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ON INTEGRATION IN PARTIALLY ORDERED GROUPS

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0. Introduction. M. Sion and T. Traynor investigated ([**15**]–[**17**]), measures and integrals having values in topological groups or semigroups. Their definition of integrability was a modification of Phillips–Rickart bilinear vector integrals, in locally convex topological vector spaces.

The purpose of this paper is to develop a good notion of an integration process in partially ordered groups, based on their order structure. The results obtained generalize some of the results of J. D. M. Wright ([**19**]–[**22**]) where the measurable functions are real-valued and the measures take values in partially ordered vector spaces.

Let H be a σ-algebra of subsets of T, X a lattice group, Y, Z partially ordered groups and $m : H \to Y$ a Y-valued measure on H. By $F(T, X)$, $M(T, X)$, $E(T, X)$ and $S(T, X)$ are denoted the lattice group of functions with domain T and with range X, the lattice group of (H, m)-measurable functions of $F(T, X)$, the lattice group of (H, m)-elementary measurable functions of $F(T, X)$ and the lattice group of (H, m)-simple measurable functions of $F(T, X)$ respectively.

First we prove that "Egoroff" convergence implies order convergence m-a.e. on T in $F(T, X)$ (without the assumption that X be a lattice) (Theorem 2.1). Moreover if X is of countable type and has the diagonal property, $S(T, X)$ is "dense" in $M(T, X)$, with respect to order convergence m-a.e. on T, and $M(T, X)$ is "closed" with respect to uniform order convergence m-a.e. on T (Theorem 2.3).

In the sequel suppose there exists a positive bi-additive function from $X \times Y$ into Z, order separately continuous. We integrate X-valued functions with respect to Y-valued measures. The integral lies in Z. First we define the lattice group I of (H, m)-elementary integrable functions in $E(T, X)$.

Next we extend the lattice group I in $M(T, X)$, using the uniform order convergence m-a.e. on T and define the lattice group $\mathscr{L}^1$ of (H, m)-integrable functions in $M(T, X)$; f belongs in $\mathscr{L}^1$ if and only if there exist $f_i \in \mathscr{L}^1$, $i = 1, 2$ such that:

$$f_1(t) \leqq f(t) \leqq f_2(t), \quad m\text{-a.e. on } T.$$

On the other hand, under mild conditions we give two convergence

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theorems. Moreover the function $\nu : H \to Z$, $\nu(A) = \int_A f(t)dm(t)$ with $f \in \mathscr{L}^1$, is σ-additive on H (with respect to order convergence in Z).

In Section 4 we have been able to obtain some connections between the given definition of measurability, the definition of partitionability (due to M. Sion [**15**]) and the classical definition, whenever X is a Banach lattice. In particular we remark that in general the space $S(T, X)$ is not sufficient to develop the space $M(T, X)$.

We close this paper with an application to a representation theorem.

1. Setting and terminology. Throughout this paper all groups are abelian and written additively. By a *partially ordered group* (p.o.g.) we mean a group X endowed with a partial ordering $\leqq$ such that the following condition is satisfied:

$$x \leqq y \text{ implies } x + z \leqq y + z, \quad \text{for all } x, y, z \text{ in } X.$$

X is a lattice group if $(x \vee y) := \sup \{x, y\}$ and $(x \wedge y) := \inf \{x, y\}$ exist for all x, y in X. In this case

$$|x| := \sup \{x, -x\}, \quad x^+ := \sup \{x, 0\} \quad \text{and} \quad x^- := \sup \{-x, 0\},$$

where 0 denotes the zero element in X.

Various concepts of order convergence can be defined in a p.o.g. X (cf. [**11**]). In this paper we shall use the following definition. The net $(x_j)_{j\in J}$ in X *o-converges* to x in X (denoted $o\text{-}\lim_j x_j = x$), if there exist an increasing net $(z_c)_{c\in C}$ and a decreasing net $(y_d)_{d\in D}$ in X such that:

(a) $\sup \{z_c : c \in C\} = x = \inf \{y_d : d \in D\}$ (denoted "$z_c \uparrow x$" and "$y_d \downarrow x$")

(b) For every $(c, d) \in C \times D$ there exists $j^* \in J$ so that:

$$z_c \leqq x_j \leqq y_d \quad \text{whenever } j \geqq j^*.$$

We define: $(x_j)_{j\in T}$ is *o-fundamental* if

$$o\text{-}\lim_{j,j'} (x_j - x_{j'}) = 0,$$

($J \times J$ is directed with the cartesian ordering). Clearly if X is a lattice group $o\text{-}\lim_j x_j = x$ if and only if there exists a decreasing net $(y_d)_{d\in D}$ in X with $y_d \downarrow 0$ and for every $d \in D$ there exists $j^* \in J$ such that: $|x_j - x| \leqq y_d$, whenever $j \geqq j^*$. The following lemma can be easily verified (cf. [**2**], Lemma 1, p. 132).

LEMMA 1.1. (i) *The o-limits are unique.*

(ii) *If* $o\text{-}\lim_j x_j = x$, *every cofinal subnet of* $(x_j)_{j\in J}$ *also converges to* x.

(iii) *If* $o\text{-}\lim_j x_j = x$ *then* $(x_j)_{j\in J}$ *is o-fundamental.*

(iv) $o\text{-}\lim_j x_j = x$ *if and only if* $o\text{-}\lim_j (x_j - x) = 0$.

(v) *If* $(x_j)_{j\in J}$ *is increasing* (*resp. decreasing*), *then* $o\text{-}\lim_j x_j = x$ *if and only if* $x_j \uparrow x$ (*resp.* $x_j \downarrow x$).

(vi) *If* $x_j \uparrow x$ (*resp.* $x_j \downarrow x$) *and* $y_d \uparrow y$ (*resp.* $y_d \downarrow y$), *then* $x_j + y_d \uparrow x + y$ (*resp.* $x_j + y_d \downarrow x + y$).

X is *monotone complete* if every majorised increasing net in X has a supremum in X. X is of *countable type* if for every decreasing net $(x_j)_{j\in J}$ in X with $x_j \downarrow 0$ there exists an increasing sequence $\{j_n : n \in \mathbf{N}\} \subseteq J$ such that: $x_{j_n} \downarrow 0$.

On the other hand X has the *diagonal property* if, whenever

$$\{x_{m,n}: (m, n) \in \mathbf{N} \times \mathbf{N}\} \subseteq X \quad \text{with } o\text{-}\lim_n x_{m,n} = x_m \in X,$$

for each $m \in \mathbf{N}$ and if

$$o\text{-}\lim_m x_m = x \in X,$$

then there exists a strictly increasing sequence $\{n_m : m \in \mathbf{N}\} \subseteq \mathbf{N}$ such that

$$o\text{-}\lim_m x_{m,n_m} = x.$$

The following lemmas will be useful in the sequel.

LEMMA 1.2. *Let X be a monotone complete* p.o.g., $(x_n)_{n\in\mathbf{N}}$ *a sequence in X with* $x_n \geqq 0$, *for every* $n \in \mathbf{N}$ *and*

$$o\text{-}\lim_n \sum_{i=1}^{n} x_i = x \in X.$$

If $(x_{k_n})_{n\in\mathbf{N}}$ *is a rearrangement of* $(x_n)_{n\in\mathbf{N}}$ *then*

$$o\text{-}\lim_n \sum_{i=1}^{n} x_{k_i} = x.$$

Proof. By Lemma 1.1 (v)

$$x = \sup \left\{ \sum_{i=1}^{n} x_i : n \in \mathbf{N} \right\}.$$

Let $n \in \mathbf{N}$. Then

$$\sum_{i=1}^{n} x_{k_i} \leqq \sum_{i=1}^{s_n} x_i \leqq x \quad \text{with} \quad s_n := \max \{k_1, k_2, k \ldots, k_n\}.$$

Hence

$$\sup \left\{ \sum_{i=1}^{n} x_{k_i} : n \in \mathbf{N} \right\} = o\text{-}\lim_n \sum_{i=1}^{n} x_{k_i} \leqq x.$$

Because the sequence $(x_n)_{n\in\mathbf{N}}$ is also a rearrangement of $(x_{k_n})_{n\in\mathbf{N}}$ we have:

$$x \leqq \sup\left\{\sum_{i=1}^{n} x_{k_i} : n \in \mathbf{N}\right\}$$

namely

$$o\text{-}\lim_n \sum_{i=1}^{n} x_{k_i} = x.$$

LEMMA 1.3. *Let X be a monotone complete* p.o.g. *and let $(x_n)_{n\in\mathbf{N}}$ be a sequence in X with $x_n \geqq 0$, for every $n \in \mathbf{N}$. Moreover, let $k: \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ be a bijection and let $y_{p,n} = x_{k(p,n)}$ for each p and each n in $\mathbf{N}$. Then the following assertions are equivalent.*

(i) $o\text{-}\lim_n \sum_{i=1}^{n} x_i = x$

(ii) $o\text{-}\lim_{m,n} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{i,j} = x.$

Proof. Suppose (i) is true. Then

$$x = \sup\left\{\sum_{i=1}^{n} x_i : n \in \mathbf{N}\right\}$$

(Lemma 1.1 (v)). Hence

$$\sum_{i=1}^{m} \sum_{j=1}^{n} y_{i,j} = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{k_{i,j}} \leqq \sum_{i=1}^{s_{m,n}} x_i \leqq o\text{-}\lim_n \sum_{i=1}^{n} x_i = x,$$

with

$$s_{m,n} := \max\{k_{i,j}: i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\},$$

whenever $(m, n) \in \mathbf{N} \times \mathbf{N}$. Therefore there exists the

$$o\text{-}\lim_{m,n} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{i,j} \leqq x.$$

On the other hand given $s \in \mathbf{N}$ there exist $(m_i, n_i) \in \mathbf{N} \times \mathbf{N}$, $i = 1, 2, \ldots, s$ so that

$$\sum_{i=1}^{s} x_i = \sum_{i=1}^{s} y_{m_i,n_i} \leqq \sum_{i=1}^{m_0} \sum_{j=1}^{n_0} y_{i,j} \leqq o\text{-}\lim_{m,n} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{i,j},$$

where

$$m_0 := \max\{m_1, m_2, \ldots, m_s\}, \quad n_0 := \max\{n_1, n_2, \ldots, n_s\}.$$

Thus

$$x = o\text{-}\lim_n \sum_{i=1}^{n} x_i \leqq o\text{-}\lim_{m,n} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{i,j},$$

namely

$$x = o\text{-}\lim_{m,n} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{i,j}.$$

The converse implication is proved similarly.

LEMMA 1.4. *Let the increasing double sequence* $\{x_{m,n}: (m, n) \in \mathbf{N} \times \mathbf{N}\}$ *in the monotone complete* p.o.g. X *such that*:

$$x = o\text{-}\lim_{m} [o\text{-}\lim_{n} x_{m,n}].$$

Then

$$x = o\text{-}\lim_{n} [o\text{-}\lim_{m} x_{m,n}] = \sup \{x_{m,n}: (m, n) \in \mathbf{N} \times \mathbf{N}\}.$$

Proof. Put

$$x_m: = o\text{-}\lim_{n} x_{m,n}, \quad m \in \mathbf{N}.$$

Evidently $x_{m,n} \leqq x_m \leqq x$, whenever $(m, n) \in \mathbf{N} \times \mathbf{N}$. Therefore there exists the

$$x_n^*: = o\text{-}\lim_{m} x_{m,n} \leqq x, \quad n \in \mathbf{N}.$$

and it is easy to see that $x_n^* \uparrow$, whence

$$x^*: = o\text{-}\lim_{n} x_n^* \leqq x.$$

On the other hand by $x_{m,n} \leqq x_n^* \leqq x^*$, $(m, n) \in \mathbf{N} \times \mathbf{N}$ we get

$$x = o\text{-}\lim_{m} [o\text{-}\lim_{n} x_{m,n}] \leqq x^*,$$

namely $x = x^*$.

Next let $y \in X$ with $x_{m,n} \leqq y$ for every $(m, n) \in \mathbf{N} \times \mathbf{N}$. Thus

$$x = o\text{-}\lim_{m} [o\text{-}\lim_{n} x_{m,n}] \leqq y.$$

Hence $x = \sup \{x_{m,n}: (m, n) \in \mathbf{N} \times \mathbf{N}\}$.

2. Partially ordered group-valued measures and measurable functions. Throughout this paper H denotes a σ-algebra of subsets of a space T, Y a p.o.g. and $m : H \rightarrow Y$ a *measure* on H ($m(A) \geqq 0$ for every $A \in H$ and

$$m\left(\bigcup_{n \in \mathbf{N}} A_n\right) = o\text{-}\lim_{n} \sum_{i=1}^{n} m(A_i),$$

whenever $(A_n)_{n \in \mathbf{N}}$ is a pairwise disjoint sequence in H).

We say that the proposition $P(t)$, $t \in T$ is true *m-almost everywhere* (denoted m-a.e.) on $S \in H$ if there exists $M \in H$ such that: $m(M) = 0$ and $P(t)$ is true whenever $t \in S - M$.

Now let X be a p.o.g. and let $F(S, X)$, $(S \subseteq T, S \neq \emptyset)$, be the p.o.g. of functions of S in X, where the group and ordering operations are defined pointwise. Evidently the function $q : X \to F(S, X)$, $q(x) = f_x$ with $f_x(t) = x$, whenever $t \in S$, is an invariant embending of the p.o.g. X in $F(S, X)$. Therefore if

$$\{x_j : j \in J\} \subseteq X \quad \text{and} \quad \sup \{x_j : j \in J\} = x$$

(resp. $\inf \{x_j : j \in J\} = y$) in X then

$$\sup \{f_{x_j} : j \in J\} = f_x$$

(resp. $\inf \{f_{x_j} : j \in J\} = f_y$) in $F(S, X)$. In the sequel we identify the p.o.g. X with its image under the embedding q.

Let the net $(f_j)_{j \in J}$ in $F(S, X)$ and $f \in F(S, X)$; $(f_j)_{j\in J}$ *o-converges to* f *on* S (resp. m-a.e. on $S \in H$), denoted $o\text{-}\lim_j f_j = f$ on S, (resp. m-a.e. on $S \in H$) if

$$o\text{-}\lim_j f_j(t) = f(t), \quad \text{for every } t \in S$$

(resp. m-a.e. on $S \in H$).

On the other hand $(f_j)_{j\in J}$ *uniformly o-converges to* f *on* S (resp. m-a.e. on $S \in H$), denoted $u\text{-}\lim_j f_j = f$ on S (resp. m-a.e. on $S \in H$), if there exist an increasing net $(z_c)_{c\in C}$ in X and a decreasing net $(y_d)_{d\in D}$ in X such that (a) is valid and:

(g) For every $(c, d) \in C \times D$ there exists $j^* \in J$, so that

$$z_c \leqq f_j(t) - f(t) \leqq y_d, \quad \text{for every } t \in S$$

(resp. m-a.e. on $S \in H$), whenever $j \geqq j^*$.

THEOREM 2.1. *Let* $(f_j)_{j\in J}$ *be a net in* $F(T, X)$ *and let* $m : H \to Y$ *be a measure on* H. *Suppose there exists a sequence* $(A_n)_{n\in \mathbf{N}}$ *in* H, *such that:*

(i) $u\text{-}\lim_j f_j = f$ *on* A_n, *for every* $n \in \mathbf{N}$ *and*

$$o\text{-}\lim_n m(T - A_n) = 0.$$

Then

$$o\text{-}\lim_j f_j = f \quad m\text{-a.e. } on \ T.$$

Proof. Let $n \in \mathbf{N}$. By (i) there exists an increasing net $(z_{n,c})_{c\in C}$ and a decreasing net $(y_{n,d})_{d\in D}$ in X such that: $z_{n,c} \uparrow 0$, $y_{n,d} \downarrow 0$ and for every $(c, d) \in C \times D$ there exist $j_n \in J$ with

$$z_{n,c} \leqq f_j(t) - f(t) \leqq y_{n,d} \quad \text{whenever } j \geqq j_n, \quad t \in A_n.$$

Next there exists a disjoint sequence $(A_n^*)_{n\in\mathbf{N}}$ in H such that

$$\bigcup_{n\in\mathbf{N}} A_n^* = \bigcup_{n\in\mathbf{N}} A_n := S.$$

Now we consider the nets $(u_c)_{c\in C}$, $(v_d)_{d\in D}$ in $F(S, X)$ with

$$u_c(t) := z_{n,c} v_d(t) := y_{n,d}, \quad \text{whenever } t \in A_n^*, \quad n \in \mathbf{N}.$$

Thus $u_c \uparrow 0$, $v_d \downarrow 0$ and for any $(c, d, t) \in C \times D \times S$ there exists $j(t) \in J$ such that:

$$u_c(t) \leqq f_j(t) - f(t) \leqq v_d(t), \quad \text{whenever } j \geqq j(t).$$

Hence $o\text{-}\lim_j f_j = f$ on S. Moreover

$$0 \leqq m(T - S) = m\left(T - \bigcup_{n\in\mathbf{N}} A_n\right)$$
$$= m\left(\bigcap_{n\in\mathbf{N}} (T - A_n)\right) \leqq m(T - A_n), \quad \text{for every } n \in \mathbf{N}$$

and

$$o\text{-}\lim_n m(T - A_n) = 0.$$

Therefore $m(T - S) = 0$ which proves the assertion.

In the following X will be a lattice group and Y, Z will be partially ordered groups.

Let $S(T, X)$ (resp. $E(T, X)$) be the set of (H, m)-simple (resp. (H, m)-elementary) measurable functions of T in X, where $m : H \to Y$ is a measure on H. By definition $S(T, X) := \{f \in F(T, X)$: there exists a finite partition $(A_i)_{1\leqq i\leqq n}$ of T such that $f(t) = a_i$, for every $t \in A_i$, $A_i \in H$, $i = 1, 2, \ldots, n\}$, (resp. $E(T, X) := \{f \in F(T, X)$: there exists a countable partition $(A_n)_{n\in\mathbf{N}}$ of T such that $f(t) = a_n$, for every $t \in A_n$, $A_n \in H$, $n \in \mathbf{N}\}$). Let $f \in F(T, X)$; f is (H, m)-measurable if there exists a net $(f_j)_{j\in J}$ in $E(T, X)$ so that: $u\text{-}\lim_j f_j = f$, m-a.e. on T. We put

$$M(T, X) := \{f \in F(T, X) : f \text{ is } (H, m)\text{-measurable}\}.$$

Evidently $M(T, X)$ (resp. $S(T, X)$, $E(T, X)$) is a lattice subgroup of the lattice group $F(T, X)$ and

$$S(T, X) \subseteq E(T, X) \subseteq M(T, X) \subseteq F(T, X).$$

THEOREM 2.2. *Let X be of countable type and $f \in M(T, X)$. Then there exists an increasing (resp. decreasing) sequence $(f_n)_{n\in\mathbf{N}}$ (resp. $(g_n)_{n\in\mathbf{N}}$) in $E(T, X)$ with $u\text{-}\lim_n f_n = f$ (resp. $u\text{-}\lim_n g_n = f$), m-a.e. on T.*

Proof. By definition there exists a net $(h_j)_{j\in J}$ in $E(T, X)$ and nets $(z_c)_{c\in C}$, $(y_d)_{d\in D}$ in X such that:

(1) $\quad z_c \uparrow 0, \quad y_d \downarrow 0.$

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Given $(c, d) \in C \times D$ there exists $j^* \in J$ with

(2) $\quad z_c \leqq h_j(t) - f(t) \leqq y_d, \quad m$-a.e. on T whenever $j \geqq j^*$.

By hypothesis for X there exist sequences $\{c_n: n \in \mathbf{N}\} \subseteq C$, $\{d_n: n \in \mathbf{N}\} \subseteq D$ so that: $z_{c_n} \uparrow 0$, $y_{d_n} \downarrow 0$. Hence by (2) for every $n \in \mathbf{N}$ there exists $j_n \in J$ with

$$z_{c_n} \leqq h_j(t) - f(t) \leqq y_{d_n}, \quad m\text{-a.e. on } T \text{ whenever } j \geqq j_n.$$

Using induction choose an increasing sequence $\{j_n : n \in \mathbf{N}\} \subseteq J$ such that:

$$z_{c_n} \leqq h_{j_k}(t) - f(t) \leqq y_{d_n}, \quad m\text{-a.e. on } T \text{ whenever } k \geqq n, \quad n \in \mathbf{N}.$$

Thus

$$u\text{-}\lim_n h_{j_n} = f, \quad m\text{-a.e. } on\ T.$$

Furthermore we put

$$f_n^*: = h_{j_n} - y_{d_n}, \quad n \in \mathbf{N}.$$

Evidently $f_n^* \in E(T, X)$ and

$$z_{c_n} - y_{d_n} \leqq f_k^*(t) - f(t) \leqq 0, \quad m\text{-a.e. on } T,$$

for every $k \geqq n$, $n \in \mathbf{N}$ with $z_{c_n} - y_{d_n} \uparrow 0$. So

$$u\text{-}\lim_n f_n^* = f, \quad m\text{-a.e. } on\ T.$$

Next we define $f_1 = f_1^*, f_{n+1}: = \sup \{f_n, f_{n+1}^*\}$, $n \in \mathbf{N}$. Then $f_n^* \leqq f_n \leqq f$, m-a.e. on T and u-$\lim_n f_n^* = f$, m-a.e. on T implies

$$u\text{-}\lim_n f_n = f, \quad m\text{-a.e. on } T \quad \text{and} \quad f_n \leqq f_{n+1}, \quad n \in \mathbf{N}.$$

For the respective case we work similarly.

THEOREM 2.3. *Let X have the diagonal property and be of countable type.*
(i) *If $(f_j)_{j \in J}$ is a net in $M(T, X)$ such that u-$\lim_j f_j = f$, m-a.e. on T then $f \in M(T, X)$.*
(ii) *For every $f \in M(T, X)$ there exists a sequence $(h_n)_{n \in \mathbf{N}}$ in $S(T, X)$ such that o-$\lim_n h_n = f$, m-a.e. on T.*

Proof. Arguing as in the preceding proof, let us choose an increasing sequence $\{j_n: n \in \mathbf{N}\} \subseteq J$ such that:

(3) $\quad u\text{-}\lim_n f_{j_n} = f, \quad m$-a.e. on T.

Furthermore there exists a double sequence $\{f_{k,n}: (k, n) \in \mathbf{N} \times \mathbf{N}\} \subseteq E(T, X)$ such that:

(4) $\quad u\text{-}\lim_k f_{k,n} = f_{j_n}, \quad m$-a.e. on T, for each $n \in \mathbf{N}$.

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Let $(k, 1) \in \mathbf{N} \times \mathbf{N}$. From (3), (4) there exist sequences

$$\{u_{k,n}: (k, n) \in \mathbf{N} \times \mathbf{N}\}, \{v_{k,n}: (k, n) \in \mathbf{N} \times \mathbf{N}\}, \{z_n: n \in \mathbf{N}\}, \{w_n: n \in \mathbf{N}\}$$

in X and $s \in \mathbf{N}$ such that:

$$u_{k,n} \leqq f_{p,n}(t) - f_{j_n}(t) \leqq v_{l,n},$$
$$z_k \leqq f_{j_n}(t) - f(t) \leqq w_l, \quad m\text{-a.e. on } T,$$

whenever $p, n \geqq s$, with

(5) $\quad u_{k,n} \uparrow 0 \ (k \to \infty), v_{l,n} \downarrow 0 \ (l \to \infty), \quad n \in \mathbf{N},$

and

$$z_n \uparrow 0, w_n \downarrow 0.$$

Now by the diagonal property for X choose sequences $\{q_n: n \in \mathbf{N}\}$, $\{r_n: n \in \mathbf{N}\}$ in $\mathbf{N}(q_n < q_{n+1}, r_n < r_{n+1}, n \in \mathbf{N})$ such that:

(6) $\quad u_{q_n,n} \uparrow 0, v_{r_n,n} \downarrow 0.$

Therefore by (5), (6) there exists $g \in N$ such that:

(7) $\quad u_{q_k,k} + z_k \leqq f_{q_n,n}(t) - f(t) \leqq v_{r_l,l} + w_l,$
m-a.e. on T, whenever $n \geqq g$.

Hence (by (7)) $u\text{-lim}_n f_{q_n,n} = f$, m-a.e. on T, with $f_{q_n,n} \in E(T, X)$, $n \in \mathbf{N}$ (similarly $u\text{-lim}_n f_{r_n,n} = f$, m-a.e. on T, with $f_{r_n,n} \in E(T, X)$, $n \in \mathbf{N}$), which implies that $f \in M(T, X)$.

(ii) Let $f \in M(T, X)$. By Theorem 2.2 there exists an increasing sequence $(f_n)_{n\in\mathbf{N}}$ in $E(T, X)$ with

$$u\text{-}\lim_n f_n = f, \quad m\text{-a.e. on } T.$$

Moreover it is easy to see that there exists a double sequence

$$\{h_{k,n}: (k, n) \in \mathbf{N} \times \mathbf{N}\}$$

in $S(T, X)$ such that

$$o\text{-}\lim_k h_{k,n} = f_n, \quad m\text{-a.e. on } T; \quad \text{for every } n \in \mathbf{N}.$$

Hence by virtue of the diagonal property choose a sequence

$$\{h_{k_n,n}: = h_n, n \in \mathbf{N}\}$$

in $S(T, X)$ $(k_n < k_{n+1}, n \in \mathbf{N})$ such that:

$$o\text{-}\lim_n h_n = f, \quad m\text{-a.e. on } T.$$

3. The integral. Let X be a lattice group and let Y, Z be partially ordered groups such that Z is monotone complete. Assume also the existence of a bi-additive function from $X \times Y$ into Z, which we denote simply by juxtaposition with the following properties:

(d) If $0 \leqq x_1 \leqq x_2$ and $0 \leqq y_1 \leqq y_2$ then $0 \leqq x_1 \cdot y_1 \leqq x_2 \cdot y_2$ whenever $x_1, x_2 \in X^+$ and $y_1, y_2 \in Y^+$.

(e) If $(y_n)_{n \in \mathbf{N}}$ is a sequence in Y with $o\text{-}\lim_n y_n = y, y \in Y$ then

$$o\text{-}\lim_n x \cdot y_n = x \cdot y, \quad \text{for any } x \in X.$$

(f) If $(x_j)_{j \in J}$ is a net in X with $o\text{-}\lim_j x_j = x, x \in X$ then

$$o\text{-}\lim_j x_j \cdot y = x \cdot y, \quad \text{for every } y \in Y.$$

Let $f \in E(T, X), f \geqq 0$. Then there exists a countable partition $(A_n)_{n \in \mathbf{N}}$ of T by elements of H such that:

$$f(t) = a_n \geqq 0, \quad \text{whenever } t \in A_n, \quad n \in \mathbf{N}.$$

Let also $m : H \rightarrow Y$ be a measure on H; f is (H, m)-integrable on T, if

$$o\text{-}\lim_n \sum_{i=1}^{n} a_i \cdot m(A_i)$$

exists in Z. In this case we put:

$$\int_T f(t) dm(t) := o\text{-}\lim_n \sum_{i=1}^{n} a_i \cdot m(A_i).$$

By Lemma 1.2 we get that the integral $\int_T f(t)dm(t)$ is independent of a rearrangement of the series

$$\left(\sum_{i=1}^{n} a_i \cdot m(A_i)\right)_{n \in \mathbf{N}}.$$

Next let $(B_n)_{n \in \mathbf{N}}$ be another countable partition of T by elements of H so that: $f(t) = b_n \geqq 0$, for every $t \in B_n$, $n \in \mathbf{N}$. Thus $a_i = b_j$, whenever $A_i \cap B_j \neq \emptyset$.

The following lemma verifies that the integral $\int_T f(t)dm(t)$ depends only on f and is independent of the particular way in which f is written as an (H, m)-elementary measurable function. Its proof is straightforward.

LEMMA 3.1. *Suppose that there exists*

$$o\text{-}\lim_n \sum_{i=1}^{n} a_i \cdot m(A_i) \quad \textit{in } Z.$$

Then

$$o\text{-}\lim_n \sum_{i=1}^{n} a_i \cdot m(A_i) = o\text{-}\lim_n \sum_{j=1}^{n} b_j \cdot m(B_j).$$

Now let $f \in E(T, X)$; f is (H, m)-integrable on T if there exist

$$\int_T f^+(t)dm(t) \quad \text{and} \quad \int_T f^-(t)dm(t).$$

Moreover as usual we define

$$\int_T f(t)dm(t) := \int_T f^+(t)dm(t) - \int_T f^-(t)dm(t).$$

Clearly the integral $\int_T f(t)dm(t)$ is well defined. We put

$$I := I(X, m, Z, H,) := \left\{ f \in E(T, X) : \text{there exists } \int_T f(t)dm(t) \right\}.$$

Also

$$\int_A f(t)dm(t) := \int_T f_A(t)dm(t)$$

with

$$f_A(t) := \begin{Bmatrix} f(t) & \text{if } t \in A \\ 0 & \text{if } t \notin A \end{Bmatrix} \quad \text{whenever } A \in H.$$

It is an easy matter to prove $-f, f_A \in I$ whenever $f \in I$, $A \in H$ and

$$-\int_T f(t)dm(t) = \int_T -f(t)dm(t).$$

The following propositions can be easily proved.

Proposition 3.2. *If* $f_i \in I$, $i = 1, 2$ *then* $(f_1 + f_2) \in I$ *and*

$$\int_T (f_1 + f_2)(t)dm(t) = \int_T f_1(t)dm(t) + \int_T f_2(t)dm(t).$$

Proposition 3.3. *If* $f \in E(T, X)$ *and* $f(t) = 0$, m-a.e. *on* T *then* $f \in I$ *and*

$$\int_T f(t)dm(t) = 0.$$

Proposition 3.4. $\int_T f(t)dm(t) \geqq 0$ *whenever* $f \in I$ *with* $f(t) \geqq 0$, m-a.e. *on* T.

Furthermore the following theorems are true.

Theorem 3.5. *The set* I *is a lattice subgroup of* $E(T, X)$.

Proof. This is obvious.

Theorem 3.6. *Let* $f \in E(T, X)$. *Then* $f \in I$ *if and only if there exists* $f_i \in I$, $i = 1, 2$ *with*

$$f_1(t) \leqq f(t) \leqq f_2(t), \quad m\text{-a.e. on } T.$$

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Proof. Let $f \in E(T, X)$ and $f_i \in I$, $i = 1, 2$ with $f_1(t) \leqq f(t) \leqq f_2(t)$, m-a.e. on T. Thus

$$|f|(t) \leqq g(t), \quad m\text{-a.e. on } T,$$

where $g: = \sup\{|f_1|, |f_2|\}$. By Theorem 3.5 we get $g \in I$.

On the other hand there exist countable partitions $(A_n)_{n\in\mathbf{N}}$, $(B_n)_{n\in\mathbf{N}}$ of T by elements of H and sequences $(a_n)_{n\in\mathbf{N}}$, $(b_n)_{n\in\mathbf{N}}$ in X such that: $f(t) = a_n$, $g(t) = b_n$, whenever $t \in (A_n \cap B_n)$, $n \in \mathbf{N}$. Therefore

$$\sum_{i=1}^{k} \sum_{j=1}^{n} a_i^+ m(A_i \cap B_j) \leqq \int_T g(t) dm(t), \quad \text{for each } (k, n) \in \mathbf{N} \times \mathbf{N}$$

implies there exist the iterated limits:

$$o\text{-}\lim_k \sum_{i=1}^{k} a_i^+ \left[o\text{-}\lim_n \sum_{j=1}^{n} m(A_i \cap B_j) \right] = o\text{-}\lim_n \sum_{i=1}^{n} a_i^+ \cdot m(A_i) = \int_T f^+(t) dm(t).$$

Similarly there exists $\int_T f^-(t) dm(t)$, hence $f \in I$.

COROLLARY 3.7. *Let $f \in E(T, X)$. The following assertions are equivalent.*
(i) $f \in I$.
(ii) $f^+, f^- \in I$
(iii) $|f| \in I$.

THEOREM 3.8. *Let $f \in I$. Then the function $v : H \to Z$, $v(A): = \int_A f(t) dm(t)$ whenever $A \in H$ is σ-additive on H.*

Proof. Let $(A_n)_{n\in\mathbf{N}}$ be a countable partition of T by elements of H such that:

$$f(t) = a_n, \quad \text{whenever } t \in A_n,\ A_n \in H,\ n \in \mathbf{N}.$$

Now let $(B_n)_{n\in\mathbf{N}}$ be an increasing sequence in H with $B_n \uparrow B$. The increasing double sequence

$$\left\{ \sum_{i=1}^{k} a_i^+ m(A_i \cap B_n) : (k, n) \in \mathbf{N} \times \mathbf{N} \right\}$$

is order bounded from above by $\int_T f^+(t) dm(t)$, hence there exists:

$$o\text{-}\lim_{k,n} \sum_{i=1}^{k} a_i^+ \cdot m(A_i \cap B_n) = \sup \left\{ \sum_{i=1}^{k} a_i^+ \cdot m(A_i \cap B_n) : (k, n) \in \mathbf{N} \times \mathbf{N} \right\} : = a.$$

By Lemma 1.4

$$a = o\text{-}\lim_n \left[o\text{-}\lim_k \sum_{i=1}^{k} a_i^+ \cdot m(A_i \cap B_n) \right]$$
$$= o\text{-}\lim_k \left[o\text{-}\lim_n \sum_{i=1}^{k} a_i^+ \cdot m(A_i \cap B_n) \right].$$

But

$$o\text{-}\lim_n \int_{B_n} f^+(t)dm(t) = o\text{-}\lim_n \left[o\text{-}\lim_k \sum_{i=1}^{k} a_i^+ m(A_i \cap B_n) \right]$$

and

$$\int_B f^+(t)dm(t) = o\text{-}\lim_k \left[o\text{-}\lim_n \sum_{i=1}^{k} a_i^+ \cdot m(A_i \cap B_n) \right].$$

Thus

$$\int_B f^+(t)dm(t) = o\text{-}\lim_n \int_{B_n} f^+(t)dm(t).$$

Similarly

$$o\text{-}\lim_n \int_{B_n} f^-(t)dm(t) = \int_B f^-(t)dm(t)$$

which proves the assertion.

THEOREM 3.9. *Let* $(f_j)_{j\in J}$ *be a net in* I *and* $f \in E(T, X)$ *such that:*

$$u\text{-}\lim_j f_j = f, \quad m\text{-a.e. on } T.$$

Then $f \in I$ *and*

$$o\text{-}\lim_j \int_T f_j(t)dm(t) = \int_T f(t)dm(t).$$

Proof. By definition there exist nets $(z_c)_{c\in C}$, $(y_d)_{d\in D}$ in X such that:

(8) $\quad z_c \leqq f_j(t) - f(t) \leqq y_d$, m-a.e. on T whenever $j \geqq j^*$, $z_c \uparrow 0$, $y_d \downarrow 0$.

Hence

$$-y_d + f_{j^*}(t) \leqq f(t) \leqq -z_c + f_{j^*}(t), \quad m\text{-a.e. on } T.$$

Therefore by Theorem 3.6, $f \in I$.

Next by (8)

$$z_c \cdot m(T) \leqq \int_T f_j(t)dm(t) - \int_T f(t)dm(t) \leqq y_d \cdot m(T),$$

whenever $j \geqq j^*$. Evidently $z_c \cdot m(T) \uparrow 0$, $y_d \cdot m(T) \downarrow 0$ and the desired conclusion follows.

THEOREM 3.10. *Let the net* $(f_j)_{j\in J}$ *in* I *with*

$$u\text{-}\lim_{j,j'} (f_j - f_{j'}) = 0, \quad m\text{-a.e. } on\ T.$$

Then the net $(\int_T f_j(t)dm(t))_{j\in J}$ *in* Z *is o-fundamental.*

Proof. This is similar to that of 3.9.

THEOREM 3.11. *Let the increasing net* $(f_j)_{j\in J}$ *in* I *with*

$$u\text{-}\lim_{j} f_j = f, \quad m\text{-a.e. } on\ T,$$

where $f \in M(T, X)$. *Then there exists the* $o\text{-}\lim_j \int_T f_j(t)dm(t)$ *in* Z.

Proof. Since there exist $z \in Z$ and $j^* \in J$ with

$$\int_T f_j(t)dm(t) \leqq z + \int_T f_{j^*}(t)dm(t), \quad \text{whenever } j \geqq j^*,$$

the increasing net $(\int_T f_j(t)dm(t))_{j\in J}$ is order bounded in Z.

COROLLARY 3.12. *Let the increasing sequences* $(f_n)_{n\in\mathbf{N}}$, $(g_n)_{n\in\mathbf{N}}$ *in* I *such that*

$$u\text{-}\lim_{n} f_n = u\text{-}\lim_{n} g_n, \quad m\text{-a.e. } on\ T.$$

Then

$$o\text{-}\lim_{n} \int_T f_n(t)dm(t) = o\text{-}\lim_{n} \int_T g_n(t)dm(t).$$

Hereafter in this paper suppose that X is of countable type. So Theorem 2.2 implies $M(T, X) = \{f \in F(T, X)$: there exists an increasing sequence $(f_n)_{n\in\mathbf{N}}$ in $E(T, X)$ with $u\text{-}\lim_n f_n = f$, m-a.e. on $T\}$. The preceding results lead to the following definition: A function $f \in M(T, X)$ is said to be (H, m)-integrable on T if there exists an increasing sequence $(f_n)_{n\in\mathbf{N}}$ in I with $u\text{-}\lim_n f_n = f$, m-a.e. on T.

The integral of f with respect to m is the element $\int_T f(t)dm(t)$ in Z defined by the equality

$$\int_T f(t)dm(t) := o\text{-}\lim_{n} \int_T f_n(t)dm(t).$$

According to Corollary 3.12 the integral $\int_T f(t)dm(t)$ does not depend on the increasing sequence $(f_n)_{n\in\mathbf{N}}$ in I.

The set of (H, m)-integrable functions $f \in M(T, X)$ is denoted by $\mathscr{L}^1$. As is easily verified if $f \in \mathscr{L}^1$ then $-f, f_A \in \mathscr{L}^1$ whenever $A \in H$ and

$$\int_T -f(t)dm(t) = -\int_T f(t)dm(t).$$

Furthermore we put

$$\int_A f(t)dm(t) := \int_T f_A(t)dm(t), \quad A \in H.$$

On the other hand the propositions and theorems of the integral of (H, m)-elementary integrable functions remain also valid for (H, m)-integrable functions.

We close the paragraph with the following:

THEOREM 3.13. *Let* $(f_n)_{n\in\mathbf{N}}$ *be a sequence in* $\mathscr{L}^1$ *such that:*

(i) *There exists* $x \in X$ *with* $0 \leqq f_{n+1}(t) \leqq f_n(t) \leqq x$, m-a.e. *on* T, $n \in \mathbf{N}$.

(ii) *There exists a sequence* $(A_n)_{n\in\mathbf{N}}$ *in* H *with*

$$u\text{-}\lim_n f_n = 0, \quad m\text{-a.e. on each } A_k,\ k \in \mathbf{N} \text{ and}$$

$$o\text{-}\lim_n m(T - A_n) = 0.$$

(iii) *Z has the diagonal property.*

Then

$$o\text{-}\lim_n \int_T f_n(t)dm(t) = 0$$

Proof. There exist a decreasing sequence $(y_n)_{n\in\mathbf{N}}$ in Y and a subsequence $(A_{k_n})_{n\in\mathbf{N}}$ of $(A_n)_{n\in\mathbf{N}}$ so that:

$$m(T - A_{k_n}) \leqq y_n, \quad n \in \mathbf{N} \quad \text{and} \quad \inf \{y_n : n \in \mathbf{N}\} = 0.$$

Therefore

$$0 \leqq \int_T f_n(t)dm(t) \leqq \int_{A_{kp}} f_n(t)dm(t) + x \cdot y_p,$$

for any $(n, p) \in \mathbf{N} \times \mathbf{N}$. Since

$$o\text{-}\lim_n \left[\int_{A_{kp}} f_n(t)dm(t) + x \cdot y_p\right] = x \cdot y_p, \quad p \in \mathbf{N} \text{ and}$$

$$o\text{-}\lim_p x \cdot y_p = 0,$$

there exists a strictly increasing sequence $(q_n)_{n\in\mathbf{N}}$ in $\mathbf{N}$ such that:

$$o\text{-}\lim_p \int_{A_{k_p}} f_{q_p}(t)dm(t) + x \cdot y_p = 0.$$

On the other hand by

$$0 \leqq \int_T f_{q_n}(t)dm(t) \leqq \int_{A_{kn}} f_{q_n}(t)dm(t) + x \cdot y_n,$$

for every $n \in N$ implies

$$o\text{-}\lim_n \int_T f_{q_n}(t)dm(t) = 0.$$

Hence the decreasing sequence

$$\left(\int_T f_n(t)dm(t)\right)_{n\in\mathbf{N}}$$

has a subsequence

$$\left(\int_T f_{q_n}(t)dm(t)\right)_{n\in\mathbf{N}}$$

with

$$o\text{-}\lim_n \int_T f_{q_n}(t)dm(t) = 0,$$

so the assertion follows.

4. Applications. (i) Measurable Banach lattice-valued functions. Let $(X, \leqq, \|\cdot\|)$ be a Banach lattice. First we need some definitions.

(a_1) The norm $\|\cdot\|$ is *order σ-continuous* on X if $o\text{-}\lim_n x_n = x$ implies $\lim_{n\to\infty} \|x_n\| = \|x\|$ whenever $(x_n)_{n\in\mathbf{N}}$ is a sequence in X and $x \in X$.

(a_2) Let H' be the σ-algebra of Borel subsets of X. A function $f : T \to X$ is *(H, H')-measurable* if $f^{-1}(F) \in H$, whenever $F \in H'$.

(a_3) A function $f : T \to X$ is *m-partitionable* if for every neighborhood V of 0 there is a partition $(A_n)_{n\in\mathbf{N}}$ of T in H such that:

$$m\left(T - \bigcup_{n\in\mathbf{N}} A_n\right) = 0 \quad \text{and} \quad f(A_n) - f(A_n) \subseteq V, \quad \text{for all } n \in \mathbf{N}.$$

(a_4) An element $e > 0$ is an *order unit* with respect to the norm $\|\cdot\|$ if $\|x\| \leqq k$ implies $|x| \leqq k \cdot e$, $k \in \mathbf{R}^+$, $x \in X$.

THEOREM 4.1. *Let $(X, \leqq, \|\cdot\|)$ be a Banach lattice. Suppose that X is separable and has an order unit e with respect to the norm $\|\cdot\|$. Let furthermore $f : T \to X$ be an (H, H')-measurable function. Then $f \in M(T, X)$.*

Proof. By hypothesis there exists a countable set

$$Q: = \{a_n: n \in \mathbf{N}\} \subseteq X \quad \text{with} \quad f(T) \subseteq \bar{Q} = X.$$

Put $A_{n,r}: = f^{-1}(S_{1/n}(a_r))$, where $S_{1/n}(a_r)$ denotes the closed sphere with center a_r and radius $1/n$, $(n, r) \in \mathbf{N} \times \mathbf{N}$.

For every $n \in \mathbf{N}$ consider the disjoint sequence in H:

$$B_{n,1}: = A_{n,1}, \quad B_{n,r}: = A_{n,r} - \bigcup_{i<r} A_{n,i}, \; r \geqq 2.$$

Thus $T = \bigcup_{r \in \mathbf{N}} B_{n,r}$, for every $n \in \mathbf{N}$. Let the sequence $(f_n)_{n \in \mathbf{N}}$ in $E(T, X)$ with $f_n(t) := a_r$, for $t \in B_{n,r}$, $(n, r) \in \mathbf{N} \times \mathbf{N}$. Then

$$\|f(t) - f_n(t)\| \leqq 1/n, \quad \text{for any } (t, n) \in T \times \mathbf{N}.$$

Hence

$$|f(t) - f_n(t)| \leqq (1/n)e, \quad \text{whenever } (t, n) \in T \times \mathbf{N},$$

namely $u\text{-}\lim_n f_n = f$ on T.

THEOREM 4.2. *Let $(X, \leqq, \|\ \ \|)$ be a Banach lattice with an order σ-continuous norm $\|\ \ \|$ on X.*
Suppose there exists a sequence $(f_n)_{n \in \mathbf{N}}$ in $E(T, X)$ and $f \in M(T, X)$ with $u\text{-}\lim_n f_n = f$ on T, Then:
(a) *f is m-partitionable.*
(b) *f is (H, H')-measurable.*

Proof. (a) By hypothesis there exists a sequence $(u_n)_{n \in \mathbf{N}}$ in X such that: $u_n \downarrow 0$ and for each $n \in N$ there exists $n_0 \in \mathbf{N}$ with:

$$|f_k(t) - f(t)| \leqq u_n \quad \text{whenever } k \geqq n_0, \ t \in T.$$

Hence

$$\|f_k(t) - f(t)\| \leqq \|u_n\| \quad \text{whenever } k \geqq n_0, \quad t \in T.$$

So f is m-partitionable (cf. [**15**] Theorem 3.2).
(b) This is a direct consequence of (a) and Theorem 2.7 in [**15**].

Remark 4.3. Cearly, if the conditions of the preceding theorem are satisfied and f is non-order bounded, m-a.e. on T, there is no sequence $(f_n)_{n \in \mathbf{N}}$ in $S(T, X)$ with $u\text{-}\lim_n f_n = f$, m-a.e. on T. Hence the space $S(T, X)$ is not sufficient in general to develop the space $M(T, X)$.

(ii) *A Riesz representation theorem.* Here suppose that $Y = Z$ and there exists an element $e \in X$ such that: $e > 0$ and $e \cdot y = y$, whenever $y \in Y$.

Now a function $U: \mathscr{L}^1 \to Z$ is *positive* if $U(f) \geqq 0$, for every $f \in \mathscr{L}^1$ with $f(t) \geqq 0$, m-a.e. on T, *additive* if $U(f_1 + f_2) = U(f_1) + U(f_2)$, whenever $f_i \in \mathscr{L}^1$, $i = 1, 2$ and *order continuous* if $o\text{-}\lim_j U(f_j) = U(f)$, for every net $(f_j)_{j \in J}$ in $\mathscr{L}^1$ with $f \in \mathscr{L}^1$, whenever $o\text{-}\lim_j f_j = f$, m-a.e. on T.

Since X is of countable type the order continuity property of U is equivalent to the following: $o\text{-}\lim_n U(f_n) = U(f)$, for every sequence $(f_n)_{n \in \mathbf{N}}$ in $\mathscr{L}^1$ with $f \in \mathscr{L}^1$, whenever $o\text{-}\lim_n f_n = f$, m-a.e. on T. The equivalence can be easily established following standard arguments (cf. [**18**], p. 220).

On the other hand if $x \in X$, $A \in H$ by $f_A{}^x$ it is denoted the element of $S(T, X)$ defined: $f_A{}^x(t) = x$ if $t \in A$ and $f_A{}^x(t) = 0$ if $t \notin A$.

THEOREM 4.4. *Let the positive additive and order continuous function* $U: \mathscr{L}^1 \to Z$, *so that:* $U(f_A{}^x) = x \cdot U(f_A{}^e)$, *for every* $(x, A) \in X \times H$. *Then there exists a unique measure* $m : H \to Y$ *with*

$$U(f) = \int_T f(t)dm(t) \quad \textit{whenever } f \in \mathscr{L}^1.$$

Proof. Define the function $m : H \to Z$, $m(A) := U(f_A{}^e)$. Evidently m is σ-additive on H, which implies that m is a measure on H. Now let $f \in I$. Then there exist a countable partition $(A_n)_{n \in \mathbf{N}}$ of T and a sequence $(a_n)_{n \in \mathbf{N}}$ in X with $f(t) = a_n$, whenever $t \in A_n$, $n \in \mathbf{N}$. Since

$$o\text{-}\lim_n \sum_{i=1}^n f_{A_i}{}^{a_i} = f,$$

we get by hypothesis

$$o\text{-}\lim_n U\left(\sum_{i=1}^n f_{A_i}{}^{a_i}\right) = U(f).$$

Moreover

$$o\text{-}\lim_n U\left(\sum_{i=1}^n f_{A_i}{}^{a_i}\right) = o\text{-}\lim_n \sum_{i=1}^n U(f_{A_i}{}^{a_i})$$

$$= o\text{-}\lim_n \sum_{i=1}^n a_i \cdot m(A_i) = \int_T f(t)dm(t).$$

Therefore $U(f) = \int_T f(t)dm(t)$.

Next let $f \in \mathscr{L}^1$. Then there exists an increasing sequence $(f_n)_{n \in \mathbf{N}}$ in I so that:

$$u\text{-}\lim_n f_n = f, \quad m\text{-a.e. on } T.$$

Thus

$$(9) \quad o\text{-}\lim_n U(f_n) = U(f).$$

Furthermore by the preceding

$$U(f_n) = \int_T f_n(t)dm(t) \quad \text{for every } n \in \mathbf{N}.$$

Hence

$$(10) \quad o\text{-}\lim_n U(f_n) = o\text{-}\lim_n \int_T f_n(t)dm(t) = \int_T f(t)dm(t).$$

By (9) and (10) it follows that $U(f) = \int_T f(t)dm(t)$.

Finally let $l : H \to Y$ be another measure such that:

$$U(f) = \int_T f(t)dl(t) \quad \text{for any } f \in \mathcal{L}^1.$$

Then

$$m(A) = U(f_A{}^e) = e \cdot l(A) = l(A) \quad \text{whenever } A \in H,$$

which completes the proof.

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