV^3 - 2x^2)\sqrt{1 - \frac{4 + V^3}{4}x + \frac{x^2}{4}}\right). \end{aligned}$$

The right-hand side appears as a formal power series in x , with constant term $(2 + V^3)^2$ and coefficients in $\mathbb{Q}[\sqrt{1 - V^3}, 1/V^3]$. Upon taking square roots, we conclude that the series (4.5) is also a series in x with coefficients in $\mathbb{Q}[\sqrt{1 - V^3}, 1/V^3]$. For $i \geq 0$, the coefficient of x^{i+1} in this series is $C_{i,i}/V^i$. We have thus proved that

$$(4.6) \quad D_i = C_{i,i} \in V^i \mathbb{Q}[\sqrt{1 - V^3}, 1/V^3], \quad \text{so that} \quad C_{i,i} \in t^i \mathbb{Q}[\sqrt{1 - V^3}, 1/V^3].$$

Since $V = t(2 + V^3)$, and $\sqrt{1 - V^3} = 1 - 2W$ by definition of W , the part of the corollary dealing with walks that end on the diagonal follows. We still retain the above more precise statement for further use. We obtain in particular the values of $C_{0,0}$ and $C_{1,1}$ given below Corollary 1.3. To obtain the claimed result for walks ending just above or below the diagonal ($j - i = \pm 1$, with $i, j \geq 0$), it suffices to note that $C_{i,i} - tC_{i-1,i-1} = 2tC_{i,i+1} + \mathbb{1}_{i=0}$ due to the choice of steps. In particular, we thus obtain the value of $C_{0,1}$ given below Corollary 1.3.

We go on with walks ending on the negative x -axis, the generating function of which is given in Theorem 1.1. By periodicity, the series

$$(4.7) \quad V \left(2tU(x/V) + 2x/V - \frac{1}{t}\right) = V \left(2t\bar{x}VC_-(x/V) + 2x/V - \frac{1}{t}\right)$$

is a series in t^3 with coefficients in $\mathbb{Q}[x]$. Using Theorem 1.1, and the definition of W , we first rewrite its square in terms of x and W :

$$\begin{aligned} V^2 \left(2t\bar{x}VC_-(x/V) + 2x/V - \frac{1}{t}\right)^2 &= V^2 I\left(\frac{x}{V}\right) \\ &= 2\bar{x} \left(x(1 - V^3)^{3/2} + (1 - x)^2(x - V^3) + (V^3 + xV^3 - 2x^2)\sqrt{1 - \frac{4 + V^3}{4}x + \frac{x^2}{4}}\right). \end{aligned}$$

The right-hand side is a formal power series in x with constant term $2(1 - V^3)^{3/2} + 2 + 5V^3 - V^6/4 = 4(1 - W^2)(1 + W)^2$, whereas the coefficients of higher powers of x are polynomials in V^3 . Given that $V^3 = 4W(1 - W)$, we conclude that (4.7) has its

coefficients in $\sqrt{1 - W^2} \mathbb{Q}[W, 1/(1 - W^2)]$. For $i \geq 1$, the coefficient of x^{i-1} in (4.7) is

$$2tV^2C_{-i,0}/V^i - \frac{V}{t} \mathbb{1}_{i=1} + 2\mathbb{1}_{i=2}.$$

Hence, using once again $V = t(2 + V^3)$, we find that

$$(4.8) \quad C_{-i,0} \in t^i \mathbb{Q} \left[\sqrt{1 - W^2}, W, 1/V^3, 1/(1 - W^2) \right].$$

We now introduce the series Z , which satisfies $\sqrt{1 - W^2} = (1 - Z^2)/(1 + Z^2)$. The corollary now follows for walks that end on the negative x -axis. We obtain in particular the expression (1.6) of $C_{-1,0}$.

Let us finally prove the corollary for any point (i, j) in the three quadrant cone. Without loss of generality, we assume $j \geq i$ and argue by induction on $j \geq -1$. If $j = -1$, then $i < 0$ and the result is obvious because $C_{i,j} = 0$. The corollary has also been proved for $j = 0, j = i$, and $j = i + 1$. Now, assume that $j \geq 1$ and $j \geq i + 2$. We clearly have

$$(4.9) \quad C_{i,j-1} = t \left(C_{i-1,j-2} + C_{i+1,j-1} + C_{i,j} \right),$$

and the result holds for the three series $C_{i,j-1}, C_{i-1,j-2}$, and $C_{i+1,j-1}$ by the induction hypothesis. Thus, it holds for $C_{i,j}$ as well. ■

4.2 Asymptotics

We go on with the determination of the harmonic function.

Proof of Corollary 1.4 Let us first assume that we have established an estimate of the form (1.7) for $i = j \geq 0$ and for $i < 0, j = 0$. For $(i, j) \in \mathbb{C}$, fix a walk w from $(0, 0)$ to (i, j) in \mathbb{C} , and let k be its length, with $k + i + j \equiv 0 \pmod{3}$. Then $c_{i,j}(3m + k) \geq c_{0,0}(3m)$ (because one can concatenate w with a walk in \mathbb{C} starting and ending at $(0, 0)$, translated so that its starting point is (i, j)). Given the estimation of $c_{0,0}(3m)$, the numbers $c_{i,j}(n)$ cannot be $o(3^n n^{-7/4})$.

Now, for $i \geq 0$,

$$c_{i,i}(n + 1) = c_{i-1,i-1}(n) + 2c_{i,i+1}(n).$$

This identity implies that an estimate of the form (1.7) also holds for $j = i + 1 \geq 1$, with

$$(4.10) \quad 3H_{i,i} = H_{i-1,i-1} + 2H_{i,i+1}.$$

Here, we use the fact that $3H_{i,i} - H_{i-1,i-1} \neq 0$; otherwise, $c_{i,i+1}(n)$ would be $o(3^n n^{-7/4})$, which we have excluded. Then the induction on j , with $j \geq i$, that we have just used in the proof of the previous corollary, establishes (1.7) for all i and j , based on the following version of (4.9):

$$c_{i,j-1}(n + 1) = c_{i-1,j-2}(n) + c_{i+1,j-1}(n) + c_{i,j}(n).$$

The identity (1.8) saying that H is S^* -harmonic similarly follows from

$$c_{i,j}(n + 1) = c_{i-1,j-1}(n) + c_{i+1,j}(n) + c_{i,j+1}(n).$$

Let us now define $\mathcal{H}(x, y)$, $\mathcal{H}_-(x)$, and $\mathcal{H}_d(y)$ in terms of the numbers $H_{i,j}$ as in Corollary 1.4. Then, using the harmonicity of H , we have

$$\begin{aligned} \mathcal{H}(x, y) &= \frac{1}{3} \sum_{j \geq 0, i \leq j} (H_{i-1, j-1} + H_{i+1, j} + H_{i, j+1}) x^{j-i} y^j \\ &= \frac{1}{3} \sum_{j \geq -1, i \leq j} H_{i, j} x^{j-i} y^{j+1} + \frac{1}{3} \sum_{j \geq 0, i \leq j+1} H_{i, j} x^{j-i+1} y^j \\ &\quad + \frac{1}{3} \sum_{j \geq 1, i \leq j-1} H_{i, j} x^{j-i-1} y^{j-1} \\ &= \frac{y}{3} \mathcal{H}(x, y) + \frac{x}{3} \left(\mathcal{H}(x, y) + \bar{x} \sum_{j \geq 0} H_{j+1, j} y^j \right) \\ &\quad + \frac{\bar{x}\bar{y}}{3} \left(\mathcal{H}(x, y) - \sum_{i > 0} H_{-i, 0} x^i - \sum_{i \geq 0} H_{i, i} y^i \right) \\ &= \frac{1}{3} \left((y + x + \bar{x}\bar{y}) \mathcal{H}(x, y) + \sum_{j \geq 0} H_{j+1, j} y^j - \bar{x}\bar{y} \mathcal{H}_-(x) - \bar{x}\bar{y} \mathcal{H}_d(y) \right) \\ &= \frac{1}{3} \left((y + x + \bar{x}\bar{y}) \mathcal{H}(x, y) + \frac{3-y}{2} \mathcal{H}_d(y) - \bar{x}\bar{y} \mathcal{H}_-(x) - \bar{x}\bar{y} \mathcal{H}_d(y) \right), \end{aligned}$$

where we have used (4.10) to express the final sum. This gives (1.10) upon regrouping terms.

At this stage, it remains to prove (1.7) for $i = j \geq 0$ and for $i < 0, j = 0$, and to establish the announced values of the corresponding generating functions $\mathcal{H}_d(y)$ and $\mathcal{H}_-(x)$. Let us begin with the numbers $c_{i,i}(n)$, for n of the form $i + 3m$. Recall the form of $C_{i,i}/t^i$ given by (4.6). An elementary singularity analysis of the series V^3 (seen as a function of t^3) shows that it has a unique dominant singularity, located at $t^3 = 1/27$. It increases on the interval $[0, 1/27]$, and we have the following singular expansions:

$$(4.11) \quad \begin{aligned} V^3 &= 1 - \sqrt{3}\sqrt{1 - 27t^3} + \mathcal{O}(1 - 27t^3), \\ \sqrt{1 - V^3} &= 3^{1/4}(1 - 27t^3)^{1/4} + \mathcal{O}((1 - 27t^3)^{3/4}). \end{aligned}$$

Moreover, the equation $V = t(2 + V^3)$ shows that V cannot vanish on its disk of convergence. In sight of (4.6), $C_{i,i}/t^i$ is thus an algebraic series in t^3 , with radius of convergence at least $1/27$, taking a finite value at this point, and we expect a singular behavior in $(1 - 27t^3)^\alpha$ for some $\alpha \geq 1/4$. Note that there cannot be any singularity other than $1/27$ on the circle of radius $1/27$, because this is the only value of t^3 for which V equals 1.

We now use Theorem 1.2 to express the series $(yD(y/t) + 1)^2$ (which is a series in t^3) in terms of V^3 . Plugging in this expression, the above expansion of V^3 then gives

$$\begin{aligned} (yD(y/t) + 1)^2 &= t^2 \frac{J(y/t)}{\Delta(y/t)} \\ &= 2 \frac{1-3y}{4-3y} + \frac{1+6y}{\sqrt{(1-3y)(4-3y)}} - 6 \cdot 3^{3/4} y \frac{(1-27t^3)^{3/4}}{(1-3y)^2(4-3y)} + \mathcal{O}(1-27t^3). \end{aligned}$$

We refer once again to the Maple session for details. It follows that as t^3 approaches $1/27$,

$$(4.12) \quad D(y/t) = c_0(y) + c_1(y)(1 - 27t^3)^{3/4} + \mathcal{O}(1 - 27t^3),$$

where $c_0(y)$ is an algebraic function of y that we do not need to make explicit, and

$$c_1(y) = -\frac{1}{3^{1/4}(1-3y)\sqrt{y(4-3y)}} \sqrt{\frac{1+6y}{1-3y} \sqrt{\frac{4-3y}{1-3y}}} - 2.$$

By extracting from (4.12) the coefficient of y^i , we obtain

$$C_{i,i} = t^i [y^i] c_0(y) + t^i [y^i] c_1(y) (1 - 27t^3)^{3/4} + \mathcal{O}(1 - 27t^3),$$

so that, for $n = i + 3m$,

$$\begin{aligned} c_{i,i}(n) &\sim \frac{[y^i] c_1(y)}{\Gamma(-3/4)} 27^m m^{-7/4} \\ &\sim \frac{3^{-i+7/4} [y^i] c_1(y)}{\Gamma(-3/4)} 3^n n^{-7/4} \\ &\sim -\frac{H_{i,i}}{\Gamma(-3/4)} 3^n n^{-7/4}, \end{aligned}$$

where $H_{i,i} = -3^{7/4} [y^i] c_1(y/3)$. The announced expression of $\mathcal{H}_d(y) = -3^{7/4} c_1(y/3)$ follows.

Note that the above argument would fail if the coefficient of y^i in $c_1(y)$ were zero. However, in this case, $c_{i,i}(n)$ would be $o(3^n n^{-7/4})$, which we have excluded at the beginning of the proof.

We proceed similarly for walks ending on the negative x -axis. Equation (4.8) gives the form of $C_{-,i,0}/t^i$ in terms of the series V and W . As V^3 , the series W has a unique dominant singularity at $t^3 = 1/27$. It increases on the interval $[0, 1/27]$ and as t^3 approaches $1/27$,

$$(4.13) \quad \begin{aligned} W &= \frac{1}{2} - \frac{3^{1/4}}{2} (1 - 27t^3)^{1/4} + \mathcal{O}((1 - 27t^3)^{3/4}), \\ 1 - W^2 &= \frac{3}{4} + \frac{3^{1/4}}{2} (1 - 27t^3)^{1/4} + \mathcal{O}(\sqrt{1 - 27t^3}). \end{aligned}$$

Since W has nonnegative coefficients, and equals $1/2$ when $t^3 = 1/27$, it remains away from the values ± 1 on the disk $|t^3| < 1/27$. Hence, by (4.8), the series $C_{-,i,0}/t^i$ has radius of convergence at least $1/27$, and no other singularity of modulus $1/27$.

Then we consider the series $2C_{-}(x/t) + 2x^2/t^3 - x/t^3 = x^2 I(x/t)/t^4$, which is a series in t^3 . We express its square in terms of V^3 using Theorem 1.1, and inject the singular expansion of V^3 . This gives

$$(4.14) \quad \begin{aligned} (2C_{-}(x/t) + 2x^2/t^3 - x/t^3)^2 &= 27x(1-3x)(1+6x)\sqrt{(1-3x)(4-3x)} \\ &- 54x(1-3x)^3 + 162 \cdot 3^{3/4} x^2 (1 - 27t^3)^{3/4} + \mathcal{O}(1 - 27t^3). \end{aligned}$$

We must be a bit careful with the sign when taking square roots. Indeed, both sides are $\mathcal{O}(x^2)$, the coefficient of x^2 in the right-hand side is

$$\frac{2,187}{4} + \mathcal{O}((1 - 27t^3)^{3/4}),$$

and this is the singular expansion of $(2C_{-1,0}/t - 1/t^3)^2$ at $t^3 = 1/27$. However, it follows from the expression (1.6) of $C_{-1,0}$, evaluated at $t = 1/3$ (where $Z = 2 - \sqrt{3}$), that at this point, $2C_{-1,0}/t - 1/t^3 = -27\sqrt{3}/2$, with a minus sign. Hence, the square root of the right-hand side of (4.14) is $-(2C_{-1,0}/t + 2x^2/t^3 - x/t^3)$, and as t^3 approaches $1/27$,

$$C_{-1,0}(x/t) = \tilde{c}_0(x) + \tilde{c}_1(x)(1 - 27t^3)^{3/4} + \mathcal{O}(1 - 27t^3),$$

where

$$\tilde{c}_1(x) = -\frac{3^{5/4}x}{2} \sqrt{\frac{1+6x}{1-3x} \sqrt{\frac{4-3x}{1-3x}}} + 2.$$

As we have just done for walks ending on the diagonal, we conclude that for n of the form $i + 3m$ and $i > 0$,

$$c_{-i,0}(n) \sim -\frac{H_{-i,0}}{\Gamma(-3/4)} 3^n n^{-7/4},$$

where $H_{-i,0} = -3^{7/4}[x^i]\tilde{c}_1(x/3)$. The announced expression of $\mathcal{H}_-(x) = -3^{7/4}\tilde{c}_1(x/3)$ follows. ■

Remark 4.1 An invariant approach for harmonic functions. Let us now explain how we can predict the relation that we observed below Corollary 1.4 between the \mathcal{S}^* -harmonic function $H_{i,j}$ in \mathbb{C} and the \mathcal{L}^* -harmonic function $h_{i,j}$ in Ω .

As soon as an asymptotic estimate of the form (1.7) holds for the numbers $c_{i,j}(n)$, the function $H_{i,j}$ is \mathcal{S}^* -harmonic in \mathbb{C} , symmetric in the first diagonal, and the series $\mathcal{H}(x, y)$ defined by (1.9) satisfies (1.10). Denoting by $c(x, y) := \frac{1}{2}(2 + xy^2 - 3xy)$ the coefficient of $\mathcal{H}_d(y)$ in this equation, we observe that

$$c(x, y)^2 = \frac{x^2\delta(y)}{4} - 3xy\mathcal{K}(x, y; 1/3) = \frac{x^2\delta(y)}{4} + (1 + xy^2 + x^2y - 3xy),$$

where $\delta(y) = y(y - 4)(y - 1)^2$ is the discriminant of $3xy\mathcal{K}(x, y; 1/3)$ with respect to x . This identity is of course closely related to Lemma 2.3. After multiplying (1.10) by

$$\bar{x}^2 (\mathcal{H}_-(x) - c(x, y)\mathcal{H}_d(y)),$$

we obtain

$$(4.15) \quad (1 + xy^2 + x^2y - 3xy)\overline{\mathcal{H}}(x, y) = \bar{x}^2 \mathcal{H}_-(x)^2 - \frac{\delta(y)}{4} \mathcal{H}_d(y)^2,$$

where

$$(4.16) \quad \overline{\mathcal{H}}(x, y) := \bar{x}^2 \mathcal{H}(x, y) (\mathcal{H}_-(x) - c(x, y)\mathcal{H}_d(y)) + \bar{x}^2 \mathcal{H}_d(y)^2.$$

Using the functional equation (1.10) satisfied by $\mathcal{H}(x, y)$, we can check that $\overline{\mathcal{H}}(x, y)$ is a formal power series in x and y , and that, moreover,

$$(4.17) \quad \overline{\mathcal{H}}(x, 0) = \bar{x}^2 \mathcal{H}_-(x)^2 \quad \text{and} \quad \overline{\mathcal{H}}(0, y) = -\frac{\delta(y)}{4} \mathcal{H}_d(y)^2 + H_{-1,0}^2.$$

In particular, if we denote $\overline{\mathcal{H}}(x, y) = \sum_{i,j \geq 0} \bar{h}_{i,j} x^i y^j$, we have $\overline{\mathcal{H}}(0, 0) = \bar{h}_{0,0} = H_{-1,0}^2$. Equation (4.15) thus reads

$$(1 + xy^2 + x^2y - 3xy)\overline{\mathcal{H}}(x, y) = \overline{\mathcal{H}}(x, 0) + \overline{\mathcal{H}}(0, y) - \overline{\mathcal{H}}(0, 0),$$

which precisely means that the function $(\bar{h}_{i,j})_{(i,j) \in \Omega}$ is \mathcal{S}^* -harmonic in the quadrant Ω .

Now, assume that there exists a unique positive \mathcal{S}^* -harmonic function $h_{i,j}$ in Ω (up to a multiplicative constant). The associated generating function is $\mathcal{H}(x, y) = \sum_{i,j \geq 0} h_{i,j} x^i y^j$, where $\mathcal{H}(x, 0) = \mathcal{H}(0, x)$ is given by (1.14). Hence, if we assume that the above series $\overline{\mathcal{H}}(x, y)$ has positive coefficients, then there exists a positive constant κ such that $\overline{\mathcal{H}}(x, y) = \kappa^2 \mathcal{H}(x, y)$. Returning to (4.17), we thus predict in particular that

$$\mathcal{H}_-(x) = \kappa x \sqrt{\mathcal{H}(x, 0)} \quad \text{and} \quad \mathcal{H}_d(y) = 2\kappa \sqrt{-\frac{\mathcal{H}(0, y) - \mathcal{H}(0, 0)}{\delta(y)}},$$

where the constant κ must be

$$\kappa = \frac{H_{-1,0}}{\sqrt{h_{0,0}}} = \frac{H_{0,0}}{\sqrt{h_{0,1}}}.$$

Using the expression (1.14) of $\mathcal{H}(x, 0) = \mathcal{H}(0, x)$, we can now check from the expressions (1.11) and (1.12) of $\mathcal{H}_-(x)$ and $\mathcal{H}_d(y)$ that this prediction indeed holds true, with $\kappa = 3$.

We finally complete this section with the enumeration of all walks avoiding the negative quadrant, regardless of their final position.

Proof of Corollary 1.6 We specialize the equations of Theorem 1.1 at $x = y = 1$. The first one gives the link between $C(1, 1)$ and $C_-(1)$ shown on the first line of (1.15) (this is in fact $I(1)/2$), and the second one yields the expression on the second line. The degree of $C(1, 1)$ is found to be 24 by elimination, first of the two square roots, and finally of V .

Now, for the asymptotics, we need more details about V than what we have used so far, which only involved V^3 (seen as a series in t^3). The series V has three dominant singularities, located at $\zeta/3$ where ζ is any cubic root of unity. The singular expansions of V at these points can be computed by combining $V = t(2 + V^3)$ with the known expansion of V^3 around $1/27$ (see (4.11)). Denoting $\zeta_0 = 1$ and $\zeta_{\pm} = (-1 \pm i\sqrt{3})/2$, we find that around any dominant singularity $\zeta/3$,

$$(4.18) \quad \frac{V}{\zeta} = 1 - (1 - 3t/\zeta)^{1/2} + (1 - 3t/\zeta)/3 - 4/9(1 - 3t/\zeta)^{3/2} + \mathcal{O}\left((1 - 3t/\zeta)^2\right).$$

Observe that in the expression (1.15) of $(1 - 3t)^2 (C(1, 1) + 1/t)^2$, the square root term involving

$$1 - V \frac{4 + V^3}{4} + \frac{V^2}{4} = \frac{(1 - V)(2 + V)(2 - V + V^2)}{4}$$

does not vanish for $|V| < 1$ (but does vanish at $V = 1$ of course). We now plug the above expansion of V in the expression of $(1 - 3t)^2 (C(1, 1) + 1/t)^2$, extract square roots, and find the following behaviors, respectively, at $t = \zeta_0/3 = 1/3$ and $t = \zeta_{\pm}/3$:

$$\begin{aligned} (1 - 3t) (C(1, 1) + 1/t) &= 3^{3/4} \sqrt{2 - \sqrt{2}} (1 - 3t)^{3/8} + \mathcal{O}((1 - 3t)^{7/8}), \\ (1 - 3t) (C(1, 1) + 1/t) &= a_{\pm} + b_{\pm} (1 - 3\zeta_{\mp} t)^{3/4} + \mathcal{O}(1 - 3\zeta_{\mp} t), \end{aligned}$$

for some nonzero constants a_{\pm} and b_{\pm} . In sight of the factor $(1 - 3t)$ on the left-hand side, we conclude that, as far as the first order of c_n is concerned, the only singularity that contributes is $1/3$, and that c_n satisfies (1.16). ■

5 Reverse Kreweras steps

In this section, we take $\mathcal{S} = \{\rightarrow, \uparrow, \swarrow\}$, so that $\mathcal{S} = \{\nearrow, \leftarrow, \downarrow\}$.

5.1 Statements of the results

We will establish the following counterpart of Theorems 1.1 and 1.2.

Theorem 5.1 *The generating function $C(x, y)$ of walks with steps in $\{\rightarrow, \uparrow, \swarrow\}$ starting from $(0, 0)$ and avoiding the negative quadrant is algebraic of degree 96. It is given by the following equation:*

$$(1 - t(x + y + \bar{x}\bar{y}))C(x, y) = 1 - t\bar{x}\bar{y}C_{-}(\bar{x}) - t\bar{x}\bar{y}C_{-}(\bar{y}) - t\bar{x}\bar{y}C_{0,0},$$

where $C_{0,0}$ is algebraic of degree 6 and $C_{-}(x)$ is algebraic of degree 24. These series can be expressed explicitly in terms of the series V, W , and Z defined in Section 1 (see Table 2). Indeed,

$$C_{0,0} = \frac{V}{t} \cdot \frac{4 - 4W - 2W^2 + 3W^3}{8(1 - W)},$$

and

$$\begin{aligned} &\left(2tC_{-}(x) + \bar{x}^2 - \frac{\bar{x}}{t} + x + tC_{0,0}\right)^2 = \\ (5.1) \quad &\left(\bar{x}^2 - \frac{\bar{x}}{t} - x\right)^2 + A_2\left(\bar{x}^2 - \frac{\bar{x}}{t} - x\right) + A_1\left(\frac{1}{x} - \frac{1}{V}\right)\sqrt{1 - xV^2} + A_0, \end{aligned}$$

where the series A_0, A_1 , and A_2 belong to $\mathbb{Q}(t, Z)$, with respective degrees 6, 12, and 6:

$$\begin{aligned} \frac{A_0}{V^2} &= -\frac{8 + 18W - 20W^2 + 5W^3 - 6W^4 + 4W^5}{8W(1 - W)}, \\ A_1 &= \frac{(2 - W)^3(1 + W)}{2} \sqrt{\frac{1 + W}{1 - W}} = \frac{(2 - W)^3(1 + W)}{2} \cdot \frac{1 + Z}{1 - Z}, \end{aligned}$$

and

$$\frac{A_2}{V} = \frac{4 - 4W - 2W^2 + 3W^3}{4(1 - W)}.$$

The generating function $D(x)$ of walks ending on the diagonal is algebraic of degree 24, given by

$$(5.2) \quad \Delta(x) \left(xD(x) + \frac{1}{tx} \right)^2 = \left(\bar{x}^2 - \frac{\bar{x}}{t} - x \right)^2 + A_2 \left(\bar{x}^2 - \frac{\bar{x}}{t} - x \right) - A_1 \left(\frac{1}{x} - \frac{1}{V} \right) \sqrt{1 - xV^2} + A_0,$$

with the above values of A_0, A_1, A_2 , and

$$(5.3) \quad \Delta(x) = (1 - t\bar{x})^2 - 4t^2x.$$

For walks ending at a prescribed position, we obtain the following result.

Corollary 5.2 *Let us define V, W , and Z as above. For any $(i, j) \in \mathbb{C}$, the generating function of walks avoiding the negative quadrant and ending at (i, j) is algebraic of degree (at most) 12 and belongs to $\mathbb{Q}(t, Z)$. More precisely, $C_{i,j}/t^{i+j}$ belongs to $\mathbb{Q}(Z)$.*

This holds obviously for the series $C_{0,0}$ given in Theorem 5.1. Note that this series coincides with the series $C_{0,0}$ obtained for Kreweras walks (it suffices to rewrite V/t in terms of W to obtain the form (1.5)): this is clear, upon reversing the direction of steps. Other examples are

$$(5.4) \quad tC_{1,1} = \frac{2Z(2Z^9 - Z^8 - 4Z^7 + 10Z^6 - 10Z^4 + 6Z^3 + 4Z^2 - 4Z + 1)}{(1 - Z)^2(1 + Z^2)^4},$$

$$tC_{-1,0} = \frac{A_1}{4} - 1 = \frac{(2 - W)^3(1 + W)}{8} \cdot \frac{1 + Z}{1 - Z} - 1.$$

The length generating function $C(1, 1)$ of walks avoiding the negative quadrant can be characterized using the first two results of Theorem 5.1.

Corollary 5.3 *The generating function $C(1, 1)$ is algebraic of degree 24 over $\mathbb{Q}(t)$. It is given by*

$$(5.5) \quad (1 - 3t)^2(1 + tC(1, 1))^2 = 1 - tA_2 + t^2A_1 \left(1 - \frac{1}{V} \right) \sqrt{1 - V^2} + t^2A_0,$$

where the series V, A_0, A_1 , and A_2 are those of Theorem 5.1. The asymptotic behavior of its n th coefficient c_n can be obtained via singularity analysis:

$$c_n \sim \frac{9}{4\Gamma(5/8)} \left(\frac{9}{2} - 3\sqrt{2} \right)^{1/4} 3^n n^{-3/8}.$$

Harmonic function

We have also derived the counterpart of Corollary 1.4, that is, an explicit harmonic function on \mathbb{C} associated with the Kreweras step set $\mathcal{S}^* = \{\leftarrow, \downarrow, \nearrow\}$. We still have an

asymptotic behavior of the form (1.7), this time with $n \equiv i + j \pmod 3$. We do not give here all details of the numbers $H_{i,j}$, but simply the values of

$$\mathcal{H}_-(x) := \sum_{i>0} H_{-i,0} x^i = \frac{27\sqrt{3}}{8} \left(\sqrt{1 + \frac{3\sqrt{3}x}{2(1-x)^{3/2}}} - 1 \right)$$

and

$$\mathcal{H}_d(y) := \sum_{i \geq 0} H_{i,i} y^i = \frac{27\sqrt{3}}{4(1-y)\sqrt{1-4y}} \sqrt{1 - \frac{3\sqrt{3}y}{2(1-y)^{3/2}}}$$

from which all numbers $H_{i,j}$ can be reconstructed via an equation that is the counterpart of (1.10). The proof is analogous to Kreweras' case. Once again, details of the calculations can be found in our Maple session. For comparison, the number of quadrant \mathcal{S} -walks of length n going from $(0, 0)$ to (i, j) is asymptotic to $h_{i,j} 3^n n^{-5/2} / \Gamma(-3/2)$, with

$$\sum_{i \geq 0} h_{i,0} x^i = \frac{9}{(1-x)^{3/2}}.$$

5.2 Proofs for reverse Kreweras steps

Proof of Theorem 5.1 The first equation of the theorem is of course the basic functional equation (2.4). The functional equations (2.8) and (2.9) defining $D(y)$ and $U(x, y)$ read

$$(5.6) \quad (1 - t\bar{y})D(y) = 1 - t\bar{y}D_0 + 2tU(0, y),$$

$$(5.7) \quad x(1 - t(\bar{x} + \bar{y} + xy))U(x, y) = txyD(y) - tx\bar{y}U(x, 0) - tU(0, y).$$

The \mathcal{S} -invariants of Proposition 3.3 are

$$(5.8) \quad I(x) = \left(2txU(x, 0) + \frac{1}{x^2} - \frac{1}{tx} + x + tD_0 \right)^2, \quad J(y) = \Delta(y) \left(yD(y) + \frac{1}{ty} \right)^2,$$

with Δ defined by (5.3). The rational invariants (I_0, J_0) are given in Table 3:

$$(5.9) \quad I_0(x) = \bar{x}^2 - \bar{x}/t - x, \quad J_0(y) = I_0(y).$$

They satisfy

$$(5.10) \quad \frac{I_0(x) - J_0(y)}{\mathcal{H}(x, y)} = \frac{x - y}{txy}.$$

The invariants related to quadrant walks with steps in \mathcal{S} , defined by (2.24), are

$$(5.11) \quad I_1(x) = tx\mathcal{Q}(x, 0) + \bar{x} - \frac{1}{2t}, \quad J_1(y) = -ty\mathcal{Q}(0, y) - \bar{y} + \frac{1}{2t} = -I_1(y).$$

They satisfy

$$(5.12) \quad \frac{I_1(x) - J_1(y)}{\mathcal{K}(x, y)} = -xy\mathcal{Q}(x, y) - \frac{1}{t}.$$

Applying the invariant lemma

As in the previous section, we want to combine the pair of invariants $(I(x), J(y))$ defined by (5.8) with those of (5.9) and (5.11) to form a pair $(\tilde{I}(x), \tilde{J}(y))$ satisfying the condition of Lemma 2.6. Since the conclusion of this lemma is that $\tilde{I}(x)$ and $\tilde{J}(y)$ belong to $\mathbb{Q}((t))$, we must look for a polynomial combination of $I(x)$, $I_0(x)$, and $I_1(x)$ that, at least, has no pole at $x = 0$. In sight of the expansions of these three series at $x = 0$, we are led to consider

$$(5.13) \quad \tilde{I}(x) := I(x) - I_0(x)^2 - A_2I_0(x) - A_1I_1(x),$$

with

$$(5.14) \quad A_2 = 2tD_0, \quad A_1 = 4(1 + tU_{0,0}),$$

which indeed has no pole at $x = 0$. Let us define accordingly

$$(5.15) \quad \tilde{J}(y) := J(y) - J_0(y)^2 - A_2J_0(y) - A_1J_1(y),$$

and examine whether $(\tilde{I}(x) - \tilde{J}(y))/\mathcal{K}(x, y)$ is a multiple of xy . Denoting by Rat , Rat_0 , and Rat_1 the right-hand sides of (3.7), (5.10), and (5.12), respectively, we have

$$(5.16) \quad \frac{\tilde{I}(x) - \tilde{J}(y)}{\mathcal{K}(x, y)} = \text{Rat} - (I_0(x) + J_0(y)) \text{Rat}_0 - A_2 \text{Rat}_0 - A_1 \text{Rat}_1.$$

This ratio has poles of bounded order at 0. We first expand it around $x = 0$, and observe that it is $\mathcal{O}(1/x)$. In order to prove that the coefficient of $1/x$ is in fact 0, we use (5.6) (see our Maple session for details). In order to prove that the coefficient of x^0 is also 0, we use moreover the first term in the expansion of (5.7) at $x = 0$, which gives an expression of $U'_x(0, y)$ in terms of $U(0, y)$, $D(y)$, and $U_{0,0}$. Similarly, in order to prove that the ratio (5.16) is a multiple of y , we inject in its y -expansion (which is $\mathcal{O}(y^0)$) the expansion at $y = 0$ of (5.6), which relates D_0 , D_1 , and $U_{0,0}$, and the first term in the expansion of (5.7) at $y = 0$, which gives an expression of $U'_y(x, 0)$ in terms of $U(x, 0)$ and $U_{0,0}$. Then we happily conclude that the pair $(\tilde{I}(x), \tilde{J}(y))$ is independent of x and y , equal to (A_0, A_0) for some series A_0 that depends on t only.

Recall the expression (5.11) of $I_1(x)$ in terms of the generating function $\mathcal{Q}(x, 0)$ of quadrant walks with Kreweras steps. This series is known, and we obtain from Proposition 13 in [10]

$$(5.17) \quad I_1(x) = \left(\frac{1}{x} - \frac{1}{V}\right) \sqrt{1 - xV^2} = -J_1(x).$$

This gives the expressions (5.1) and (5.2) announced in Theorem 5.1, which are those of $I(x)$ and $J(x)$, but we still need to determine the three series $A_0, A_1,$ and A_2 .

The series $A_0, A_1,$ and A_2

In the Kreweras case (Section 4), we only had one unknown series to determine, denoted by A (see (4.4)). We derived it using the (only) root of $\Delta(y)$ that was finite at $t = 0$ for this model. Now, with the new value of $\Delta(y)$, we have two such roots—but three series to determine. The third identity between the A_i 's will follow from the fact that $I(x)$ has a double root (see (5.8)).

We begin with a useful property, which tells us that $I_0(x)$ is (almost) the square of $I_1(x)$. Indeed, the expressions of $I_0(x)$ and $I_1(x)$, together with the definition (1.3) of V , imply that

$$(5.18) \quad I_1(x)^2 = I_0(x) + \frac{1}{V^2} + 2V.$$

The series $J_1(y) = -I_1(y)$ and $J_0(y) = I_0(x)$ are related by the same identity. (If we did not know the explicit expression (5.17) of $I_1(x)$, we could still derive this identity from the invariant lemma, which gives a polynomial relation between the pairs $(I_1(x), J_1(y))$ and $(I_0(x), J_0(y))$; this is how quadrant walks with steps in \mathcal{S} are solved in [2].)

Using this, and the definitions (5.13) and (5.15) of \tilde{I} and \tilde{J} , the fact that $\tilde{I}(x) = \tilde{J}(y) = A_0$ translates into

$$(5.19) \quad I(x) = P_4(I_1(x)), \quad J(y) = P_4(J_1(y)),$$

where

$$P_4(u) = \left(u^2 - \frac{1}{V^2} - 2V\right)^2 + A_2 \left(u^2 - \frac{1}{V^2} - 2V\right) + A_1u + A_0.$$

Note that the identities of (5.19) are those that we would obtain from the invariant lemma by playing with the pairs $(I(x), J(y))$ and $(I_1(x), J_1(y))$ only, with no reference to $(I_0(x), J_0(y))$.

As already mentioned, two roots of $\Delta(y)$ are finite at $t = 0$. We denote them by Y_+ and Y_- :

$$Y_{\pm} = t \pm 2t^{5/2} + 6t^4 \pm 21t^{11/2} + \mathcal{O}(t^7).$$

By replacing t by its expression in terms of V in $\Delta(y)$, we see that Y_{\pm} are the roots of the following quadratic polynomial in Y :

$$4Y^2 - V(V^3 + 4)Y + V^2.$$

The third root of Δ is $1/V^2$. From this, and the expression (5.17) of $I_1(x) = -J_1(x)$, we conclude that the values $J_1(Y_{\pm})$ are roots of

$$\tilde{P}_4(u) = 4u^4V^4 + V^2(V^6 - 20V^3 - 8)u^2 - 4V^9 + 12V^6 - 12V^3 + 4.$$

Since $J(y)$ contains a factor $\Delta(y)$ and equals $P_4(J_1(y))$, the values $J_1(Y_{\pm})$ are also roots of $P_4(u)$, and hence of the following quadratic polynomial:

$$\begin{aligned} P_2(u) &= 4V^2P_4(u) - \tilde{P}_4(u)/V^2 \\ &= V^2(4A_2 - V^4 + 4V)u^2 + 4V^2A_1u \\ &\quad + 4(V^7 - 2A_2V^3 + V^4 + A_0V^2 - A_2 + 7V). \end{aligned}$$

Hence, $P_4(u)$ contains a factor $P_2(u)$.

Now, consider the equation $I(X) = 0$, that is,

$$X = t(1 + X^3 + tX^2D_0 + 2tX^3U(X, 0)).$$

From the form of this equation, we see that it admits a unique solution X in the ring $\mathbb{Q}[[t]]$. The first terms in its expansion are

$$X = t + 2t^4 + 16t^7 + \mathcal{O}(t^{10}).$$

Since $I(X) = 0$ and $I'(X) = 0$, it follows from (5.19) that $I_1(X)$ is a root of $P_4(u)$, and even a double root, unless $I'_1(X) = 0$. However, $I'_1(X) = -1/t^2 + \mathcal{O}(t)$, and hence $P_4(u)$ admits indeed $I_1(X)$ as a double root. Moreover, we easily check that $I_1(X) \neq J_1(Y_{\pm})$. Thus, we have found all roots of $P_4(u)$, and we have the following factorization:

$$P_4(u) = \frac{1}{V^2(4A_2 - V^4 + 4V)}(u - I_1(X))^2P_2(u).$$

(The multiplicative constant is adjusted using the leading coefficients of P_2 and P_4 .) By extracting the coefficients of u^3, \dots, u^0 in this identity, we obtain four polynomial equations that relate $I_1(X)$, A_0 , A_1 , and A_2 . By eliminating the first three series, we obtain a polynomial equation for A_2 . We determine which of its factors vanishes thanks to the first coefficients of $A_2 = 2tD_0$ (see (5.14)), and, after introducing the series W , we obtain the value of A_2 given in Theorem 5.1. The values of A_1 and A_0 follow similarly. We refer to our Maple session for details. Finally, the expression of $C_{0,0}$ follows from $C_{0,0} = D_0 = A_2/(2t)$.

Degrees

Let us now discuss the algebraic degrees of our generating functions. The expression of $C_{0,0}$ shows that it has degree at most 6, and this is easily checked to be tight by elimination of W and V . The expression of $I(x)$ shows that it belongs to an extension of degree 12 of $\mathbb{Q}(t, x)$: this can be seen by extending this field first by V (degree 3), then W (degree 2), and finally by $A_1\sqrt{1 - xV^2}$; the square of the latter series being in $\mathbb{Q}(x, t, V, W)$ (because $\frac{1+Z}{1-Z} = \sqrt{\frac{1+W}{1-W}}$), this yields another extension of degree 2, resulting in total into a degree 12 at most for $I(x)$. Since $C_{0,0} \in \mathbb{Q}(W)$, this implies that $U(x, 0)$ has degree at most 24. An effective elimination procedure shows that these bounds are tight. The same argument shows that $J(x)$ has degree 12, and $D(x)$ degree

24. In fact, $I(x)$ and $J(x)$ satisfy the same equation over $\mathbb{Q}(t, x)$, as shown by their respective expressions. Finally, the basic functional equation implies that the degree of $C(x, y)$ is the degree of $C_-(x) + C_-(y) + C_{0,0}$. We have seen that $C_{0,0}$ belongs to $\mathbb{Q}(t, V, W)$, and that $C_-(x)$ belongs to an extension of degree 4 of $\mathbb{Q}(t, x, V, W)$. Hence, the above sum belongs to an extension of $\mathbb{Q}(t, x, V, W)$ of degree at most $4 \times 4 = 16$, and thus has degree at most $16 \times 6 = 96$. We compute its minimal polynomial over $\mathbb{Q}(t)$ for $x = 2$ and $y = 3$, and find this bound to be tight. ■

Next, we establish the results that deal with walks ending at a prescribed point (i, j) .

Proof of Corollary 5.2 We begin with walks ending on the diagonal, the generating function of which is given in Theorem 5.1. We consider the series

$$(5.20) \quad xD(x/V^2) + \frac{V^4}{tx},$$

which, due to the periodicity of the model and the properties of V , is a series in t^3 (or equivalently, in V^3) with coefficients in $\mathbb{Q}[\tilde{x}, x]$. Using the second part of Theorem 5.1, we first rewrite its square in terms of x and the series Z (this series is only needed because of the term A_1 , otherwise we could do with W as in Kreweras' case). We thus obtain a (Laurent) series in x with rational coefficients in Z . The coefficient of x^{-2} in this series is V^8/t^2 . Hence, the square root of this series, which is (5.20), is also a series in x with coefficients in $\mathbb{Q}(Z)$. The coefficient of x^i in (5.20) is $C_{i-1,i-1}/V^{2i-2}$ as soon as $i > 0$. The corollary thus follows for walks ending on the diagonal. In particular, for $i = 2$, we obtain the expression (5.4) of $C_{1,1}$.

The same argument, starting now from

$$2C_-\left(\frac{x}{V^2}\right) + \frac{V^4}{tx^2} - \frac{V^2}{t^2x} + \frac{x}{tV^2} + C_{0,0},$$

establishes the result for walks ending on the negative x -axis, thanks to the expression of $I(x)$. Indeed, one can show that this is a (Laurent) series in x with rational coefficients in Z , and the coefficient of x^i in this series, for $i > 0$, is $2C_{-i,0}/V^{2i} + 1/(tV^2)\mathbb{1}_{i=1}$. We obtain in particular the expression of $C_{-1,0} = U_{0,0}$ given below Corollary 5.2 (but it also follows from (5.14) and the expression of A_1).

We can now prove the corollary for any point (i, j) in the three-quadrant cone. Without loss of generality, we assume that $j \geq i$, and argue by induction on $j \geq -1$. The result is obvious when $j = -1$ (because then $i < 0$ and $C_{i,j} = 0$), and has just been established for $j = 0$. Since it has also been proved for $j = i$, we assume that $j \geq i + 1$ and $j \geq 1$. We have

$$C_{i-1,j-1} = t(C_{i-1,j-2} + C_{i-2,j-1} + C_{i,j}),$$

and the result holds for the three series $C_{i-1,j-1}$, $C_{i-1,j-2}$, and $C_{i-2,j-1}$ by the induction hypothesis. Thus, it holds for $C_{i,j}$ as well. ■

We conclude this section with the series $C(1, 1)$.

Proof of Corollary 5.3 The value of $C(1, 1)$ is obtained by specializing at $x = y = 1$ the first and third identities of Theorem 5.1. The fact that the degree of $C(1, 1)$ is 24 follows either by direct elimination, or by using the connection (3.8) between

$I(1) = R(1)^2$ and $C(1, 1)$ and the polynomial equation (of degree 12) over $\mathbb{Q}(t, x)$ that we have established for $I(x)$ in the proof of Theorem 5.1.

The asymptotic result is obtained through singularity analysis, using the expressions of $A_0, A_1,$ and $A_2,$ and the singular properties of V and W (see (4.11), (4.13), (4.18), and the proof of Corollary 1.6 at the end of Section 4). ■

6 Double Kreweras steps

In this section, we take $\mathcal{S} = \{\rightarrow, \uparrow, \swarrow, \nearrow, \leftarrow, \downarrow\}$, so that $\mathcal{S} = \mathcal{S}$.

6.1 Statements of the results

In order to state the counterparts of Theorems 1.1 and 1.2, we first need to introduce an extension of $\mathbb{Q}(t)$ of degree 16, illustrated in Figure 3.

First, we define $M \equiv M(t)$ as the unique formal power series in t satisfying

$$M = t(1 + 2M + 4M^2).$$

The coefficient of t^{n+1} in M is 2^n times the n th Motzkin number, hence the notation M . The series M/t is known to count walks with steps in \mathcal{S} confined to the North-West quadrant $\{(i, j) : i \leq 0, j \geq 0\}$ (see [10, Proposition 10]). Note that $\mathbb{Q}(t, M) = \mathbb{Q}(M)$. Then we consider an extension of degree 4 of $\mathbb{Q}(M)$, generated by the series $P_1 = t + \mathcal{O}(t^2)$ satisfying

$$P_1 = \frac{M(1+M)^3}{(1+4M)^3} \cdot \frac{(1+P_1)^4}{(1-P_1)^2}.$$

Finally, we define

$$A_1 = 4(1+M)\sqrt{\frac{(1+M)P_1}{M}} = \frac{4(1+M)^3}{(1+4M)^{3/2}} \cdot \frac{(1+P_1)^2}{1-P_1},$$

which has degree 8 above $\mathbb{Q}(M)$ and degree 16 above $\mathbb{Q}(t)$. Between $\mathbb{Q}(M)$ and $\mathbb{Q}(M, A_1) = \mathbb{Q}(t, A_1)$, there are three extensions of $\mathbb{Q}(M)$ of degree 2, respectively, generated by $\sqrt{1+4M}, \sqrt{1-4M^2},$ and $\sqrt{(1+4M)(1-4M^2)}$. We shall consider, in particular, a generator of $\mathbb{Q}(\sqrt{1+4M})$, denoted by N , defined by $N = \mathcal{O}(t)$ and

$$(6.1) \quad N = t \frac{N^4 - 2N^3 + 6N^2 - 2N + 1}{(1-N)^2}, \quad \text{or equivalently} \quad N = M(1-N)^2.$$

Figure 3 gives a complete description of $\mathbb{Q}(t, A_1)$ and its subfields. The series denoted by P_2 is the only solution of

$$P_2^4 - (1+4M)^3 P_2^2 + 4M(1+M)^3(1+4M)^3 = 0,$$

satisfying $P_2 = 1 + \mathcal{O}(t)$. It is related to $M, P_1,$ and A_1 by

$$(6.2) \quad P_2 = (1+4M)^{3/2} \frac{1-P_1}{1+P_1} = \frac{MA_1}{4} + 4 \frac{(1+M)^3}{A_1}.$$

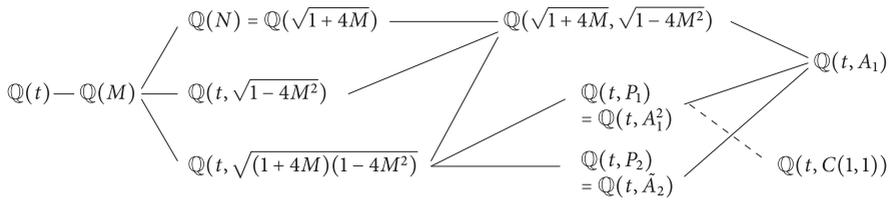


Figure 3: The extension of $\mathbb{Q}(t)$ of degree 16 generated by A_1 . All elementary extensions have degree 2. The series $C(1,1)$ has degree 16 over $\mathbb{Q}(t)$, as A_1 , but lies in a different extension of $\mathbb{Q}(t)$.

Theorem 6.1 *The generating function $C(x, y)$ of walks with steps in $\{\rightarrow, \uparrow, \swarrow, \nearrow, \leftarrow, \downarrow\}$ starting from $(0, 0)$ and avoiding the negative quadrant is algebraic of degree 256. It is given by the following equation:*

$$(1 - t(x + y + xy + \bar{x}\bar{y} + \bar{x} + \bar{y}))C(x, y) = 1 - t\bar{y}(1 + \bar{x})C_-(\bar{x}) - t\bar{x}(1 + \bar{y})C_-(\bar{y}) - t\bar{x}\bar{y}C_{0,0},$$

where $C_{0,0}$ is algebraic of degree 16 and $C_-(x)$ is algebraic of degree 64. These series can be expressed in terms of the series M, N , and A_1 defined above. Indeed,

$$(6.3) \quad tC_{0,0} = 1 + \frac{(1 + 2M)^2}{2M} - \frac{3A_1}{8} - \frac{2(1 + M)^3}{MA_1},$$

and

$$(6.4) \quad \left(2t(1 + \bar{x})C_-(x) + \bar{x} + 1 + x - \frac{1 + 2t}{t(1 + x)} + tC_{0,0} \right)^2 = \left(\frac{1}{x} - x - \frac{1 + 2t}{t(x + 1)} \right)^2 + \tilde{A}_2 \left(\frac{1}{x} - x - \frac{1 + 2t}{t(x + 1)} \right) + A_1 \frac{N + 2xN/(1 - N) - x^2}{2x(1 + x)N} \sqrt{\Delta_+(x)} + \tilde{A}_0,$$

where $\Delta_+(x)$ is the following polynomial in x :

$$(6.5) \quad \Delta_+(x) = 1 - \frac{2N(1 + N^2)}{(1 - N)^2}x + N^2x^2,$$

and \tilde{A}_2 and \tilde{A}_0 are algebraic series of respective degree 8 and 16, both in $\mathbb{Q}(t, A_1)$:

$$\tilde{A}_2 = \frac{(1 + 2M)^2}{M} - \frac{A_1}{4} - \frac{4(1 + M)^3}{MA_1} = \frac{(1 + 2M)^2}{M} - \frac{P_2}{M},$$

$$\tilde{A}_0 = \frac{A_1^2}{8} + \left(\frac{A_1}{8M} + 2 \frac{(M + 1)^3}{A_1 M^2} \right) \left((1 + 4M)^{3/2} - (1 + 2M)^2 \right) + \frac{2M^3 - 14M^2 - 12M - 3}{M}.$$

The generating function $D(x)$ of walks ending on the diagonal is algebraic of degree 64, given by

$$(6.6) \quad \Delta(x) \left(xD(x) + \frac{1}{t(1+x)} \right)^2 = \left(\frac{1}{x} - x - \frac{1+2t}{t(x+1)} \right)^2 + \tilde{A}_2 \left(\frac{1}{x} - x - \frac{1+2t}{t(x+1)} \right) - A_1 \frac{N + 2xN/(1-N) - x^2}{2x(1+x)N} \sqrt{\Delta_+(x)} + \tilde{A}_0,$$

with the above values of $\Delta_+(x)$, \tilde{A}_0 , A_1 , \tilde{A}_2 , and

$$(6.7) \quad \Delta(x) = (1 - t(x + \bar{x}))^2 - 4t^2\bar{x}(1+x)^2.$$

For walks ending at a prescribed position, we obtain the following result.

Corollary 6.2 *Let us define A_1 as above. For any $(i, j) \in \mathbb{C}$, the generating function of walks avoiding the negative quadrant and ending at (i, j) is algebraic of degree (at most) 16 and belongs to $\mathbb{Q}(t, A_1)$.*

The first example is provided by the expression of $C_{0,0}$ in (6.3). A simpler one is

$$(6.8) \quad tC_{-1,0} = \frac{A_1}{4} - 1.$$

Now, for the generating function $C(1,1)$ that counts all walks avoiding the negative quadrant, it follows from Theorem 6.1 that

$$(1 - 6t)^2 \left(C(1,1) + \frac{1}{2t} \right)^2 = \left(\frac{1+2t}{2t} \right)^2 - \tilde{A}_2 \frac{1+2t}{2t} + A_1 \frac{N + 2N/(1-N) - 1}{4N} \sqrt{\Delta_+(1)} + \tilde{A}_0.$$

Upon noticing that $\Delta_+(1) = (1 - N)^2(1 - 4M^2)$, one sees that the right-hand side belongs to $\mathbb{Q}(t, A_1)$, and could thus be expected to have degree 16 (which would give an expected degree 32 for $C(1,1)$). However, the right-hand side belongs to $\mathbb{Q}(t, A_1^2)$ and has degree 8 only, as made explicit in the following corollary. Details of the derivation are available in our Maple session.

Corollary 6.3 *The generating function $C(1,1)$ counting by their length all walks with steps in \mathcal{S} avoiding the negative quadrant is algebraic of degree 16 over $\mathbb{Q}(t)$, and of degree 2 over $\mathbb{Q}(t, A_1^2)$. More precisely,*

$$(6.9) \quad (1 - 6t)^2 \left(C(1,1) + \frac{1}{2t} \right)^2 = \frac{MA_1^4}{512(M+1)^3} + \frac{(10M^4 - 34M^3 - 18M^2 - 2M - 1)A_1^2}{32M(M+1)^3} - \frac{14M^4 - 22M^3 - 6M^2 + 2M - 1}{4M^2}.$$

It has radius $1/6$, with a unique dominant singularity at $1/6$. As t approaches this point,

$$C(1,1) \sim \kappa(1 - 6t)^{-5/8},$$

with

$$\kappa = 2^{1/4} 3^{9/8} (\sqrt{2} - 1).$$

Hence, the number of walks of length n avoiding the negative quadrant satisfies

$$c_n \sim \frac{\kappa}{\Gamma(5/8)} 6^n n^{-3/8}.$$

Harmonic function

We have also derived the counterpart of Corollary 1.4, that is, a harmonic function on \mathbb{C} associated with the double Kreweras step set $\mathcal{S} = \mathcal{S}^* = \{\rightarrow, \nearrow, \uparrow, \leftarrow, \swarrow, \downarrow\}$. We still have an asymptotic behavior of the form (1.7), with the factor 3^n replaced of course by 6^n , and without any periodicity condition. We do not give here all details of the numbers $H_{i,j}$, but simply the values of

$$\mathcal{H}_-(x) := \sum_{i>0} H_{-i,0} x^i = \frac{3^{7/4} x \sqrt{\sqrt{2}-1}}{\sqrt{2}(1+x)} \left(\sqrt{\sqrt{2} + \frac{2-\sqrt{3}+x}{1-x}} \sqrt{\frac{7+4\sqrt{3}-x}{1-x}} - \sqrt{\sqrt{2}-1} \right)$$

and

$$\mathcal{H}_d(y) := \sum_{i \geq 0} H_{i,i} y^i = \frac{3^{7/4} \sqrt{2} \sqrt{\sqrt{2}-1}}{(1-y)\sqrt{1-14y+y^2}} \sqrt{\sqrt{2} - \frac{2-\sqrt{3}+y}{1-y}} \sqrt{\frac{7+4\sqrt{3}-y}{1-y}}.$$

The proof is analogous to Kreweras' case, and in fact a bit easier since the model \mathcal{S} is aperiodic. Once again, details of the calculations can be found in our Maple session. The number of quadrant \mathcal{S} -walks of length n going from $(0,0)$ to (i,j) in the first quadrant is now asymptotic to $h_{i,j} 6^n n^{-5/2} / \Gamma(-3/2)$, with

$$\sum_{i \geq 0} h_{i,0} x^i = \frac{3}{2(1+x)} \left(\frac{2-\sqrt{3}+x}{1-x} \sqrt{\frac{7+4\sqrt{3}-x}{1-x}} + 1 \right)$$

(and in this case uniqueness of the positive harmonic function is proved [7]).

6.2 Proofs for double Kreweras steps

Proof of Theorem 6.1 *Proof.* The first equation of the theorem is of course the basic functional equation (2.4). The functional equations (2.8) and (2.12) defining $D(y)$ and $U(x,y)$ read

$$(6.10) \quad (1 - t(y + \bar{y}))D(y) = 1 - t\bar{y}D_0 + 2t(1 + \bar{y})U(0, y) - 2t\bar{y}U_{0,0},$$

$$2xy(1 - t(\bar{x}\bar{y} + x + y + \bar{x} + \bar{y} + xy))U(x, y) = y + y(t(y + \bar{y}) + 2tx(1 + y) - 1)D(y) - 2t(1 + x)U(x, 0) - tD_0.$$

The invariants of Proposition 3.3 are

$$I(x) = \left(2t(1+x)U(x,0) + \frac{1}{x} + 1 + x - \frac{1+2t}{t(1+x)} + tD_0 \right)^2,$$

$$J(y) = \Delta(y) \left(yD(y) + \frac{1}{t(1+y)} \right)^2,$$

with Δ defined by (6.7). Note that they have now poles at 0 and at -1 . The rational invariants I_0, J_0 are given by Table 3:

$$I_0(x) = \bar{x} - x - \frac{1+2t}{t(1+x)}, \quad J_0(y) = I_0(y).$$

They satisfy

$$(6.11) \quad I_0(x) - J_0(y) = \frac{x-y}{t(1+x)(1+y)} \mathcal{K}(x, y).$$

The invariants (2.24) related to quadrant walks with steps in \mathcal{S} are

$$I_1(x) = t(1+x)\mathcal{Q}(x,0) - \frac{x-t-tx^2}{t(1+x)}, \quad J_1(y) = -t(1+y)\mathcal{Q}(0,y) + t\mathcal{Q}_{0,0} - \bar{y}.$$

They satisfy

$$(6.12) \quad I_1(x) - J_1(y) = -\mathcal{K}(x, y) \left(xy\mathcal{Q}(x, y) + \frac{x}{t(1+x)} \right).$$

Applying the invariant lemma

Again, we want to combine these three pairs of invariants to form a pair satisfying the condition of Lemma 2.6. As in the previous section, we look for a polynomial combination of $I(x), I_0(x)$, and $I_1(x)$ that, at least, has no pole at $x = 0$ nor at $x = -1$. We first eliminate poles at 0 using $I_0(x)$, and then poles at -1 using $I_1(x)$. We are thus led to consider

$$\tilde{I}(x) := I(x) - I_0(x)^2 - A_2I_0(x) - A_1I_1(x),$$

with

$$(6.13) \quad A_2 = 2(1 + tD_0 + 2tU_{0,0}), \quad A_1 = 4(1 + tU_{0,0}),$$

which indeed has no pole at $x = 0$ nor at $x = -1$. Let us define accordingly

$$\tilde{J}(y) = J(y) - J_0(y)^2 - A_2J_0(y) - A_1J_1(y),$$

and examine whether the ratio $(\tilde{I}(x) - \tilde{J}(y))/\mathcal{K}(x, y)$ is a multiple of xy . We proceed as in the previous section. Denoting by Rat, Rat_0 , and Rat_1 the right-hand sides of (3.7), (6.11), and (6.12), the expression (5.16) of $(\tilde{I}(x) - \tilde{J}(y))/\mathcal{K}(x, y)$ still holds (because $\tilde{I}(x)$ and $\tilde{J}(y)$ have the same form for both models). In order to see that this ratio is indeed a multiple of x , we need to inject (6.10). Proving that it is a multiple of y requires no additional information. We conclude that the pair $(\tilde{I}(x), \tilde{J}(x))$ is constant, equal to (A_0, A_0) for some series A_0 that depends on t only.

Recall that $I_1(x)$ is expressed in terms of the generating function $\mathcal{Q}(x, 0)$ of quadrant walks with double Kreweras steps. This series is known, and we obtain from Proposition 15 in [10]

$$I_1(x) = -\frac{1}{2}I_0(x) + I_2(x) - \frac{2N^4 + N^3 + 3N^2 - N + 1}{2N(1-N)^2},$$

where N is defined by (6.1) (this series is denoted by Z in [10]) and

$$I_2(x) := \frac{N + 2xN/(1-N) - x^2}{2x(1+x)N} \sqrt{\Delta_+(x)},$$

with $\Delta_+(x)$ given by (6.5). In what follows, we use $I_2(x)$ rather than $I_1(x)$. That is, we rewrite the identity $\tilde{I}(x) = A_0$, which reads

$$I(x) = I_0(x)^2 + A_2I_0(x) + A_1I_1(x) + A_0$$

as

$$(6.14) \quad I(x) = I_0(x)^2 + \tilde{A}_2I_0(x) + A_1I_2(x) + \tilde{A}_0,$$

where the two new series \tilde{A}_2 and \tilde{A}_0 are

$$(6.15) \quad \begin{aligned} \tilde{A}_2 &= A_2 - A_1/2 = 2t(D_0 + U_{0,0}), \\ \tilde{A}_0 &= A_0 - A_1 \frac{2N^4 + N^3 + 3N^2 - N + 1}{2N(1-N)^2}. \end{aligned}$$

Similarly, we find that the quadrant invariant $J_1(y)$ is given by

$$J_1(y) = -\frac{1}{2}J_0(y) - I_2(y) - \frac{2N^4 + N^3 + 3N^2 - N + 1}{2N(1-N)^2},$$

and write

$$(6.16) \quad J(y) = J_0(y)^2 + \tilde{A}_2J_0(y) - A_1I_2(y) + \tilde{A}_0,$$

with the above values of \tilde{A}_2 and \tilde{A}_0 .

This gives the expressions (6.4) and (6.6) announced in Theorem 6.1, which are those of $I(x)$ and $J(x)$. However, we still need to determine the three series \tilde{A}_0, A_1 , and \tilde{A}_2 .

The series \tilde{A}_0, A_1 , and \tilde{A}_2

We cannot follow exactly the same approach as in the previous section, where we had written all invariants as polynomials in $I_1(x)$ using the identity (5.18) (it is still possible to write $I(x)$ as a rational function of $I_1(x)$; see Section 9.1). Indeed, the counterpart of this identity relates now I_0 and I_2 , and is no longer linear in I_0 :

$$I_2(x)^2 = \frac{1}{4}I_0(x)^2 + c_1I_0(x) + c_0,$$

with

$$c_1 = \frac{1 + N + N^2 - N^3}{2N(1 - N)^2} \quad \text{and} \quad c_0 = \frac{(N^2 + 1)(4N^4 - 9N^3 + 13N^2 - N + 1)}{4N^2(1 - N)^3}.$$

However, compare the expressions (6.14) and (6.16) of $I(x)$ and $J(y)$, and recall that $J_0(y) = I_0(y)$. This, as well as the above expression of $I_2(x)^2$, suggests to form the product $I(x)J(x)$, which will be a polynomial in $I_0(x)$:

$$\begin{aligned} I(x)J(x) &= (I_0(x)^2 + \tilde{A}_2 I_0(x) + \tilde{A}_0)^2 - A_1^2 I_2(x)^2 \\ &= (I_0(x)^2 + \tilde{A}_2 I_0(x) + \tilde{A}_0)^2 - A_1^2 \left(\frac{1}{4} I_0(x)^2 + c_1 I_0(x) + c_0 \right) \\ &=: P_4(I_0(x)), \end{aligned}$$

where

$$P_4(u) = (u^2 + \tilde{A}_2 u + \tilde{A}_0)^2 - A_1^2 \left(\frac{1}{4} u^2 + c_1 u + c_0 \right).$$

We can now apply the same ideas as in the previous case. The product $I(x)J(x)$ vanishes:

- when x is one of the two roots of $\Delta(x)$, denoted by X_+ and X_- , that are finite at $t = 0$;
- when $x = X = t + 3t^3 + \mathcal{O}(t^4)$ is the (only) root of $I(x)$ lying in $\mathbb{Q}[[t]]$, that is,

$$X = t \frac{1 + X}{1 + 2t} (2t(1 + X)C_-(X) + 1 + X + X^2 + tXC_{0,0}).$$

The values $I_0(X_\pm)$ are found to be the two roots of

$$\begin{aligned} P_2(u) &= N(1 - N)^3 u^2 + 2N(1 - N)(N^3 + N^2 + N - 1)u \\ &\quad - (N^2 + 1)(N^4 - N^3 + 13N^2 - 9N + 4). \end{aligned}$$

(This polynomial should not be mixed up with the series P_2 of (6.2)!) They expand as

$$I_0(X_\pm) = \pm \frac{2}{\sqrt{t}} + 1 \pm \sqrt{t} + \mathcal{O}(t).$$

Now, the series $I_0(X) = -1 + \mathcal{O}(t)$ is also a root of $P_4(u)$, distinct from $I_0(X_\pm)$, and in fact is even a double root of P_4 (as before, one has to check that $I'_0(X) \neq 0$). Hence, we have the following factorization:

$$P_4(u) = \frac{1}{N(1 - N)^3} (u - I_0(X))^2 P_2(u).$$

We now equate the coefficients of u^3, \dots, u^0 in both sides of this identity. From the coefficients of u^3 , we obtain

$$I_0(X) = \frac{N^3 + N^2 + N - 1}{(1 - N)^2} - \tilde{A}_2.$$

We then replace every occurrence of $I_0(X)$ by this expression. The coefficients of u^2 give

$$(6.17) \quad \tilde{A}_0 = \frac{(N^3 + N^2 + N - 1)\tilde{A}_2}{(1 - N)^2} + \frac{A_1^2}{8} - \frac{N^7 + 4N^6 - 3N^5 + 12N^4 - 15N^3 + 10N^2 - 5N + 2}{N(1 - N)^4}.$$

We proceed with the coefficients of u^1 , after replacing every occurrence of \tilde{A}_0 by the above expression. This gives

$$(6.18) \quad \tilde{A}_2 = -2 \frac{(N^3 - N^2 - N - 1)(1 - N)^4 A_1^2 - 16(N^3 + N^2 + N - 1)(N^2 - N + 1)^3}{(1 - N)^2(A_1^2 N(1 - N)^4 + 16(N^2 - N + 1)^3)},$$

and we finally derive by comparing constant terms that

$$(6.19) \quad N^2(1 - N)^8 A_1^4 + 4N(N + 1)^3(1 - N)^7 A_1^3 + 32N(N^2 - N + 1)^3(1 - N)^4 A_1^2 - 64(1 - N)^3(N + 1)^3(N^2 - N + 1)^3 A_1 + 256(N^2 - N + 1)^6 = 0.$$

One can now check, using this equation and the first terms $A_1 = 4 + 4t^2 + \mathcal{O}(t^3)$, that A_1 is indeed the series described at the beginning of the section, and that the values of \tilde{A}_2 and \tilde{A}_0 given in the theorem in terms of the series M and A_1 coincide with those derived from (6.18) and (6.17).

It also follows from (6.13) and (6.15) that

$$tC_{0,0} = tD_0 = 1 - \frac{A_1}{4} + \frac{\tilde{A}_2}{2},$$

and the expression (6.3) can again be checked using (6.18) and (6.19).

Degrees

From the identities in this theorem, it follows that $C_-(x, 0)$ and $D(x)$ have degree at most $16 \times 4 = 64$ above $\mathbb{Q}(t, x)$. To prove that this bound is tight, we compute by elimination the minimal polynomials of $C_-(2, 0)$ and $D(2)$. We proceed similarly for the degree of $C(x, y)$, for which the natural upper bound is $16 \times 4 \times 4 = 256$. In this case, we specialize not only x and y , but also t , in such a way that $xyt < 1/6$, which guarantees the convergence of all series under consideration. ■

Remark 6.4 It may be interesting for some readers to know how one can derive from the polynomial equation (6.19) the description of the series A_1 , and of the subfields of $\mathbb{Q}(t, A_1)$, given at the beginning of the section (see Figure 3). This is fully documented in the Maple session that accompanies this paper, but here are a few details.

Recall that the series N arises from the earlier paper [10]. The first step is to investigate the subextensions of $\mathbb{Q}(t, N)$ and to discover the series M . This can be done, for instance, using the Subfields command.

Then, it is easy to prove, by elimination of N , that A_1 has degree 16 over $\mathbb{Q}(t)$. The idea of introducing the generator P_1 comes by looking at the denominator of \tilde{A}_2 in (6.18), and observing that the defining equation (6.19) of A_1 can also be written in the form:

$$(A_1^2 N(1 - N)^4 + 16(N^2 - N + 1)^3)^2 + 4(1 + N)^3(1 - N)^3 A_1 (A_1^2 N(1 - N)^4 - 16(N^2 - N + 1)^3) = 0,$$

or, in terms of M ,

$$(MA_1^2 + 16(1 + M)^3)^2 + 4A_1(1 + 4M)^{3/2} (MA_1^2 - 16(1 + M)^3) = 0.$$

Once P_1 is thus defined, one explores the subfields of $\mathbb{Q}(t, A_1)$ using the Subfields command. ■

We now consider number of walks ending at (i, j) .

Proof of Corollary 6.2 This model is aperiodic. The right-hand side of (6.4) is a (Laurent) series in x with coefficients in $\mathbb{Q}(t, A_1)$, with first term $1/x^2$. So its square root is a Laurent series in x , starting with $1/x$, with coefficients in $\mathbb{Q}(t, A_1)$ again. This proves the corollary for the coefficients of $U(x, 0)$, that is, for the series $C_{i,0}$ with $i < 0$. The value of $C_{-1,0}$ given in (6.8) comes directly from (6.13).

A similar argument, based on (6.6), proves the result for the series $C_{i,i}$. We then prove the corollary for the points $(i, i + 1)$, by induction on $i \geq 0$, by writing

$$C_{i,i} = \mathbb{1}_{i=0} + t(C_{i,i+1} + C_{i-1,i} + C_{i-1,i-1} + C_{i,i-1} + C_{i+1,i} + C_{i+1,i+1}) = \mathbb{1}_{i=0} + t(2C_{i,i+1} + 2C_{i-1,i} + C_{i-1,i-1} + C_{i+1,i+1}).$$

We finally prove it for all points (i, j) with $j \geq i$ by induction on $2j - i \geq 0$, using

$$C_{i,j-1} = t(C_{i,j-2} + C_{i,j} + C_{i-1,j-1} + C_{i+1,j-1} + C_{i-1,j-2} + C_{i+1,j}). \quad \blacksquare$$

7 A D-algebraic model

Let us now consider the 6th model of Table 1, with step polynomial $S(x, y) = x + \bar{x} + y + \bar{y} + xy$. The companion model is given by $\mathcal{S}(x, y) = x + \bar{x} + xy + \bar{x}\bar{y} + y$. In this case, there are no rational \mathcal{S} -invariants, but there exists a pair of quadrant invariants of the form (2.24) (see Table 4), namely $(I_1(x), J_1(y))$, with

$$(71) \quad I_1(x) = t\mathcal{Q}(x, 0) + \bar{x}, \quad J_1(y) = -t(1 + y)\mathcal{Q}(0, y) + t\mathcal{Q}_{0,0} + \frac{y(1 - ty)}{t(1 + y)}.$$

This pair satisfies

$$I_1(x) - J_1(y) = -\mathcal{K}(x, y) \left(xy\mathcal{Q}(x, y) + \frac{y}{t(1 + y)} \right).$$

The series $\mathcal{Q}(x, y)$ is known to be D-algebraic, and admits an explicit expression in terms of elliptic functions [2, Theorem 5.7]. Our approach relates $C(x, y)$ to this series.

7.1 Generating functions

Theorem 7.1 Let $I_1(x)$ and $J_1(y)$ be defined as above. The generating function $C(x, y)$ of walks with steps in $\{\rightarrow, \nearrow, \uparrow, \leftarrow, \downarrow\}$ starting from $(0, 0)$ and avoiding the negative quadrant is given by the following equation:

$$(1 - t(x + y + \bar{x} + \bar{y} + xy))C(x, y) = 1 - t\bar{y}C_-(\bar{x}) - t\bar{x}C_-(\bar{y}),$$

where $C_-(x)$ can be expressed in terms of I_1 and J_1 by

$$(7.2) \quad \left(t\bar{x}C_-(x) + x + \bar{x} - \frac{1}{2t} \right)^2 = \frac{(I_1(x) - A)^2 (I_1(x) - J_1(Y))}{I_1(x)}.$$

In the above expression, Y is the only root of

$$(7.3) \quad \Delta(y) := (1 - ty)^2 - 4t^2\bar{y}(1 + y)^2$$

that is a formal power series in t . Then,

$$A = \sqrt{\frac{1 - t^2\mathcal{Q}_{0,0} - t^2\mathcal{Q}_{0,1}}{tJ_1(Y)}}.$$

The generating function $D(y)$ of walks ending on the diagonal satisfies

$$(7.4) \quad \frac{\Delta(y)}{4} \left(yD(y) + \frac{1 - y}{t(1 + y)} \right)^2 = \frac{(J_1(y) - A)^2 (J_1(y) - J_1(Y))}{J_1(y)}.$$

In particular, $C_-(x)$, $D(y)$, and $C(x, y)$ are D -algebraic in all variables.

Our D -algebraicity result is not effective, and it seems hard to obtain an explicit differential equation in t for, say, the series $C(1, 1)$. This is already true in the quadrant case [2]. However, we indicate in Section 7.2 how one can write a differential equation in y for $D(y)$.

Proof The invariants of Proposition 3.3 are

$$(7.5) \quad I(x) = \left(2tU(x, 0) + 2x + 2\bar{x} - \frac{1}{t} \right)^2, \quad J(y) = \Delta(y) \left(yD(y) + \frac{1 - y}{t(1 + y)} \right)^2,$$

where $\Delta(y)$ is defined by (7.3). We want to combine them polynomially with the invariants $(I_1(x), J_1(y))$ given by (7.1) to form a pair of invariants $(\tilde{I}(x), \tilde{J}(y))$ to which Lemma 2.6 would apply, thus proving that $\tilde{I}(x) = \tilde{J}(y)$ is a series in t .

Observe that $J(y)$ has a pole at $y = 0$ (coming from $\Delta(y)$) and at $y = -1$. The modified invariant $\tilde{J}(y)$ should have no poles. Observing that $J_1(0) = 0$, we first form the product $J(y)J_1(y)$ to remove the pole of $J(y)$ at 0. The resulting series now has a triple pole at -1 , which can be removed by subtracting a cubic polynomial in $J_1(y)$. This leads us to introduce

$$\tilde{J}(y) = J(y)J_1(y) - 4J_1(y)^3 - A_2J_1(y)^2 - A_1J_1(y),$$

and to define accordingly

$$\tilde{I}(x) = I(x)I_1(x) - 4I_1(x)^3 - A_2I_1(x)^2 - A_1I_1(x),$$

for values of A_2 and A_1 that involve $D(-1), D'(-1)$ and various by-products of $\mathcal{Q}(x, y)$ (we refer to our Maple session for details). Then, we use the functional equations satisfied by $U(x, y), D(y)$ and $\mathcal{Q}(x, y)$ to check that the ratio

$$\frac{\tilde{I}(x) - \tilde{J}(y)}{\mathcal{K}(x, y)}$$

is a multiple of xy , and conclude that $\tilde{I}(x) = \tilde{J}(y) = A_0$, for some series $A_0 \in \mathbb{Q}((t))$. Hence, denoting $P(u) = 4u^3 + A_2u^2 + A_1u + A_0$, we have

$$(7.6) \quad I(x)I_1(x) = P(I_1(x)), \quad \text{and} \quad J(y)J_1(y) = P(J_1(y)).$$

Since the left-hand sides of (7.2) and (7.4) in Theorem 7.1 are $I(x)/4$ and $J(y)/4$, the proof will be complete if we can prove that

$$P(u) = 4(u - A)^2(u - J_1(Y)).$$

We will identify the roots of $P(u)$ by looking at the values of y that cancel $J(y)$, and thus $P(J_1(y))$. First, $\Delta(y)$ has a (unique) solution $Y = \mathcal{O}(t^2)$ in $\mathbb{Q}[[t]]$. Thus, $J_1(Y) = 4t + \mathcal{O}(t^3)$ must be a root of P . Then, another root of $J(y)$ arises from the factor of $J(y)$ involving $D(y)$. Indeed, the (unique) series $Y_0 = 1 + 2t + \mathcal{O}(t^2)$ such that $Y_0 = 1 + tY_0(1 + Y_0)D(Y_0)$ is a double root of $J(y)$, and thus $J_1(Y_0) = 1/(2t) + \mathcal{O}(1)$ is a double root of $P(u)$, unless $J'_1(Y_0) = 0$, which we readily check not to be the case. At this stage, we can write

$$P(u) = 4(u - J_1(Y_0))^2(u - J_1(Y)).$$

However, we would like to make $A := J_1(Y_0)$ more explicit. This can be done as follows: we return to the second identity of (7.6), replace $J(y)$ and $J_1(y)$ by their expressions in terms of $D(y)$ and $\mathcal{Q}(0, y)$, replace P by the above expression, and expand the result around $y = 0$. This gives

$$tJ_1(Y)J_1(Y_0)^2 = 1 - t^2\mathcal{Q}_{0,0} - t^2\mathcal{Q}_{0,1},$$

which concludes our derivation after choosing the correct (positive) sign for the square root.

The fact that all series under consideration are D-algebraic comes from the closure properties of D-algebraic series [2, Section 6.1], and from the fact that $\mathcal{Q}(x, y)$ is D-algebraic [2, Section 6.4]. ■

We can now specialize the first two equations of Theorem 7.1 to relate the series $C(1, 1)$ to the quadrant series \mathcal{Q} .

Corollary 7.2 *The generating function $C(1, 1)$ counting walks with steps in $\{\rightarrow, \nearrow, \uparrow, \leftarrow, \downarrow\}$ that start from $(0, 0)$ and avoid the negative quadrant is given by*

$$(7.7) \quad \frac{1}{4}(1 - 5t)^2 \left(C(1, 1) + \frac{1}{t} \right)^2 = \left(tC_{-}(1) + 2 - \frac{1}{2t} \right)^2 = \frac{(I_1(1) - A)^2(I_1(1) - J_1(Y))}{I_1(1)},$$

where $I_1(x)$ and $J_1(y)$ are given by (7.1), and Y and A are defined in Theorem 7.1; in particular, $I_1(1) = 1 + t\mathcal{Q}(1, 0)$.

7.2 An expression in terms of a weak invariant

Instead of using the quadrant-related invariants I_1 and J_1 of (7.1), one can use the analytic approach of [2, Section 5] and express $D(y)$ (say) in terms of an explicit *weak invariant* $w(y)$ (expressed itself in terms of elliptic functions). Let us give a few details, using the notation of [2], which we will not redefine. First, the model denoted by \mathcal{S} here is model #5 in [2, Table 5]. Then, for t a small positive real number, the series $D(y)$ can be defined analytically in the domain $\mathcal{G}_{\mathcal{L}}$, with finite limits on the boundary curve \mathcal{L} [31, Lemma 5]. This curve is bounded because \mathcal{S} contains a West step [2, Lemma 5.2]. The points 0 and -1 both belong to $\mathcal{G}_{\mathcal{L}}$, by [2, Lemmas 5.2 and 5.8]. Hence, the invariant $J(y)$ defined by (7.5), which is a simple variation on $D(y)$, is meromorphic in $\mathcal{G}_{\mathcal{L}}$, with finite limits on \mathcal{L} , and exactly two poles in $\mathcal{G}_{\mathcal{L}}$, at 0 and -1 . The first pole is simple, and the second is double. Thus, $J(y)$ is a *weak invariant* for the model \mathcal{S} , in the sense of [2, Definition 5.4]. We can apply the (analytic) invariant lemma [2, Lemma 5.6]: there exists a polynomial $P(u)$ (with complex coefficients that depend on the value t), of degree at most 3, such that

$$J(y) = \Delta(y) \left(yD(y) + \frac{1-y}{t(1+y)} \right)^2 = \frac{P(w(y))}{(w(y) - w(0))(w(y) - w(-1))^2}.$$

As before, we can obtain some information on the roots of $P(u)$ by examining the roots of $J(y)$. First, the formal power series $Y = \mathcal{O}(t^2)$ that cancels $\Delta(y)$ specializes to the value denoted by y_2 in [2, Lemma 5.1] (the value y_1 is zero for this model). This is precisely the (unique) pole of $w(y)$ in $\mathcal{G}_{\mathcal{L}}$, by [2, Proposition 5.5], and we conclude that $P(u)$ has degree 2 at most. The series $Y_0 = 1 + 2t + \mathcal{O}(t^3)$ that cancels the factor of $J(y)$ involving $D(y)$ can also be shown to specialize to a value y_0 of $\mathcal{G}_{\mathcal{L}}$ (because the positive point where the curve \mathcal{L} intersects the real axis, denoted by $Y(x_2)$ in [2], is $1/\sqrt{t}$ at first order, so that y_0 is smaller), so that $P(u)$ admits $w(y_0)$ as a double root. Finally,

$$(7.8) \quad J(y) = \Delta(y) \left(yD(y) + \frac{1-y}{t(1+y)} \right)^2 = \frac{\alpha(w(y) - a)^2}{(w(y) - w(0))(w(y) - w(-1))^2}$$

with $a = w(y_0)$. One can determine α and a in terms of $w(0)$, $w(-1)$, $w'(0)$, and $w'(-1)$ by expanding this identity at first order around $y = 0$ and $y = -1$.

Alternatively, the above identity can be derived by combining the expression of $J(y)/4$ given in (7.4) in terms of $J_1(y)$ (or equivalently, in terms of $\mathcal{Q}(0, y)$) with the following expression, borrowed from [2, Theorem 5.7]:

$$(7.9) \quad y(1+t)\mathcal{Q}(0, y) = -y - \frac{1+t}{t(1+y)} - 1 + \frac{1+t}{t} \left(\frac{w'(-1)}{w(y) - w(-1)} + \frac{w''(-1)}{2w'(-1)} \right).$$

Indeed, the expression (7.1) of $J_1(y)$ then yields

$$J_1(y) = \frac{1+t}{t} \cdot \frac{w'(-1)}{w(0) - w(-1)} \cdot \frac{w(y) - w(0)}{w(y) - w(-1)},$$

and, since $Y = y_2$ is a pole of $w(y)$,

$$J_1(Y) = \frac{1+t}{t} \cdot \frac{w'(-1)}{w(0) - w(-1)}.$$

We plug the latter two expressions in (7.4), and thus obtain the form (7.8), with values of a and α expressed in terms of $w(0)$, $w(-1)$, $w'(-1)$, and A . Another by-product of this approach is an expression for the other series involved in the expression (7.4) of $J(y)/4$:

$$A^2 = \frac{w'(0)}{w(0) - w(-1)}.$$

Remarks 1. In [2, Section 6.5], we worked out an explicit differential equation in y for $\mathcal{Q}(0, y)$, of order 4, with polynomial coefficients in t and y , by combining the expression (7.9) and a first-order differential equation satisfied by $w(y)$. The same procedure could be applied to (7.8) to write down a differential equation, still of fourth order, for $J(y)$ or $D(y)$.

2. It is interesting to note that, while quadrant generating functions, when they can be expressed in terms of $w(y)$, are in fact homographic functions of $w(y)$, the above expression of $J(y)$ has degree 3 in $w(y)$. Moreover, the three-quadrant generating function $D(y)$ is no longer rational, but algebraic in $w(y)$. This is another sign of the higher difficulty of three-quadrant problems.

7.3 Harmonic functions

As already sketched in Remark 4.1 in the Kreweras case, we can relate the asymptotic behaviors of \mathcal{S} -walks in \mathbb{C} and \mathcal{S} -walks in \mathbb{Q} , under highly plausible assumptions.

Let us begin with quadrant \mathcal{S} -walks. The number $\tilde{q}_{i,j}(n)$ of \mathcal{S} -walks of length n in the quadrant \mathbb{Q} ending at (i, j) is known to have the following asymptotic form [6, 14]:

$$\tilde{q}_{i,j}(n) \sim \frac{h_{i,j}}{\Gamma(-\alpha)} \mu^n n^{-1-\alpha},$$

where $\mu \simeq 4.729$ is the positive solution of $\mu^3 + \mu^2 - 18\mu - 43 = 0$, and $\alpha = \pi / \arccos(-c) \simeq 1.39$, where $c \simeq 0.626$ satisfies $64c^6 - 64c^4 + 28c^2 - 5 = 0$. The numbers $h_{i,j}$ satisfy a $1/\mu$ -harmonic relation

$$h_{i,j} = \frac{1}{\mu} (h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i-1,j-1} + h_{i+1,j+1}),$$

from which one derives the functional equation

$$(7.10) \quad (1 + y + xy^2 + x^2y + x^2y^2 - \mu xy) \mathcal{H}(x, y) = \mathcal{H}(x, 0) + (1 + y)\mathcal{H}(0, y) - \mathcal{H}(0, 0),$$

where $\mathcal{H}(x, y) = \sum_{i,j \geq 0} h_{i,j} x^i y^j$. As discussed in [30, Section 6], no simple expression is known for $\mathcal{H}(x, y)$ (the results of [30] only hold for walks with no drift; the same is true for the results of [32] on harmonic functions in \mathbb{C}).

The number $c_{i,j}(n)$ of \mathcal{S} -walks of length n in the three-quadrant cone \mathcal{C} ending at (i, j) is widely believed to have the following asymptotic form:

$$c_{i,j}(n) \sim -\frac{H_{i,j}}{\Gamma(-\alpha/2)} \mu^n n^{-1-\alpha/2},$$

with μ and α as above. More precisely, what is known is a pair of lower and upper bounds on $c_{i,j}(n)$ that have the above form and differ simply by a multiplicative constant [28]. The minus sign has been chosen so that $H_{i,j} > 0$.

If this holds, then

$$H_{i,j} = H_{j,i} = \frac{1}{\mu} (H_{i-1,j} + H_{i+1,j} + H_{i,j-1} + H_{i,j+1} + H_{i-1,j-1}),$$

which yields for the generating function $\mathcal{H}(x, y) = \sum_{j \geq 0, i \geq j} H_{i,j} x^i y^j$ the functional equation

(7.11)

$$(1 + y + xy^2 + x^2y + x^2y^2 - \mu xy) \mathcal{H}(x, y) = \mathcal{H}_-(x) + \frac{1}{2}(2 + 2y + xy^2 - \mu xy) \mathcal{H}_d(y),$$

where as before $\mathcal{H}_-(x) = \sum_{i > 0} H_{-i,0} x^i$ and $\mathcal{H}_d(y) = \mathcal{H}(0, y)$.

We can now adapt the argument of Remark 4.1. The term $c(x, y) = (2 + 2y + xy^2 - \mu xy)/2$ satisfies

$$\begin{aligned} c(x, y)^2 &= \frac{\delta(y)}{4} x^2 - \mu xy(1 + y) \mathcal{H}(x, y; 1/\mu) \\ &= \frac{\delta(y)}{4} x^2 + (1 + y) (1 + y + xy^2 + x^2y + x^2y^2 - \mu xy), \end{aligned}$$

where $\delta(y) = (\mu^2 y - 2\mu y^2 + y^3 - 4y^2 - 8y - 4)y$ is the discriminant of $\mu xy \mathcal{H}(x, y; 1/\mu)$ with respect to x . Upon multiplying (7.11) by $\bar{x}^2(\mathcal{H}_-(x) - c(x, y)\mathcal{H}_d(y))$, we conclude that the series

$$\overline{\mathcal{H}}(x, y) := \bar{x}^2 \mathcal{H}(x, y) (\mathcal{H}_-(x) - c(x, y)\mathcal{H}_d(y)) + \bar{x}^2 (1 + y) \mathcal{H}_d(y)^2$$

is a formal power series in x and y , which satisfies the same equation (7.10) as $\mathcal{H}(x, y)$, as well as

$$\overline{\mathcal{H}}(x, 0) = \bar{x}^2 \mathcal{H}_-(x)^2, \quad \text{and} \quad \overline{\mathcal{H}}(0, y) = -\frac{\delta(y)}{4(1+y)} \mathcal{H}(0, y)^2 + \frac{H_{-1,0}^2}{1+y}.$$

In particular, $\overline{\mathcal{H}}(0, 0) = H_{-1,0}^2$. Let us now assume that:

- $\overline{\mathcal{H}}(x, y)$ is uniquely determined (up to a multiplicative factor) by equation (7.10) and the positivity of its coefficients;
- the series $\overline{\mathcal{H}}(x, y)$ has positive coefficients.

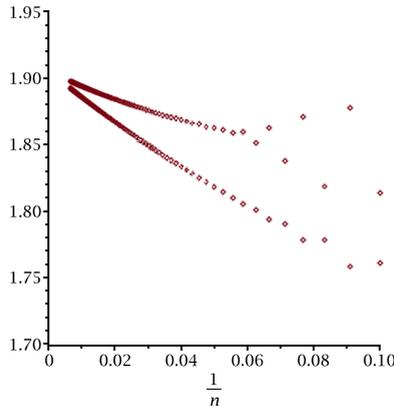


Figure 4: Plots of the sequences (7.13) against $1/n$, for $n \leq 150$.

Then there exists a positive constant κ^2 such that $\overline{\mathcal{H}}(x, y) = \kappa^2 \mathcal{H}(x, y)$, which implies in particular that

(7.12)

$$\mathcal{H}_-(x) = \kappa x \sqrt{\mathcal{H}(x, 0)} \quad \text{and} \quad \mathcal{H}_d(y) = 2\kappa \sqrt{-\frac{(1+y)\mathcal{H}(0, y) - \mathcal{H}(0, 0)}{\delta(y)}}.$$

Expanding these predictions at $x = 0$ and $y = 0$ gives

$$\kappa = \frac{H_{-1,0}}{\sqrt{h_{0,0}}} = \frac{H_{0,0}}{\sqrt{h_{0,0} + h_{0,1}}},$$

so that we expect

$$\frac{H_{0,0}}{H_{-1,0}} = \sqrt{1 + \frac{h_{0,1}}{h_{0,0}}}.$$

This seems to hold, as shown by Figure 4, where we have plotted against $1/n$ the sequences

(7.13)

$$\frac{c_{0,0}(n)}{c_{-1,0}(n)} \quad \text{and} \quad \sqrt{1 + \frac{\tilde{q}_{0,1}(n)}{\tilde{q}_{0,0}(n)}}.$$

We have checked numerically more consequences of the prediction (7.12), like

$$\frac{H_{-1,0}}{H_{-2,0}} = 2 \frac{h_{0,0}}{h_{1,0}} \quad \text{and} \quad \frac{H_{1,1}}{H_{0,0}} = \frac{\mu^2}{8} - 1 + \frac{1}{2} \cdot \frac{h_{0,1} + h_{0,2}}{h_{0,0} + h_{0,1}},$$

each time with perfect agreement.

8 The simple and diagonal models

In this section, we explain how the use of invariants also give new solutions of three-quadrant walks when $\mathcal{S}_1 = \{\rightarrow, \uparrow, \leftarrow, \downarrow\}$ (simple model) and $\mathcal{S}_2 = \{\nearrow, \nwarrow, \swarrow, \searrow\}$ (diagonal model). This may seem unexpected, for the following reason.

For both models, the basic functional equation of Lemma 2.1 holds (with a slightly different definition of U and D for the model \mathcal{S}_2); but by Proposition 3.1, there is no way to decouple y in the desired form. Moreover, if we could still construct a pair of invariants involving $U(x, 0)$, and relate it polynomially to the invariants $(I_1(x), J_1(y))$ that involve quadrant generating functions with steps in \mathcal{S}_1 or \mathcal{S}_2 (the so-called Gessel model for \mathcal{S}_1 , or its reflection in the main diagonal for \mathcal{S}_2), then $U(x, 0)$ would be algebraic, because quadrant walks with steps in \mathcal{S}_1 (or \mathcal{S}_2) are algebraic. However, it is known that $U(x, 0)$ is transcendental for both models [9].

However, for both models ($\mathcal{S} = \mathcal{S}_1$ or \mathcal{S}_2), it is natural [9, 12] to introduce the series $A(x, y)$ defined by

$$(8.1) \quad A(x, y) = C(x, y) - \frac{1}{3} (Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y})),$$

where $Q(x, y)$ counts quadrant \mathcal{S} -walks. The series $A(x, y)$ is easily shown to satisfy the same functional equation (2.4) as $C(x, y)$, but with the initial term 1 replaced by $(2 + \bar{x}^2 + \bar{y}^2)/3$. And then the heart of [9] is to prove that $A(x, y)$ is algebraic, for both models.

Hence, a natural idea is to try to apply the tools developed in this paper to $A(x, y)$ rather than $C(x, y)$. We shall see that the new series $U(x, y)$, defined from $A(x, y)$ rather than $C(x, y)$, satisfies an equation similar to the one of Lemma 2.1, but with an initial term $y(1 + x^e)$ instead of y (the exponent e is 2 for the simple model, and 1 for the diagonal model). And now *this term decouples*, whereas y does not. This yields new \mathcal{S} -invariants, involving $A(x, y)$, and ultimately a relation between $A(x, y)$ and the generating function of quadrant walks with Gessel's steps.

We finally derive from this the harmonic functions $H_{i,j}$ for the simple and diagonal models in the three-quadrant plane. The generating functions $\sum_{(i,j) \in \mathbb{C}} H_{i,j} x^i y^j$ are algebraic in both cases, even though the generating functions $C(x, y)$ are D-finite but transcendental. We do not work out more exact results (for instance, the degree of $A(x, y)$ or of the series $A_{i,j}$), as such results already appear in [9].

8.1 The simple model

When $\mathcal{S} = \{\rightarrow, \uparrow, \leftarrow, \downarrow\}$, the series $A(x, y)$ defined by (8.1) satisfies

$$(8.2) \quad (1 - t(x + \bar{x} + y + \bar{y}))A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y}),$$

where the series $A_-(\bar{x})$ is defined in a similar fashion as $C_-(\bar{x})$ (see (2.3)). The series $A(x, y)$ has a simple combinatorial interpretation: it counts weighted walks in the three-quadrant cone \mathbb{C} , starting either from $(0, 0)$, or from $(-2, 0)$, or from $(0, -2)$, with a weight $2/3$ in the first case, and $1/3$ in the other two. We now define two series $U(x, y)$ and $D(y)$ by

$$(8.3) \quad A(x, y) = \bar{x}U(\bar{x}, xy) + D(xy) + \bar{y}U(\bar{y}, xy).$$

Note that they do *not* have the same combinatorial meaning as in the previously solved four models. We can reproduce the step-by-step arguments that led to Lemma 2.1, and we thus obtain

$$(8.4) \quad 2xy\mathcal{K}(x, y)U(x, y) = \frac{2}{3}y(1+x^2) + y(2tx(1+y) - 1)D(y) - 2tU(x, 0),$$

where $\mathcal{K}(x, y) = 1 - t\mathcal{S}(x, y) = 1 - t(x + \bar{x} + xy + \bar{x}\bar{y})$ is the step polynomial of Gessel’s model. Compared with the original functional equation (2.12), the only thing that has changed is the initial term y , which has become $2y(1+x^2)/3$.

For the model \mathcal{S} , we know two pairs of invariants (see Table 3): one is rational, and will not be used; the other takes the form (2.24) and involves quadrant generating functions:

$$(8.5) \quad I_1(x) = t\mathcal{Q}(x, 0) + \bar{x}, \quad J_1(y) = -t(1+y)\mathcal{Q}(0, y) + t\mathcal{Q}_{0,0} + \frac{y}{t(1+y)}.$$

These two series are algebraic, and we refer to [5, 8] for explicit expressions. They satisfy

$$I_1(x) - J_1(y) = -y\mathcal{K}(x, y) \left(x\mathcal{Q}(x, y) + \frac{1}{t(1+y)} \right).$$

We will obtain expressions for $U(x, 0)$ and $D(y)$ in terms of $I_1(x)$ and $J_1(y)$, which are reminiscent of those of the previous section (Theorem 7.1). Given that a polynomial identity relates $(I_0(x), J_0(y))$ and $(I_1(x), J_1(y))$, of degree 3 in the latter pair, we could also decrease to 2 the degree in $I_1(x)$ and $J_1(y)$ of our expressions, upon introducing $I_0(x)$ and $J_0(y)$.

Theorem 8.1 *Let $I_1(x)$ and $J_1(y)$ be defined as above. The generating function $A(x, y)$ defined by (8.1) satisfies equation (8.2), where $A_-(x)$ can be expressed in terms of I_1 and J_1 by*

$$(8.6) \quad \left(3t\bar{x}A_-(x) + 1 + \bar{x}^2 - \frac{\bar{x}}{t} \right)^2 = I_1(x) (I_1(x) - B)^2 (I_1(x) - J_1(Y)).$$

In the above expression, Y is the only root of

$$(8.7) \quad \Delta(y) = 1 - 4t^2\bar{y}(1+y)^2$$

that is a formal power series in t , and

$$B = \frac{1}{t} + 2t\mathcal{Q}_{0,0} - \frac{J_1(Y)}{2}.$$

The generating function $D(y)$ defined by (8.3) satisfies

$$(8.8) \quad \frac{9}{4}\Delta(y) \left(yD(y) + \frac{2y}{3t^2(1+y)^2} \right)^2 = J_1(y) (J_1(y) - B)^2 (J_1(y) - J_1(Y)).$$

In particular, the series $A_-(x)$, $D(y)$, and $A(x, y)$ are algebraic.

Remark 8.2 **Degrees and rational parametrizations of $A_-(x)$ and $D(y)$.** In [8], the series $\mathcal{Q}(xt, 0)$ and $\mathcal{Q}(0, y)$ (involved in the expressions of $I_1(xt)$ and $J_1(y)$, respectively) are expressed as rational functions in four algebraic series denoted by

$T, Z = \sqrt{T}, U \equiv U(x)$, and $V \equiv V(y)$. Here, T has degree 4 over $\mathbb{Q}(t)$, whereas $U(x)$ is cubic over $\mathbb{Q}(x, T)$ and $V(y)$ is cubic over $\mathbb{Q}(y, T)$. All these series are series in t^2 . Moreover, $t^2 \in \mathbb{Q}(T)$, $x \in \mathbb{Q}(T, U)$, and $y \in \mathbb{Q}(T, V)$. One can derive from these results rational expressions of $I_1(xt)/t$ and $J_1(y)/t$ in terms of \sqrt{T}, U , and V , and well as the following simple expressions:

$$J_1(Y) = \frac{256tT\sqrt{T}}{(T+3)^3}, \quad B = \frac{64tT\sqrt{T}(T^2+4T-1)}{(T-1)(T+3)^3}.$$

Details of these calculations are given in our Maple session. Using (8.6) (resp. (8.8)), we then obtain a rational expression of

$$\left(3\bar{x}A_-(xt) + 1 + \frac{\bar{x}^2}{t^2} - \frac{\bar{x}}{t^2}\right)^2 \quad \left(\text{resp.} \quad \left(yD(y) + \frac{2y}{3t^2(1+y)^2}\right)^2\right)$$

in terms of T and U (resp. T and V), showing that this series has degree 12 only (as \sqrt{T} is not involved here). Moreover, this rational expression is of the form $T \text{Rat}(T, U)^2$ (resp. $T \text{Rat}(T, V)^2$) for some rational function Rat , so that

$$3\bar{x}A_-(xt) + 1 + \frac{\bar{x}^2}{t^2} - \frac{\bar{x}}{t^2} \quad \left(\text{resp.} \quad yD(y) + \frac{2y}{3t^2(1+y)^2}\right)$$

belongs to $\sqrt{T}\mathbb{Q}(T, U)$ (resp. $\sqrt{T}\mathbb{Q}(T, V)$), and hence to the same extension of degree 24 of $\mathbb{Q}(t, x)$ (resp. $\mathbb{Q}(t, y)$) as $\mathcal{Q}(xt, 0)$ (resp. $\mathcal{Q}(0, y)$). This is in contrast with the Kreweras-like models solved in Sections 4–6, for which the degree of $C_-(x)$ and $D(y)$ is twice the degree of $\mathcal{Q}(x, 0)$ (or $\mathcal{Q}(0, y)$).

Proof of Theorem 8.1 We start from the functional equation (8.4), and observe that we have a decoupling relation:

$$y(1+x^2) = (2tx(1+y) - 1)G(y) + F(x) + H(x, y)\mathcal{K}(x, y),$$

where

$$F(x) = -1 - \bar{x}^2 + \frac{1}{tx}, \quad G(y) = \frac{y}{t^2(1+y)^2}, \quad H(x, y) = \frac{y(1-t(x+\bar{x}+\bar{x}y+xy))}{t^2(1+y)^2}.$$

This, together with Lemma 2.3, leads us to define a new pair of \mathcal{S} -invariants $(I(x), J(y))$:

$$I(x) = \left(2tU(x, 0) + \frac{2}{3}\left(1 + \bar{x}^2 - \frac{\bar{x}}{t}\right)\right)^2, \quad J(y) = \Delta(y) \left(yD(y) + \frac{2y}{3t^2(1+y)^2}\right)^2,$$

where $\Delta(y)$ is given by (8.7). We want to combine this new pair with the pair $(I_1(x), J_1(y))$ given by (8.5) to form a pair of invariants $(\tilde{I}(x), \tilde{J}(y))$ to which Lemma 2.6 would apply, thus proving that $\tilde{I}(x) = \tilde{J}(y)$ is a series in t .

Observe that $I_1(x)$ and $I(x)$ have poles at $x = 0$, whereas $J_1(y)$ and $J(y)$ have poles at $y = -1$ (only). More precisely, the series $I(x)$ has a (quadruple) pole at 0, which can

be removed by subtracting a well-chosen quartic polynomial in $I_1(x)$. This leads us to introduce

$$\tilde{I}(x) = I(x) - \frac{4}{9}I_1(x)^4 - B_3I_1(x)^3 - B_2I_1(x)^2 - B_1I_1(x),$$

and analogously

$$\tilde{J}(y) = J(y) - \frac{4}{9}J_1(y)^4 - B_3J_1(y)^3 - B_2J_1(y)^2 - B_1J_1(y),$$

for values of $B_3, B_2,$ and B_1 that involve $U_{0,0}, U_{1,0}$ and various by-products of $\mathcal{Q}(x, y)$ (we refer to our Maple session for details; we have denoted the auxiliary series B_i instead of A_i to avoid any confusion with the series $A(x, y)$ or $A_-(x)$). Then, we use the functional equations satisfied by $U(x, y), D(y),$ and $\mathcal{Q}(x, y)$ to check that the ratio

$$\frac{\tilde{I}(x) - \tilde{J}(y)}{\mathcal{K}(x, y)}$$

is a multiple of $xy,$ and conclude that $\tilde{I}(x) = \tilde{J}(y) = B_0,$ for some series $B_0 \in \mathbb{Q}((t)).$ By expanding $\tilde{J}(y)$ at $y = 0,$ we realize that $B_0 = 0.$ Hence, denoting by $P(u) = 4u^3/9 + B_3u^2 + B_2u + B_1,$ we have

$$(8.9) \quad I(x) = I_1(x)P(I_1(x)) \quad \text{and} \quad J(y) = J_1(y)P(J_1(y)).$$

We can express the roots of P in terms of the series \mathcal{Q} by looking at the formal power series $X \in \mathbb{Q}[[t]]$ or $Y \in \mathbb{Q}[[t]]$ that cancel $I(x)$ or $J(y).$ First, $\Delta(y)$ has a unique solution in $\mathbb{Q}[[t]],$ which we denote by $Y = 4t^2 + \mathcal{O}(t^4).$ Thus, $J_1(Y) = 4t + \mathcal{O}(t^3)$ must be a root of $P.$ Then, another root of P arises from the series $X = t + \mathcal{O}(t^3)$ such that $I(X) = 0,$ that is, $X = 3t^2U(X, 0)X^2 + tX^2 + t.$ This series is a double root of $I(x),$ and thus $I_1(X) = 1/t + \mathcal{O}(t^3)$ is a double root of $P(u),$ unless $I'_1(X) = 0,$ which we readily check not to be the case. At this stage, we can write

$$P(u) = \frac{4}{9}(u - J_1(Y))(u - I_1(X))^2,$$

but we still have to relate the double root $B := I_1(X)$ to $\mathcal{Q}.$ We expand around $x = 0$ the first identity of (8.9), and obtain the expression of the root B given in the statement of the theorem.

The expressions (8.6) and (8.8), which are those of $9I(x)/4$ and $9J(y)/4,$ are now proved. ■

Here is now our description of the harmonic function associated with simple walks in $\mathbb{C}.$

Corollary 8.3 *For $(i, j) \in \mathbb{C},$ there exists a positive constant $H_{i,j}$ such that, as $n \rightarrow \infty$ with $n \equiv i + j \pmod{2},$*

$$(8.10) \quad c_{i,j}(n) \sim -\frac{H_{i,j}}{\Gamma(-2/3)} 4^n n^{-5/3}.$$

The generating function

$$\mathcal{H}(x, y) := \sum_{j \geq 0, i \leq j} H_{i,j} x^{j-i} y^j,$$

which is a formal power series in x and y , is algebraic of degree 9, given by

$$(1 + y + x^2 y + x^2 y^2 - 4xy) \mathcal{H}(x, y) = \mathcal{H}_-(x) + (1 + y - 2xy) \mathcal{H}_d(y),$$

where

$$\mathcal{H}_-(x) := \sum_{i > 0} H_{-i,0} x^i \quad \text{and} \quad \mathcal{H}_d(y) := \sum_{i \geq 0} H_{i,i} y^i.$$

Each of these series is algebraic of degree 3. Let us define $L \equiv L(x) = \sqrt{3} + \mathcal{O}(x)$ and $P \equiv P(y) = 1/3 + \mathcal{O}(y)$ by

$$x = 9 \frac{3 - L^2}{(2L - 3)(L^2 - 12L + 9)}, \quad y = \frac{1 - 3P}{P^2(P - 3)}.$$

Then

$$(8.11) \quad \mathcal{H}_-(x) = \frac{128 \sqrt{3} x (2L - 3)}{9 (L - 3)^2} \quad \text{and} \quad \mathcal{H}_d(y) = \frac{64 \sqrt{3} (P + 1) (P - 3)^2 P^3}{27 (1 - P)^5}.$$

Note that an explicit expression of $\mathcal{H}_d(y)$ was given in [32, equation (53)] in terms of radicals. The fact that the degree is only 3 is not obvious on this alternative expression.

(sketched) The principle is the same as in the proof of Corollary 1.4, where we determined a harmonic function for (reverse) Kreweras' walks in three quadrants. We focus on the asymptotic behavior of the coefficients $a_{i,j}(n)$ of the series $A(x, y)$ defined by (8.1), and establish for them the estimate (8.10): since the coefficients of $Q(x, y)$ grow like $4^n n^{-3}$ only, the coefficients $a_{i,j}(n)$ and $c_{i,j}(n)$ are asymptotically equivalent.

We start from the expressions of $A_-(x)$ and $D(y)$ given in Theorem 8.1, and use the results of [8] to express $A_-(xt)$ and $D(y)$ rationally in terms of three algebraic series denoted by \sqrt{T} , $U(x)$, and $V(y)$, as described in Remark 8.2.

We then perform the singular analysis of T , $U(x)$, and $V(y)$ in the neighborhood of their dominant singularity, located at $t^2 = 1/16$. The series $L(x)$ (resp. $P(y)$) occurs in the singular expansion of $U(x)$ (resp. $V(y)$). The three series T , $U(x)$, and $V(y)$ have a singular behavior in $(1 - 16t^2)^{1/3}$, but we need to work out more terms because cancellations occur when moving from these series to $A_-(xt)$ and $D(y)$, which are found to have a singular behavior in $(1 - 16t^2)^{2/3}$. This is how we finally obtain the expressions of $\mathcal{H}_-(x)$ and $\mathcal{H}_d(y)$. The expression of $\mathcal{H}(x, y)$ simply comes from the harmonicity of $H_{i,j}$. ■

Remark 8.4 In passing, we have also determined the harmonic function for Gessel's walks in the first quadrant: for $(i, j) \in \mathcal{Q}$ and $n \equiv i \pmod{2}$, the number of n -step walks

ending at (i, j) is asymptotic to

$$\frac{h_{i,j}}{\Gamma(-4/3)} 4^n n^{-7/3},$$

where

$$\sum_{i \geq 0} h_{i,0} x^i = 48\sqrt{3} \frac{(2L-3)^2}{(L-3)^4} \quad \text{and} \quad \sum_{j \geq 0} h_{0,j} y^j = \frac{32}{\sqrt{3}} \frac{P^3(3-P)}{(P+1)(P-1)^4}.$$

We observe that these two series are cubic over $\mathbb{Q}(\sqrt{3})$, as the two univariate series describing the harmonic function of simple walks in three quadrants (see (8.11)). This is in contrast with the previously solved models, where the degree of $\mathcal{H}_-(x)$ and $\mathcal{H}_d(y)$ is twice the degree of the corresponding quadrant harmonic function (see, for instance, (1.11) and (1.14)). As already mentioned in Remark 8.2, a similar property holds at the level of (counting) generating functions: $\mathcal{Q}(x, 0)$ and $\mathcal{Q}(0, y)$ have degree 24, and the same holds for $A_-(x)$ and $D(y)$.

8.2 The diagonal model

When $\mathcal{S} = \{ \nearrow, \nwarrow, \swarrow, \searrow \}$, the series $A(x, y)$ defined by (8.1) satisfies

$$(1 - t(x + \bar{x})(y + \bar{y}))A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}(x + \bar{x})A_-(\bar{x}) - t\bar{x}(y + \bar{y})A_-(\bar{y}) - t\bar{x}\bar{y}A_{0,0}. \tag{8.12}$$

As before, this series counts weighted walks in the three-quadrant cone starting either from $(0, 0)$, or from $(-2, 0)$, or from $(0, -2)$, with a weight $2/3$ in the first case, and $1/3$ in the other two. As discussed in Lemma 2.1 for the corresponding series $C(x, y)$, it makes sense to define two series $U(x, y)$ and $D(y)$ by

$$A(x, y) = \bar{x}^2 U(\bar{x}^2, xy) + D(xy) + \bar{y}^2 U(\bar{y}^2, xy). \tag{8.13}$$

Now, we can reproduce the step-by-step arguments that led to (2.14) for the series $C(x, y)$, and we thus obtain

$$2xy\mathcal{K}(x, y)U(x, y) = \frac{2}{3}y(1+x) + y(t(y+\bar{y}) + 2xyt - 1)D(y) - 2t(1+x)U(x, 0) - tD_0, \tag{8.14}$$

where $\mathcal{K}(x, y) = 1 - t\mathcal{S}(x, y) = 1 - t(y + \bar{y} + xy + \bar{x}\bar{y})$ is the kernel of Gessel’s model, reflected in the first diagonal. Compared with the original functional equation (2.14), the only thing that has changed is the initial term y , which has become $2y(1+x)/3$.

For the model \mathcal{S} , we know two pairs of invariants (see Table 3): one is rational, and will not be used; the other takes the form (2.24) and involves quadrant generating functions:

$$I_1(x) = t(1+x)\mathcal{Q}(x, 0) - \frac{x}{t(1+x)}, \quad J_1(y) = -t\mathcal{Q}(0, y) + t\mathcal{Q}_{0,0} - \bar{y}. \tag{8.15}$$

These two series are algebraic, and we refer to [5, 8] for explicit expressions. Denoting by $(I_1^g(x), J_1^g(y))$ the pair of invariants (8.5) that we used for Gessel walks, we observe

that

$$I_1(x) = -J_1^g(x) + t\mathcal{Q}_{0,0}, \quad J_1(y) = -I_1^g(y) + t\mathcal{Q}_{0,0},$$

in accordance with the fact that the two models differ by an x/y -symmetry.

Theorem 8.5 Let $\mathcal{S} = \{\nearrow, \nwarrow, \swarrow, \searrow\}$, and let $\mathcal{S} = \{\uparrow, \nearrow, \downarrow, \swarrow\}$ be the set of reflected Gessel's steps. Let $C(x, y)$ be the generating function of walks with steps in \mathcal{S} , confined to the three-quadrant cone \mathcal{C} . Let us define $A(x, y)$ as (8.1).

Let $I_1(x)$ and $J_1(y)$ be the quadrant \mathcal{S} -invariants defined by (8.15). The generating function $A(x, y)$, which counts (weighted) walks in \mathcal{C} with steps in \mathcal{S} , satisfies equation (8.12), where $A_-(x)$ can be expressed in terms of I_1 and J_1 by

$$(8.16) \quad \frac{9}{4} \left(2t(\bar{x} + 1)A_-(\sqrt{x}) + tD_0 - \frac{2}{3t(1+x)} \right)^2 = (I_1(x) - J_1(Y_0))(I_1(x) - J_1(Y_1)).$$

In this expression, the series $Y_{0,1}$ are the two roots of

$$(8.17) \quad \Delta(y) = (1 - t^2\bar{y}(1+y)^2)(1 - t^2\bar{y}(1-y)^2)$$

that are formal power series in t .

The generating function $D(y)$ defined by (8.13) satisfies

$$(8.18) \quad \frac{9}{4}\Delta(y) \left(yD(y) + \frac{2}{3t} \right)^2 = (I_1(y) - J_1(Y_0))(I_1(y) - J_1(Y_1)).$$

In particular, the series $D_0 = D(0)$ involved in (8.16) satisfies

$$3t^2D_0 = 2 + t(J_1(Y_0) + J_1(Y_1)).$$

The series $A_-(x)$, $D(y)$, and $A(x, y)$ are algebraic.

Remark 8.6 Degrees and rational parametrizations of $A_-(\sqrt{x})$ and $D(y)$. We observe for this model the same phenomenon as for the simple model: the series $\mathcal{Q}(x, 0)$ and $\mathcal{Q}(0, y)$ have degree 24, but $A_-(\sqrt{x})$ and $D(y)$ have also degree 24 (only). More precisely, using again the results and notation of [8], one can express $\mathcal{Q}(x, 0)$ and $\mathcal{Q}(0, yt)$ (involved in the expressions of $I_1(x)$ and $J_1(yt)$, respectively) as rational functions in the four algebraic series $T, Z = \sqrt{T}, U \equiv U(y)$, and $V \equiv V(x)$ (note the exchange of x and y , which comes from the diagonal reflection of steps). One then derives from these results rational expressions of $I_1(x)/t$ and $J_1(yt)/t$ in terms of \sqrt{T}, U , and V , as well as the following expressions:

$$J_1(Y_0) + J_1(Y_1) = \frac{64tZ^3(Z^3 + 3Z^2 - Z + 1)}{(1 - Z)(Z^2 + 3)^3},$$

$$J_1(Y_0)J_1(Y_1) = 4 \frac{Z^7 + 7Z^6 - 3Z^5 + 19Z^4 - 45Z^3 + 53Z^2 - Z + 1}{(Z - 1)(Z^2 + 3)^3}.$$

Details of these calculations are given in our Maple session. Using (8.16) (resp. (8.18)), we then obtain a rational expression of

$$2(\tilde{x} + 1)A_-(\sqrt{x}) + D_0 - \frac{2}{3t^2(1+x)} \quad \left(\text{resp. } D(yt) + \frac{2}{3t^2y} \right)$$

of the form $\sqrt{T} \text{Rat}(T, V(x))$ (resp. $\sqrt{T} \text{Rat}(T, U(y))$).

Proof of Theorem 8.5 We start from the functional equation (8.14), and observe that we have a decoupling relation:

$$y(1+x) = (t(y + \tilde{y}) + 2txy - 1)G(y) + F(x) + H(x, y)\mathcal{K}(x, y),$$

where

$$F(x) = \frac{1}{t(1+x)}, \quad G(y) = \frac{1}{t}, \quad H(x, y) = \frac{x}{t(1+x)}.$$

This, together with Lemma 2.3, leads us to define a new pair of invariants:

$$I(x) = \left(2t(1+x)U(x, 0) + tD_0 - \frac{2}{3t(1+x)} \right)^2, \quad J(y) = \Delta(y) \left(yD(y) + \frac{2}{3t} \right)^2,$$

where $\Delta(y)$ is given by (8.17). We want to combine them polynomially with the invariants $(I_1(x), J_1(y))$ given by (8.15) to form a pair of invariants $(\tilde{I}(x), \tilde{J}(y))$ to which Lemma 2.6 applies.

Observe that $I_1(x)$ and $I(x)$ have poles at $x = -1$, whereas $J_1(y)$ and $J(y)$ have poles at $y = 0$. More precisely, the series $J(y)$ has a (double) pole at 0, which can be removed by subtracting a well-chosen quadratic polynomial in $J_1(y)$. This leads us to introduce

$$\tilde{J}(y) = J(y) - \frac{4}{9}J_1(y)^2 - B_1J_1(y),$$

and to define accordingly

$$\tilde{I}(x) = I(x) - \frac{4}{9}I_1(x)^2 - B_1I_1(x),$$

for a series B_1 that involves D_0 (we refer to our Maple session for details). Then, we use the functional equations satisfied by $U(x, y)$ and $\mathcal{Q}(x, y)$ to check that the ratio

$$\frac{\tilde{I}(x) - \tilde{J}(y)}{\mathcal{K}(x, y)}$$

is a multiple of xy , and conclude that $\tilde{I}(x) = \tilde{J}(y) = B_0$, for some series $B_0 \in \mathbb{Q}((t))$. Hence, denoting by $P(u) = 4u^2/9 + B_1u + B_0$, we have

$$I(x) = P(I_1(x)), \quad J(y) = P(J_1(y)).$$

We can express the roots of P in terms of the series \mathcal{Q} (and more precisely, in terms of J_1) by looking at the two roots of $\Delta(y)$, denoted by Y_i , with $i = 0, 1$, that lie in $\mathbb{Q}[[t]]$. For each of them, $J_1(Y_i)$ is a root of $P(u)$. This gives the expressions of the theorem, which are those of $9I(x)/4$ and $9J(y)/4$. ■

Let us finish with the harmonic function associated with diagonal walks in \mathbb{C} (known to be unique by [29]).

Corollary 8.7 For $(i, j) \in \mathbb{C}$, there exists a positive constant $H_{i,j}$ such that, as $n \rightarrow \infty$ with $n \equiv i + j \pmod 2$,

$$c_{i,j}(n) \sim -\frac{H_{i,j}}{\Gamma(-2/3)} 4^n n^{-5/3}.$$

Clearly, $H_{i,j} = 0$ if $i \not\equiv j \pmod 2$. The generating function

$$\mathcal{H}(x, y) := \sum_{j \geq 0, i \leq j} H_{i,j} x^{(j-i)/2} y^j,$$

which is a formal power series in x and y , is algebraic of degree 9, given by

$$(1 + x + xy^2 + x^2y - 4xy) \mathcal{H}(x, y) = \frac{x}{2} H_{0,0} + (1 + x) \mathcal{H}_-(x) + \left(1 - 2xy + \frac{x}{2}(1 + y^2)\right) \mathcal{H}_d(y),$$

where

$$\mathcal{H}_-(x) := \sum_{i > 0} H_{-i,0} x^{i/2} \quad \text{and} \quad \mathcal{H}_d(y) := \sum_{i \geq 0} H_{i,i} y^i.$$

Each of these series is algebraic of degree 3. Let $L(x)$ and $P(y)$ be defined as in Corollary 8.3. Then

$$\mathcal{H}_-(x) = \frac{32\sqrt{3}(3P(x) - 1)}{9(P(x) + 1)(P(x) - 1)^2} \quad \text{and} \quad \mathcal{H}_d(y) = \frac{144\sqrt{3}L(y)(L(y)^2 - 3)^2}{y^2(L(y)^2 + 6L(y) - 9)(3 - L(y))^5}.$$

(The exchange of x and y is intentional.)

Proof As in the proof of Corollary 8.3, one starts from the rational expressions of $A_-(\sqrt{x})$ and $D(y)$ in terms of \sqrt{T} , $U(y)$ and $V(x)$ derived from Theorem 8.5 and [8], and then plugs in them the singular expansions of T , $U(y)$, and $V(x)$ around $t^2 = 1/16$. This gives $\mathcal{H}_-(x)$ and $\mathcal{H}_d(y)$. The expression of $\mathcal{H}(x, y)$ simply comes from the harmonicity of $H_{i,j}$. ■

9 Discussion and perspectives

9.1 Generic form of the results

For the last three models that we have solved (6th model of Table 1 in Section 7, and simple and diagonal models in Section 8), we have directly written the invariant $I(x)$, involving $C_-(x)$ or $A_-(x)$, as a rational function in the quadrant invariant $I_1(x)$ involving $\mathcal{Q}(x, 0)$; and analogously for $J(y)$ and $J_1(y)$ (see Theorems 7.1, 8.1, and 8.5). The coefficients of these rational functions are series in t . This is also possible for the three models of the Kreweras trilogy, although we have (sometimes) also used the pair

$(I_0(x), J_0(y))$ to determine $(I(x), J(y))$. For the Kreweras model and the reverse Kreweras model, $I(x)$ is a polynomial in $I_1(x)$ (see (4.4) and (5.19)). For the double Kreweras model, one can ignore the pair $(I_0(x), J_0(y))$ as well, and prove that

$$I(x) = \frac{P(I_1(x))}{(I_1(x) - I_1(0))^2} \quad \text{and} \quad J(y) = \frac{P(J_1(y))}{(J_1(y) - I_1(0))^2}$$

for some polynomial $P(u)$ of degree 4.

9.2 Solving more models via invariants?

Let us finally discuss if, and how, one could go further in the solution of three-quadrant walks using the approach of this paper. Let us first recall that this approach, when it works, relates invariants involving $C(x, y)$ to pre-existing invariants (or to *weak* invariants, in the sense of [2]), which so far are systematically D-algebraic. Hence, there is little hope to solve with this approach models that would not be (at least) D-algebraic. This means that for the nine models of Table 1, invariants have done for us all that we could hope for: they have solved the six models that were not already known to be *hypertranscendental* (that is, non-D-algebraic).

If we want to go further, two different difficulties arise: the model may contain NW and/or SE steps, and it may not be diagonally symmetric. Let us consider two examples.

Diagonally symmetric models with NW and SE steps

Apart from the diagonal model, which we have actually solved via invariants, there are eight models in this class, as shown in Table 6. The first two are associated with a finite group (orders 4 and 6) and the other six with an infinite group. Let us consider, for instance, the fourth model of the table, $\mathcal{S} = \{ \nearrow, \nwarrow, \leftarrow, \downarrow, \searrow \}$ (the *scarecrow*), and write as before

$$C(x, y) = \bar{x}U(\bar{x}, xy) + D(xy) + \bar{y}U(\bar{y}, xy),$$

with $D(y) \in \mathbb{Q}[y][[t]]$ and $U(x, y) \in \mathbb{Q}[x, y][[t]]$.

We can write functional equations, first for $D(xy)$ and $\bar{x}U(\bar{x}, xy)$, as we did in Section 2.1. A step-by-step construction of walks gives

$$D(xy) = 1 + tx y D(xy) + 2t \bar{y} (\bar{x}U(0, xy) - \bar{x}U_{0,0}) + 2tx \bar{y} (\bar{x}^2 U_1(xy) - \bar{x}^2 U_{1,0}),$$

where $U_1(y)$ is the coefficient of x in $U(x, y)$. The term involving U_1 is new, and corresponds to walks that jump from the lines $j - i = \pm 2$ to the diagonal, using an NW or SE step. For walks ending above the diagonal, we find

$$\begin{aligned} \bar{x} (1 - tS(x, y)) U(\bar{x}, xy) &= t\bar{x}(1 + y)D(xy) + t\bar{x}y (\bar{y}U(0, xy) - \bar{y}U_{0,0}) \\ &\quad - t\bar{y}(1 + x) (\bar{x}U(\bar{x}, 0) + \bar{x}U(0, xy) - \bar{x}U_{0,0}) - tx\bar{y} (\bar{x}^2 U_1(xy) - \bar{x}^2 U_{1,0}), \end{aligned}$$

with $S(x, y) = xy + \bar{x}y + \bar{x} + \bar{y} + x\bar{y}$. The second term on the first line accounts for walks jumping from the line $j = i - 1$ to the line $j = i + 1$ by an NW step (we forbid steps from $(0, -1)$ to $(-1, 0)$). The last term of the second line accounts for walks that would jump from the line $j = i + 2$ to the diagonal.

S	Diagonal	King	Double tandem	
	$C_S(x, y; t)$ DF Sec. 8.2 [9]	DF [12]	DF ? [12]	

Table 6: The nine models with x/y -symmetry and steps \nwarrow and \searrow . The first three have a finite group and are *Weyl models* in the sense of [12]; the others have an infinite group.

We can eliminate the series $U_1(xy)$ by taking a linear combination of the two equations; but we still have $U(0, xy)$ and $D(xy)$, while we only had one of them when the steps NW and SE were forbidden. Upon performing the change of variables $(x, y) \mapsto (\bar{x}, xy)$, we finally obtain the following counterpart of (2.12):

$$2xy\mathcal{K}(x, y)U(x, y) = y + y(ty + 2tx(1 + xy) - 1)D(y) - 2t(1 + \bar{x})U(x, 0) + 2t(xy - \bar{x})(U(0, y) - U_{0,0}),$$

with $\mathcal{K}(x, y) := 1 - tS(\bar{x}, xy) = 1 - t(x + \bar{x}y + y + \bar{x}^2y + x^2y)$. The main two novelties are the facts that the kernel has degree 2 and valuation -2 in x , and that the right-hand side involves two series in y , namely $D(y)$ and $U(0, y)$. Hence, it seems that the first step toward solving this three-quadrant model via invariants would be to learn how to solve quadrant models with large steps via invariants—a question already raised in [3].

Asymmetric models

There are $74 - 8 - 9 = 57$ models that do not have the x/y -symmetry. Exactly 16 have a finite group, among which 4 (at least) are conjectured to have a D-finite (even algebraic, in one case) generating function [12]. They are shown in Table 7.

To illustrate the new difficulties that arise, let us work out functional equations for Gessel’s model. Since symmetry is lost, we now write

$$C(x, y) = \bar{x}U(\bar{x}, xy) + D(xy) + \bar{y}L(\bar{y}, xy),$$

with $D(y) \in \mathbb{Q}[y][[t]]$ and $U(x, y), L(x, y) \in \mathbb{Q}[x, y][[t]]$. The equation for walks ending on the diagonal reads

$$(1 - t(xy + \bar{x}\bar{y}))D(xy) = 1 - t\bar{x}\bar{y}D_0 + tx(\bar{x}U(0, xy)) + t\bar{x}(\bar{y}L(0, xy) - \bar{y}L_{0,0}).$$

	Diabolo	Tandem	Gouyou-Beauchamps	Gessel
\mathcal{S}				
$C_S(x, y; t)$	DF ?	DF ?	DF ?	Algebraic ?

Table 7: Four interesting asymmetric models with a finite group. The first three are *Weyl models* in the sense of [12].

Now, for walks ending above the diagonal, we obtain

$$(1 - tS(x, y)) \bar{x}U(\bar{x}, xy) = t\bar{x}D(xy) - t\bar{x}\bar{y}(\bar{x}U(\bar{x}, 0)) - tx(\bar{x}U(0, xy)),$$

with $S(x, y) = x + \bar{x} + xy + \bar{x}\bar{y}$. Finally, for walks ending below the diagonal, we find

$$(1 - tS(x, y)) \bar{y}L(\bar{y}, xy) = txD(xy) - t\bar{x}(1 + \bar{y})\bar{y}L(\bar{y}, 0) - t\bar{x}(\bar{y}L(0, xy) - \bar{y}L_{0,0}).$$

In the first two equations, we perform the same change of variables as before, namely $(x, y) \mapsto (\bar{x}, xy)$. In the third equation, we apply $(x, y) \mapsto (xy, \bar{x})$. This gives

$$\begin{aligned} (1 - tS(\bar{x}, xy)) xU(x, y) &= txD(y) - tx\bar{y}U(x, 0) - tU(0, y), \\ (1 - t(y + \bar{y})) D(y) &= 1 - t\bar{y}D_0 + tU(0, y) + t\bar{y}(L(0, y) - L_{0,0}), \\ (1 - tS(xy, \bar{x})) xL(x, y) &= txyD(y) - t\bar{y}(1 + x)L(x, 0) - t\bar{y}(L(0, y) - L_{0,0}). \end{aligned}$$

We now have a system with *two* trivariate series $U(x, y)$ and $L(x, y)$. In addition, the kernels

$$1 - tS(\bar{x}, xy) = 1 - t(x + \bar{x} + y + \bar{y}) \quad \text{and} \quad 1 - tS(xy, \bar{x}) = 1 - t(y + \bar{y} + xy + \bar{x}\bar{y})$$

are not the same.

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References

- [1] O. Bernardi, M. Bousquet-Mélou, and K. Raschel, *Counting quadrant walks via Tutte’s invariant method (extended abstract)*. In: DMTCS Proceedings of the 28th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2016), Discrete Mathematics and Theoretical Computer Science, Association of Discrete Mathematics and Theoretical Computer Science, Nancy, 2016, pp. 203–214. <https://doi.org/10.46298/dmtcs.6416>
- [2] O. Bernardi, M. Bousquet-Mélou, and K. Raschel, *Counting quadrant walks via Tutte’s invariant method*. *Combin. Theory* 1(2021), Article no. 3. <https://doi.org/10.5070/C61055360>
- [3] A. Bostan, M. Bousquet-Mélou, and S. Melczer, *Walks with large steps in an orthant*. *J. Eur. Math. Soc. (JEMS)* 23(2021), no. 7, 2221–2297. <https://doi.org/10.4171/jems/1053>
- [4] A. Bostan, F. Chyzak, M. van Hoeij, M. Kauers, and L. Pech, *Hypergeometric expressions for generating functions of walks with small steps in the quarter plane*. *Eur. J. Comb.* 61 (2017), 242–275. <https://doi.org/10.1016/j.ejc.2016.10.010>
- [5] A. Bostan and M. Kauers, *The complete generating function for Gessel walks is algebraic*. *Proc. Amer. Math. Soc.* 138(2010), no. 9, 3063–3078. <https://doi.org/10.1090/S0002-9939-2010-10398-2>
- [6] A. Bostan, K. Raschel, and B. Salvy, *Non-D-finite excursions in the quarter plane*. *J. Combin. Theory Ser. A*, 121(2014), 45–63. <https://doi.org/10.1016/j.jcta.2013.09.005>

- [7] A. Bouaziz, S. Mustapha, and M. Sifi, *Discrete harmonic functions on an orthant in \mathbb{Z}^d* . Electron. Commun. Probab. 20(2015), no. 52, 1–13. <https://doi.org/10.1214/ecp.v20-4249>
- [8] M. Bousquet-Mélou, *An elementary solution of Gessel's walks in the quadrant*. Adv. Math. 303(2016), 1171–1189. <https://doi.org/10.1016/j.aim.2016.08.038>
- [9] M. Bousquet-Mélou, *Square lattice walks avoiding a quadrant*. J. Combin. Theory Ser. A 144(2016), 37–79. <https://doi.org/10.1016/j.jcta.2016.06.010>
- [10] M. Bousquet-Mélou and M. Mishna, *Walks with small steps in the quarter plane*. In: Algorithmic probability and combinatorics, Contemporary Mathematics, 520, American Mathematical Society, Providence, RI, 2010, pp. 1–39. <https://doi.org/10.1090/conm/520/10252>
- [11] M. Bousquet-Mélou and M. Wallner, *More models of walks avoiding a quadrant*. In: 31st International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA 2020), Leibniz International Proceedings in Informatics, 159, Schloss Dagstuhl—Leibniz-Zentrum für Informatik, Dagstuhl, German, 2020. <https://doi.org/10.4230/LIPIcs.AofA.2020.8>
- [12] M. Bousquet-Mélou and M. Wallner, *Walks avoiding a quadrant and the reflection principle, to appear in European J. Combin.* arXiv:2110.07633
- [13] T. Budd, *Winding of simple walks on the square lattice*. J. Combin. Theory Ser. A 172(2020), Article no. 105191, 59 pp. <https://doi.org/10.1016/j.jcta.2019.105191>
- [14] D. Denisov and V. Wachtel, *Random walks in cones*. Ann. Probab. 43(2015), no. 3, pp. 992–1044. <https://doi.org/10.1214/13-AOP867>
- [15] T. Dreyfus, C. Hardouin, J. Roques, and M. F. Singer, *On the nature of the generating series of walks in the quarter plane*. Invent. Math. 213(2018), no. 1, pp. 139–203. <https://doi.org/10.1007/s00222-018-0787-z>
- [16] T. Dreyfus, C. Hardouin, J. Roques, and M. F. Singer, *Walks in the quarter plane: genus zero case*. J. Combin. Theory Ser. A 174(2020), Article no. 105251, 25 pp. <https://doi.org/10.1016/j.jcta.2020.105251>
- [17] T. Dreyfus and A. Trotignon, *On the nature of four models of symmetric walks avoiding a quadrant*. Ann. Comb. 25(2021), no. 3, 617–644. <https://doi.org/10.1007/s00026-021-00541-8>
- [18] J. Duraj, K. Raschel, P. Tarrago, and V. Wachtel, *Martin boundary of random walks in convex cones*. Ann. H. Lebesgue 5(2022), 559–609. <https://doi.org/10.5802/ahl.130>
- [19] A. Elvey Price, *Counting lattice walks by winding angle*. In FPSAC 2020 (Formal Power Series and Algebraic Combinatorics), Séminaire Lotharingien de Combinatoire, 84B, 2020, Article no. 43, 12 pp. <https://www.mat.univie.ac.at/~slc/wpapers/FPSAC2020>
- [20] A. Elvey Price, *Enumeration of three quadrant walks with small steps and walks on other M-quadrant cones*. Preprint, 2022. arXiv:2204.06847
- [21] P. Flajolet and R. Sedgewick, *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009 (available online).
- [22] C. Hardouin and M. F. Singer, *On differentially algebraic generating series for walks in the quarter plane*. Sel. Math. New Ser. 27(2021), no. 5, Article no. 89, 49 pp. <https://doi.org/10.1007/s00029-021-00703-9>
- [23] I. Ignatiouk-Robert, *Harmonic functions of random walks in a semigroup via ladder heights*. J. Theor. Probab. 34(2021), no. 1, 34–80. <https://doi.org/10.1007/s10959-019-00974-1>
- [24] I. Kurkova and K. Raschel, *On the functions counting walks with small steps in the quarter plane*. Publ. Math. Inst. Hautes Études Sci. 116(2012), 69–114. <https://doi.org/10.1007/s10240-012-0045-7>
- [25] S. Melczer and M. Mishna, *Singularity analysis via the iterated kernel method*. Comb. Probab. Comput. 23(2014), no. 5, 861–888. <https://doi.org/10.1017/S0963548314000145>
- [26] M. Mishna, *Classifying lattice walks restricted to the quarter plane*. J. Combin. Theory Ser. A 116(2009), no. 2, 460–477. <https://doi.org/10.1016/j.jcta.2008.06.011>
- [27] M. Mishna and A. Rechnitzer, *Two non-holonomic lattice walks in the quarter plane*. Theor. Comput. Sci. 410(2009), nos. 38–40, 3616–3630. <https://doi.org/10.1016/j.tcs.2009.04.008>
- [28] S. Mustapha, *Non-D-finite walks in a three-quadrant cone*. Ann. Comb. 23(2019), no. 1, 143–158.
- [29] S. Mustapha and M. Sifi, *Discrete harmonic functions in Lipschitz domains*. Electron. Commun. Probab. 24(2019), no. 58, 1–15. <https://doi.org/10.1214/19-ecp259>
- [30] K. Raschel, *Random walks in the quarter plane, discrete harmonic functions and conformal mappings*. Stoch. Process. Appl. 124(2014), no. 10, 3147–3178, with an appendix by S. Franceschi. <https://doi.org/10.1016/j.spa.2014.04.013>

- [31] K. Raschel and A. Trotignon, *On walks avoiding a quadrant*. Electron. J. Combin 26(2019), no. 3, Article no. 3.31, 34 pp. <https://doi.org/10.37236/8019>
- [32] A. Trotignon, *Discrete harmonic functions in the three-quarter plane*. Potential Anal. 56(2022), no. 2, 267–296. <https://doi.org/10.1007/s11118-020-09884-y>
- [33] W. T. Tutte, *Chromatic sums revisited*. Aequationes Math. 50(1995), nos. 1–2, 95–134.

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