ON THE REPRESENTATION OF A NUMBER AS A SUM OF SQUARES

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1. Introduction. It is well known that the number $A_k(m)$ of representations of a positive integer m as the sum of k squares of integers can be expressed in the form

$$A_k(m) = P_k(m) + R_k(m), \tag{1}$$

where $P_k(m)$ is a divisor function, and $R_k(m)$ is a remainder term of smaller order. (1) is a consequence of the fact that

$$\vartheta(z) = \sum_{-\infty}^{\infty} e^{\pi i n^2 z}$$

is a modular form for a certain congruence subgroup of the modular group, and

$$\vartheta^k(z) = 1 + \sum_{m=1}^{\infty} A_k(m) e^{\pi i m z} = E_k(z) + E_k^+(z)$$

with

$$E_k(z) = 1 + \sum_{m=1}^{\infty} P_k(m) e^{\pi i m z}, \qquad E_k^+(z) = \sum_{m=1}^{\infty} R_k(m) e^{\pi i m z}, \tag{2}$$

where $E_k(z)$ is an Eisenstein series and $E_k^+(z)$ is a cusp form (as was first pointed out by Mordell [**9**]). The result (1) remains true if m is taken to be a totally positive integer from a totally real number field K and $A_k(m)$ is the number of representations of m as the sum of k squares of integers from K (at least for $2 \mid k$, $k > 2$, and for those cases with $2 \nmid k$ which have been investigated). ϑ, E_k, E_k^+ then are replaced by modular forms for a subgroup of the Hilbert modular group with Fourier expansions of the form (10) (see section 2).

For $2 \mid k$, $k = 2r$,

$$P_k(m) = \kappa_r a_r(m), \tag{3}$$

where $a_r(m)$ is a simple finite sum (see sections 5 and 6) (e.g.

$$a_r(m) = (-1)^m \sum_{d \mid m, d \in \mathbb{N}} (-1)^d d^{r-1}$$

for $4 \mid r$) and κ_r does not depend on m (κ_r stems from normalizing to 1 the constant term in the Fourier expansion (2) of E_k). For $2 \nmid k$, $P_k(m)$ is, in general, more complicated, the evaluation of $P_k(m)$ involving the calculation of a value of an L-series, depending on m.

$E_k^+(z)$ is an element of a vector space of functions of dimension $[(k-1)/8]$ over $\mathbb{C}$, a basis of which has been given by Mordell [**9**] (see section 4). In particular, $[(k-1)/8] = 0$

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for $k \leq 8$, hence E_k^+ is identically zero for $k \leq 8$. Rankin [**14**] has shown that, if $k \geq 9$, E_k^+ does not vanish identically. For $9 \leq k \leq 16$, E_k^+ is, up to a constant factor, the basis element given by Mordell; for $k \geq 17$, E_k^+ is a linear combination of those basis elements. Rankin [**12**] has pointed out that, from the point of view of calculating the $R_k(m)$, i.e. of calculating the Fourier coefficients of the basis elements, it is not clear whether, for $k \geq 17$, the basis elements given by Mordell and others are the most suitable ones. It seemed to be a good idea to look for basis elements whose Fourier coefficients have multiplicative properties (for a detailed discussion see [**12**], [**13**]). But, even for such functions, it remains desirable to find simple expressions for the Fourier coefficients, at least for coefficients $b(p)$, p a prime (if the coefficients $b(m)$, m not a prime, can be calculated from the $b(p)$, p prime, with the help of the multiplicative properties). The object of this paper is to show how such expressions can be obtained by using Eisenstein series for suitable Hilbert modular groups.

If a modular form of weight r for a Hilbert modular group of a totally real number field K, $[K:\mathbb{Q}]=n$, defined on a product $H_1 \times \ldots \times H_1$ of n upper half-planes $H_1 = \{z \mid z \in \mathbb{C}, \operatorname{Im} z > 0\}$, is restricted to the diagonal, one gets a modular form of weight nr for the (rational) modular group (see section 3, Theorem 4). This has been used to calculate values of zeta functions and L-functions of K, which occur as constant terms in the Fourier expansions of Eisenstein series (see e.g. [**5**, p. 112, p. 134; **6**]). It can also be used to obtain modular forms for certain congruence subgroups of the modular group [**6**]. To give a typical example of the procedure, let

$$\hat{G}_{12}(z) = 1 + \frac{2^4 . 3^2 . 5 . 7 . 13}{691} \sum_{m=1}^{\infty} \sum_{d \mid m} d^{11} e^{2\pi i m z} = 1 + \frac{2^4 . 3^2 . 5 . 7 . 13}{691} e^{2\pi i z} + \ldots$$

be the normalized Eisenstein series of weight 12 for the modular group (see e.g. [**15**, (7), p. 55]). Take the normalized Eisenstein series of weight 6 for Hilbert's modular group of the field $\mathbb{Q}(\sqrt{2})$, defined on the product $H_1 \times H_1$ of two upper half-planes (see [**5**]). The restriction to the diagonal is given by (see [**5**] (p. 134, using the value of κ_6 on p. 135))

$$G_{-6}(z, z) = 1 + \frac{2^4 . 3^2 . 7}{19^2} \sum_{m=1}^{\infty} \sum_{\substack{\mathscr{S}\nu/2\sqrt{2}=m \\ \nu\sqrt{2}>0}} \sum_{(\mu) \mid (\nu)} |\mathscr{N}(\mu)|^5 e^{2\pi i m z}$$

$$= 1 + \frac{2^4 . 3^2 . 5 . 7}{19^2} e^{2\pi i z} + \ldots .$$

Here ν is an integer from K, $\nu\sqrt{2} > 0$ stands for $\nu\sqrt{2}$ totally positive, $\mathscr{S}$ denotes the trace, $\mathscr{N}$ the norm, and the summation for the inner sum is over the integral ideals (μ). $\hat{G}_{12}$ and G_{-6} are modular forms of weight 12, and their difference is a cusp form, hence a multiple of the discriminant

$$\Delta(z) = e^{2\pi i z} \prod_{m=1}^{\infty} (1 - e^{2\pi i m z})^{24} = \sum_{m=1}^{\infty} \tau(m) e^{2\pi i z} = e^{2\pi i m z} + \ldots .$$

Comparing the coefficients of $e^{2\pi iz}$ one finds that

$$19^2 \,.\, 691(\hat{G}_{12}(z) - G_{-6}(z, z)) = -2^8 \,.\, 3^4 \,.\, 5 \,.\, 7 \,.\, \Delta(z).$$

Comparing the coefficients of $e^{2\pi imz}$, one gets that

$$2^4 \,.\, 3^2 \,.\, 5 \,.\, \tau(m) = -5 \,.\, 13 \,.\, 19^2 \sum_{d \mid m} d^{11} + 691 \sum_{\substack{\mathcal{S}\nu/2\sqrt{2} = m \\ \nu\sqrt{2} > 0}} \sum_{(\mu) \mid (\nu)} |\mathcal{N}(\mu)|^5 \tag{4}$$

for Ramanujan's function $\tau(n)$. In (4), the second term on the right-hand side too is a divisor function, this time in the quadratic number field $\mathbb{Q}(\sqrt{2})$. The integers ν from $\mathbb{Q}(\sqrt{2})$ with $\mathcal{S}\nu/2\sqrt{2} = m$, $\nu\sqrt{2} > 0$, of course, are the numbers $\nu = a + m\sqrt{2}$, $a \in \mathbb{Z}$, $a^2 < 2m^2$.

In order to obtain expressions similar to (4) for $R_k(m)$, we shall use Eisenstein series of weight r for certain subgroups of a Hilbert modular group of a field K, $[K:\mathbb{Q}] = n$. Taking Eisenstein series, the Fourier expansions of which have non-zero constant terms only in cusps not equivalent to cusps on the diagonal (such groups usually have cusps of this kind), by restriction to the diagonal one immediately gets cusp forms of weight nr for a subgroup of the (rational) modular group (see section 3, Theorem 5). From Eisenstein series of integral weight r for $\mathbb{Q}(\sqrt{2})$, we obtain the necessary cusp forms of weight $2r = 6, 8, 10, 12$ (see section 7), i.e. E_k^+ and $R_k(m)$ for $k = 12, 16, 20, 24$. For $k = 28, 32$, $\mathbb{Q}(\sqrt{2})$ does not supply sufficiently many cusp forms from Eisenstein series whose restrictions are cusp forms. But additional cusp forms, and with them E_{28}^+, E_{32}^+, can be constructed by this method from Eisenstein series for $\mathbb{Q}(\sqrt{3})$. To reach, for instance, $k = 18$ with Eisenstein series from a quadratic number field, one would have to take series of half integral weight $\frac{9}{2}$ (E_{18}^+ is a modular form of weight $9 = 2.\frac{9}{2}$). The series of half integral weight are not very pleasant **[10]**. We therefore used Eisenstein series of weight 3 for a cubic number field (see section 8) to construct E_{18}^+. For the resulting formula for $A_{18}(m)$, see section 9.

The paper is organized as follows. Section 2 presents general information about Hilbert modular groups and Eisenstein series. We use non-normalized Eisenstein series having the simpler Fourier coefficients $a_r(m)$ instead of $\kappa_r a_r(m)$ in the case of (3) and are not concerned with the exact value of the constant term in the Fourier expansion, since this term, for modular forms of positive weight, is uniquely determined by the remaining coefficients and, therefore, not needed for proving linear relations between modular forms. In section 3 the restriction to the diagonal is discussed. Section 4 brings the necessary background about theta functions. Then, starting from some Eisenstein series for subgroups of Hilbert's modular groups for various fields K, sufficiently many modular forms are constructed to find linear combinations of them equal to certain powers of $\vartheta(z)$, and the Fourier coefficients of these functions are calculated. The functions for $K = \mathbb{Q}$ are treated in sections 5 and 6, the functions for $\mathbb{Q}(\sqrt{2})$ in section 7, and the functions for the cubic field of discriminant 7^2 in section 8. In section 9 the resulting values of $A_k(m)$ for certain k are given.

The method, developed in this paper, for calculating $A_k(m)$ is, of course, not confined to the case of rational integers m but can also be applied to the case of integers m of a totally real number field.

2. Hilbert's modular groups and Eisenstein series. Let

(i) K be a totally real number field with class number 1, $[K:\mathbb{Q}]=n$,
(ii) (ν), for $\nu \in K$, denote the ideal generated by ν,
(iii) d_K be the discriminant of K, (δ) the different,
(iv) $\rho \in K$ be an algebraic integer,
(v) $r \in \mathbb{N}$ with $\mathcal{N}_{K/\mathbb{Q}}(\varepsilon)^r = 1$ for every unit $\varepsilon \equiv 1 \bmod (\rho)$.

Hilbert's modular group for K is the group

$$\Gamma_K = \left\{ L \mid L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \text{ integers from } K, \det L = 1 \right\};$$

the principal congruence subgroup of level (ρ) is defined by

$$\Gamma_K(\rho) = \left\{ L \mid L \in \Gamma_K, L \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod (\rho) \right\}.$$

The n different injections of K into $\mathbb{R}$ map K onto the conjugate fields $K^{(1)}, \ldots, K^{(n)}$. To each $K^{(j)}$ one assigns a complex variable $\tau^{(j)}$, the jth conjugate of $\tau = (\tau^{(1)}, \ldots, \tau^{(n)})$. The canonical isomorphisms of $K(\tau)$ onto $K^{(j)}(\tau^{(j)})$ (with $\tau \mapsto \tau^{(j)}$), for $j = 1, \ldots, n$, map a rational function $R(\tau) \in K(\tau)$ onto its conjugates $R^{(j)}(\tau^{(j)})$. Calculations with elements from $K(\tau)$ always stand for simultaneous calculations with the conjugates in $K^{(j)}(\tau^{(j)})$ $(1 \le j \le n)$. For $R(\tau) \in K(\tau)$, trace and norm are defined by

$$\mathcal{S}R(\tau) = \sum_{j=1}^{n} R^{(j)}(\tau^{(j)}), \qquad \mathcal{N}(R(\tau)) = \prod_{j=1}^{n} R^{(j)}(\tau^{(j)}).$$

To each matrix $L \in \Gamma_K$ one assigns a transformation

$$\tau \mapsto L(\tau) = (a\tau + b)(c\tau + d)^{-1} \qquad \left(\text{for } L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right), \tag{5}$$

i.e. a simultaneous transformation

$$\tau^{(j)} \to L^{(j)}(\tau^{(j)}) = (a^{(j)}\tau^{(j)} + b^{(j)})(c^{(j)}\tau^{(j)} + d^{(j)})^{-1} \qquad (1 \le j \le n).$$

By (5), Γ_K acts as a group of analytic automorphisms on a product $H = H_1 \times \ldots \times H_1$ of n upper half-planes $H_1 = \{z \mid z \in \mathbb{C}, \operatorname{Im} z > 0\}$. $L \in \Gamma_K$ acts as a linear operator on the space of functions f holomorphic on H by

$$f(\tau) \mid L = f(\tau) \mathop{|}_{r} L = \mathcal{N}(c\tau + d)^{-r} f(L(\tau)). \tag{6}$$

The set $M_r(\Gamma_K(\rho), v)$ of entire modular forms of weight r with a multiplier system v of modulus 1 for $\Gamma_K(\rho)$ consists of those functions f on H that satisfy the following conditions:

(7) f is holomorphic on H;
(8) $f|_r L = v(L)f, |v(L)| = 1$, for $L \in \Gamma_K(\rho)$;
(9) The Fourier expansions of f at the cusps of $\Gamma_K(\rho)$ (see (10)) contain only terms with $\nu/\delta\rho \ge 0$.

If $n>1$, (9) is a consequence of (7), (8) (see [**1**], p. 323). $M_r(\Gamma_K(\rho), v)$ is a finite-dimensional vector space over $\mathbb{C}$. The cusps of $\Gamma_K(\rho)$ are the points $(\xi^{(1)}, \ldots, \xi^{(n)})$ of the boundary of H for $\xi \in K \cup \{\infty\}$. Since K is supposed to have class number 1, for each $\xi \in K \cup \{\infty\}$ there exists an $L \in \Gamma_K$ with $\xi = L(\infty)$ (if the class number is greater than 1, one has to take $L \in \mathrm{SL}(2, K)$). If $f \in M_r(\Gamma_K(\rho), 1)$ (for simplicity, we assume that $v \equiv 1$), $f \mid L$ has a Fourier expansion

$$f(\tau) \mid L = a(0; f, L) + \sum_{(1)\mid \nu, \nu/\rho\delta > 0} a(\nu/\rho\delta; f, L) e^{2\pi i \mathcal{S}(\nu/\rho\delta)\tau}, \tag{10}$$

the Fourier expansion of f at the cusp $\xi = L(\infty)$. The number $a(0; f, L)$ depends only on the cusp ξ and not on the special choice of L. f is called a cusp form if $a(0; f, L) = 0$ for every $\xi \in K \cup \{\infty\}$.

To define the Eisenstein series (for details see [**8**]), let $\rho_1, \rho_2 \in K$ be algebraic integers with $(\rho, \rho_1, \rho_2) = (1)$, $\tau \in H$, $s \in \mathbb{C}$, and

$$G_r^*(\tau, s; (\rho), \rho_1, \rho_2; K) = \sum_{\substack{\kappa_1 \equiv \rho_1 \bmod (\rho) \\ \kappa_2 \equiv \rho_2 \bmod (\rho) \\ (\kappa_1, \kappa_2)_\rho}} \mathcal{N}(\kappa_1\tau + \kappa_2)^{-r} |\mathcal{N}(\kappa_1\tau + \kappa_2)|^{-s}. \tag{11}$$

Here κ_1, κ_2 are integers from K; $(\kappa_1, \kappa_2)_\rho$ indicates that, from each equivalence class mod the units $\equiv 1 \bmod (\rho)$ ($(\kappa_1, \kappa_2) \sim (\mu_1, \mu_2)$ iff there exists a unit $\varepsilon \equiv 1 \bmod (\rho)$ with $\kappa_j \equiv \varepsilon\mu_j \bmod (\rho)$, $j = 1, 2$), only one element is to be taken and $(0, 0)$ is to be omitted. The series is convergent for $r + \mathrm{Re}\, s > 2$. The Fourier series

$$G_r^*(\tau, s; (\rho), \rho_1, \rho_2; K) = \sum_{(1)\mid \nu} a_r^*(\nu/\rho\delta; s, \mathrm{Im}\,\tau; (\rho), \rho_1, \rho_2; K) e^{2\pi i \mathcal{S}(\nu/\rho\delta)\tau}$$

is used for analytic continuation as a function of s into a region containing $s = 0$. Except for $n = 1$, $r = 2$, $\nu = 0$,

$$a_r^*(\nu/\rho\delta; 0, \mathrm{Im}\,\tau; (\rho), \rho_1, \rho_2; K) \text{ is } \begin{cases} 0 \text{ for } \nu/\rho\delta \text{ not} \geq 0, \\ \text{independent of } \mathrm{Im}\,\tau \text{ for } \nu/\rho\delta \geq 0. \end{cases}$$

Put

$$\begin{aligned} \tilde{G}_r(\tau; (\rho), \rho_1, \rho_2; K) &= \tilde{\kappa}_r^{-1}(\rho, K) G_r^*(\tau, 0; (\rho), \rho_1, \rho_2; K), \\ \tilde{\kappa}_r(\rho, K) &= (-2\pi i)^{nr}/[(r-1)!]^n d_K^{r-1/2} |\mathcal{N}(\rho)|^{r-1}. \end{aligned} \tag{12}$$

Then we have [**8**]

THEOREM 1. *Except for $n = 1$, $r = 2$, $\tilde{G}_r(\tau; (\rho), \rho_1, \rho_2; K) \in M_r(\Gamma_K(\rho), 1)$, and*

$$\tilde{G}_r(\tau; (\rho), \rho_1, \rho_2; K) = \tilde{a}_r(0; (\rho), \rho_1, \rho_2; K) + \sum_{\nu/\rho\delta > 0} \tilde{a}_r(\nu/\rho\delta; (\rho), \rho_1, \rho_2; K) e^{2\pi i \mathcal{S}(\nu/\rho\delta)\tau}.$$

The summation is over the integers $\nu \in K$. For $\nu \neq 0$, one has

$$\tilde{a}_r(\nu/\rho\delta; (\rho), \rho_1, \rho_2; K) = \sum_{\substack{\kappa \equiv \rho_1 \operatorname{mod}(\rho) \\ \kappa \mid \nu, (\kappa)_\rho}} \operatorname{sgn} \mathcal{N}(\kappa)^r \left| \mathcal{N}\left(\frac{\nu}{\kappa}\right) \right|^{r-1} e^{2\pi i \mathcal{S}(\nu\rho_2/\rho\delta\kappa)}.$$

If $r > 1$, $\tilde{a}_r(0; (\rho), \rho_1, \rho_2; K) = 0$ for $\rho \nmid \rho_1$.

The set $M_r^E(\Gamma_K(\rho), 1)$ of linear combinations of the Eisenstein series (for $n = 1, r = 2$ the set of holomorphic linear combinations) is a subspace of $M_r(\Gamma_K(\rho), 1)$, orthogonal with respect to the Petersson scalar product to the subspace $M_r^+(\Gamma_K(\rho), 1)$ of cusp forms. If $n = 1$, $M_r^E(\Gamma_K(\rho), 1)$ is the full orthogonal complement of $M_r^+(\Gamma_K(\rho, 1))$ (for $n > 1$, see [**3**]).

By (6) every $L \in \Gamma$ acts as a linear operator on $M_r(\Gamma_K(\rho), 1)$. We shall need the following lemma, which easily follows from (11).

Lemma 1. *Put*

$$(\hat{\rho}_1, \hat{\rho}_2) = (\rho_1, \rho_2)L = (\rho_1 a + \rho_2 c, \rho_1 b + \rho_2 d)$$

for $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$. Then

$$\tilde{G}_r(\tau; (\rho), \rho_1, \rho_2; K) \mid L = \tilde{G}_r(\tau; (\rho), \hat{\rho}_1, \hat{\rho}_2, K).$$

If ε is a unit,

$$\tilde{G}_r(\tau; (\rho), \varepsilon\rho_1, \varepsilon\rho_2; K) = \mathcal{N}(\varepsilon)^r \tilde{G}_r(\tau; (\rho), \rho_1, \rho_2; K).$$

Suppose that every residue class prime to (ρ) contains a unit. Then, according to Lemma 1, if ρ_1 is prime to (ρ), one may assume that $\rho_1 = 1$. If $\kappa \equiv \rho_1 \equiv 1 \operatorname{mod}(\rho)$,

$$e^{2\pi i \mathcal{S}(\nu\rho_2/\rho\delta\kappa)} = e^{2\pi i \mathcal{S}(\nu\rho_2/\rho\delta)} \qquad (\nu \text{ an integer from } K)$$

no longer depends on r or κ. If $(\rho) = (\gamma)^2$, $(\rho, \rho_1) = (\gamma)$, one may assume that $\rho_1 = \gamma$. If $\kappa \equiv \gamma \operatorname{mod}(\gamma)^2$, then $\gamma \mid \kappa$; it follows that

$$\tilde{a}_r(\nu/\rho\delta; (\gamma)^2, \gamma, \rho_2; K) = 0$$

for $\gamma \nmid \nu$. If $\gamma \mid \nu$, put $\nu = \nu_1\gamma$, $\kappa = \varepsilon\kappa_1\gamma$. Then, summing over κ with

$$\kappa \equiv \gamma \operatorname{mod}(\gamma)^2, \qquad \kappa \mid \nu, \qquad (\kappa)_\rho$$

is equivalent to summing over κ_1 and ε with

$$\kappa_1 \equiv 1 \operatorname{mod}(\gamma)^2, \qquad \kappa_1 \mid \nu_1, \qquad \varepsilon \text{ a unit, } \varepsilon \operatorname{mod}(\gamma)^2, \qquad \varepsilon \equiv 1 \operatorname{mod}(\gamma).$$

Hence we have

Theorem 2. *Suppose that every residue class prime to (ρ) contains a unit. For integers $\nu \neq 0$ from K, define*

$$c_r(\nu; \rho; K) = \sum_{\substack{\kappa \equiv 1 \operatorname{mod}(\rho) \\ \kappa \mid \nu, (\kappa)_\rho}} \operatorname{sgn} \mathcal{N}(\kappa)^r \left| \mathcal{N}\left(\frac{\nu}{\kappa}\right) \right|^{r-1}$$

and

$$w(\nu/\rho\delta;\rho_2;K)=e^{2\pi i\mathcal{S}(\nu\rho_2/\rho\delta)}.$$

Then

$$\tilde{a}_r(\nu/\rho\delta;(\rho),1,\rho_2;K)=w(\nu/\rho\delta;\rho_2;K)c_r(\nu;\rho;K).$$

If $(\rho)=(\gamma)^2,\ \gamma\mid\nu$,

$$w_r(\nu/\gamma^2\delta;\gamma,\rho_2;K)=\sum_{\substack{\varepsilon\equiv 1\,\mathrm{mod}(\gamma)\\ \varepsilon\,\mathrm{mod}(\gamma)^2,\varepsilon\text{ unit}}}\operatorname{sgn}\mathcal{N}(\varepsilon\gamma)^r e^{2\pi i\mathcal{S}(\nu\varepsilon\rho_2/\gamma^3\delta)}$$

depends on r only mod(2), *and*

$$\tilde{a}_r(\nu/\gamma^2\delta;(\gamma)^2,\gamma,\rho_2;K)=\begin{cases}0 & \text{for } \gamma\nmid\nu,\\ w_r(\nu/\gamma^2\delta;\gamma,\rho_2;K)c_r\nu/\gamma;\gamma^2;K) & \text{for } \gamma\mid\nu.\end{cases}$$

Suppose $\rho\in\mathbb{N}$, $\rho>2$. Then

$$\kappa\equiv 1\ \mathrm{mod}(\rho)\Rightarrow\mathcal{N}(\kappa)\equiv 1\ \mathrm{mod}(\rho).$$

If, under the assumption that every residue class prime to (ρ) contains a unit, γ is a prime in K, $\gamma\nmid\rho$, there is a unit ε in K with $\varepsilon\gamma\equiv 1\ \mathrm{mod}(\rho)$. Since $|\mathcal{N}(\gamma)|=p^l$, a power of a rational prime p, one deduces that

$$p^l=\pm 1\ \mathrm{mod}(\rho),\qquad \mathcal{N}(\varepsilon\gamma)=\begin{cases}p^l & \text{if } p^l\equiv 1\ \mathrm{mod}(\rho),\\ -p^l & \text{if } p^l\equiv -1\ \mathrm{mod}(\rho).\end{cases}$$

Now $c_r(\nu;\rho;K)$ can easily be calculated.

THEOREM 3. *Suppose that every residue class prime to* (ρ) *contains a unit. Then* $c_r(\nu;\rho;K)$ *is a multiplicative function of* ν, *i.e.*

$$c_r(\nu\mu;\rho;K)=c_r(\nu;\rho;K)c_r(\mu;\rho;K)\quad\text{for}\quad(\nu,\mu)=1.$$

Suppose further that $\rho\in\mathbb{N}$, $\rho>2$, *that* γ *is a prime in K, that* $|\mathcal{N}(\gamma)|=p^l$ *with p a rational prime, and that m is a positive integer. If* $\gamma\nmid\rho$, *then* $p^l\equiv\pm 1\ \mathrm{mod}(\rho)$, *and*

$$c_1(\gamma^m;\rho;K)=\begin{cases}m+1 & \text{for } p^l\equiv 1\ \mathrm{mod}(\rho),\\ 0 & \text{for } p^l\equiv -1\ \mathrm{mod}(\rho),\ 2\nmid m,\\ 1 & \text{for } p^l\equiv -1\ \mathrm{mod}(\rho),\ 2\mid m.\end{cases}$$

If $\gamma\nmid\rho$, $r>1$,

$$c_r(\gamma^m;\rho;K)=\begin{cases}\displaystyle\sum_{k=0}^{m}p^{kl(r-1)}=\frac{p^{l(m+1)(r-1)}-1}{p^{l(r-1)}-1} & \text{for } p^l\equiv 1\ \mathrm{mod}(\rho) \text{ or } 2\mid r,\\ \displaystyle\sum_{k=0}^{m}(-1)^{m-k}p^{kl(r-1)}=\frac{p^{l(m+1)(r-1)}+(-1)^m}{p^{l(r-1)}+1} & \text{for } p^l\equiv -1\ \mathrm{mod}(\rho) \text{ and } 2\nmid r.\end{cases}$$

If $\gamma\mid\rho$, *then*

$$c_r(\gamma^m;\rho,K)=p^{lm(r-1)}.$$

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If $2|r$ or $2|\mathcal{S}_v$ for every integer $v\in K$, these results remain valid for $\rho=2$ if the congruences are taken mod 4.

3. The restriction to the diagonal. In accordance with [4, §5] (with $\xi=1$), for a function f on H, and $z\in H_1=\{z \mid z\in\mathbb{C}, \operatorname{Im} z>0\}$, we define

$$\mathcal{S}_1 f(z)=f(\tau)|_{\tau^{(1)}=\ldots=\tau^{(n)}=z}. \tag{13}$$

Since $\Gamma_{\mathbb{Q}}\subset\Gamma_K$, and every $L\in\Gamma_{\mathbb{Q}}$ transforms the diagonal $\tau^{(1)}=\ldots=\tau^{(n)}$ of H into itself, we have (see (6))

$$\mathcal{S}_1(f\underset{r}{|}L)=(\mathcal{S}_1 f)\underset{nr}{|}L \quad \text{for} \quad L\in\Gamma_{\mathbb{Q}}. \tag{14}$$

THEOREM 4. *Suppose that $\rho\in\mathbb{N}$, $f\in M_r(\Gamma_K(\rho),1)$. Then*

$$\mathcal{S}_1 f\in M_{nr}(\Gamma_{\mathbb{Q}}(\rho),1),$$

and the coefficients of the Fourier expansion

$$\mathcal{S}_1 f(z)\mid L=a(0;\mathcal{S}_1 f,L)+\sum_{m=1}^{\infty}a(m/\rho;\mathcal{S}_1 f,L)e^{2\pi i(m/\rho)z}$$

of $\mathcal{S}_1 f$ at a cusp $L(\infty)$, $L\in\Gamma_{\mathbb{Q}}$, are obtained from the expansion (10) *of $f\mid L$ by*

$$a(0;\mathcal{S}_1 f,L)=a(0;f,L),$$

$$a(m/\rho;\mathcal{S}_1 f,L)=\sum_{\substack{\nu/\rho\delta>0\\ \mathcal{S}(\nu/\delta)=m}}a(\nu/\rho\delta;f,L).$$

Proof. (7) obviously remains valid for $\mathcal{S}_1 f$. (8) for $\mathcal{S}_1 f$ follows from (8), (14). The Fourier expansion of $\mathcal{S}_1 f\mid L$ is obtained from (10) by putting

$$\mathcal{S}(\nu/\delta)=m, \qquad \mathcal{S}(\nu/\rho\delta)\tau|_{\tau^{(1)}=\ldots=\tau^{(n)}=z}=(m/\rho)z.$$

THEOREM 5. *Suppose that $\rho\in\mathbb{N}$, $n>1$. Put $(\hat\rho_1,\hat\rho_2)=(\rho_1,\rho_2)L$ for $L\in\Gamma_{\mathbb{Q}}$. Then*

$$(\mathcal{S}_1\tilde G_r(z;(\rho),\rho_1,\rho_2;K))\mid L=\mathcal{S}_1\tilde G_r(z;(\rho),\hat\rho_1,\hat\rho_2;K) \qquad \text{for } L\in\Gamma_{\mathbb{Q}}.$$

If there is an automorphism of K, taking ρ_j to σ_j $(j=1,2)$, then

$$\mathcal{S}_1\tilde G_r(z;(\rho),\sigma_1,\sigma_2;K)=\mathcal{S}_1\tilde G_r(z;(\rho),\rho_1,\rho_2;K). \tag{15}$$

For every $r\in\mathbb{N}$ we have

$$\mathcal{S}_1\tilde G_r(z;(\rho),\rho_1,\rho_2;K)=\tilde G_r(\tau;(\rho),\rho_1,\rho_2;K)|_{\tau^{(1)}=\ldots=\tau^{(n)}=z}\in M_{nr}(\Gamma_{\mathbb{Q}}(\rho),1).$$

If $r>1$, and ρ_1, ρ_2 are linearly independent mod(ρ) *over $\mathbb{Z}$, $\mathcal{S}_1\tilde G_r(z;(\rho),\rho_1,\rho_2;K)$ is a cusp form.*

Proof. The transformation formula is obvious from (14) and Lemma 1. An automorphism of K only permutes the conjugates. Since $\rho\in\mathbb{N}$, from (11) it is clear that

$\mathcal{S}_1\tilde{G}_r(z;(\rho),\rho_1,\rho_2;K)$ remains unaltered by such a permutation. That $\mathcal{S}_1\tilde{G}_r\in M_{nr}(\Gamma_{\mathbb{Q}}(\rho),1)$ follows from Theorems 1, 4. If ρ_1,ρ_2 are linearly independent mod(ρ) over $\mathbb{Z}$, then $\rho\nmid\hat{\rho}_1$ for $(\hat{\rho}_1,\hat{\rho}_2)=(\rho_1,\rho_2)L$, $L\in\Gamma_{\mathbb{Q}}$, and by Theorem 4 and Theorem 1 for $r>1$,

$$a(0;\mathcal{S}_1\tilde{G}_r(\ ;(\rho),\rho_1,\rho_2;K),L)=\tilde{a}_r(0;(\rho),\hat{\rho}_1,\hat{\rho}_2;K)=0$$

for every $L\in\Gamma_{\mathbb{Q}}$, i.e., $\mathcal{S}_1\tilde{G}_r(z;(\rho),\rho_1,\rho_2;K)$ is a cusp form.

4. Theta functions. In this section, r is a positive integer, $f\mid L$ always stands for $f\mid_r L$ (see (6)). In the usual notation, for $z\in H_1=\{z\mid z\in\mathbb{C},\ \operatorname{Im} z>0\}$, write

$$\vartheta(z)=\vartheta_3(z)=\sum_{m=-\infty}^{\infty}e^{\pi i m^2 z}=1+2\sum_{m=1}^{\infty}e^{\pi i m^2 z}, \tag{16}$$

$$\vartheta_2(z)=2\sum_{m=0}^{\infty}e^{\pi i(m+1/2)^2 z,}\qquad \vartheta_4(z)=1+2\sum_{m=1}^{\infty}(-1)^m e^{\pi i m^2 z}, \tag{17}$$

$$E=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad U=\begin{pmatrix}1&1\\0&1\end{pmatrix},\qquad T=\begin{pmatrix}0&-1\\1&0\end{pmatrix}. \tag{18}$$

For $u\in\mathbb{R}$, $z\in H_1$, we have the well-known transformation formula

$$\vartheta(z;u)=\sum_{m=-\infty}^{\infty}e^{\pi i(m+u)^2 z}=e^{-\pi i/4}\sqrt{z}^{-1}\sum_{m=-\infty}^{\infty}e^{\pi i m^2(-1/z)+2\pi i m u} \tag{19}$$

$(0<\arg\sqrt{z}<\pi/2)$. From (16), (17), (19) we deduce that

$$\vartheta^{2r}\mid U=\vartheta_4^{2r},\qquad \vartheta_2^{2r}\mid U=i^r\vartheta_2^{2r},\qquad \vartheta_4^{2r}\mid U=\vartheta^{2r}, \tag{20}$$

$$\vartheta^{2r}\mid T=(-i)^r\vartheta^{2r},\qquad \vartheta_2^{2r}\mid T=(-i)^r\vartheta_4^{2r},\qquad \vartheta_4^{2r}\mid T=(-i)^r\vartheta_2^{2r}, \tag{21}$$

$$\vartheta^{2r}\mid U^2=\vartheta^{2r},\qquad \vartheta^{2r}\mid T=(-i)^r\vartheta^{2r}. \tag{22}$$

U^2 and T generate a subgroup Γ_ϑ of index 3 in $\Gamma_{\mathbb{Q}}$ (see e.g. [**15**]). If $\rho\in\mathbb{N}$, $2\mid\rho$, then $\Gamma_{\mathbb{Q}}(\rho)$ is a normal subgroup of Γ_ϑ. Γ_ϑ has two (inequivalent) cusps, $\infty=E(\infty)$ and $1=UT(\infty)$. By (20) and (21) we have that

$$\vartheta^{2r}(z)\mid UT=(-i)^r\vartheta_2^{2r}(z)=(-i)^r 2^{2r}e^{2\pi i(r/4)z}+\dots. \tag{23}$$

Lemma 2. *Let r be a positive integer. Then $\vartheta^{2r}\in M_r(\Gamma_\vartheta,v^{2r})$; the multiplier system v^{2r} is given by*

$$v^{2r}(U^2)=1,\qquad {}^{2r}(T)=(-i)^r.$$

ϑ^{2r} vanishes at the cusp $1=UT(\infty)$. We have $M_r(\Gamma_\vartheta,v^{2r})\subset M_r(\Gamma_{\mathbb{Q}}(4),1)$. If $2\mid r$, then $M_r(\Gamma_\vartheta,v^{2r})\subset M_r(\Gamma_{\mathbb{Q}}(2),1)$.

Proof. The conditions (7), (8), (9) of section 2 for ϑ^{2r} (and Γ_ϑ instead of $\Gamma_K(\rho)$) are derived as follows: the condition (7) is a consequence of (16); the condition (8) and the values given for v^{2r} are a consequence of (22) and the fact that U^2,T generate Γ_ϑ; the condition (9) follows from (16), (23) and the fact that $\infty=E(\infty)$ and $1=UT(\infty)$ are the (inequivalent) cusps of Γ_ϑ. By (23), the constant term in the Fourier expansion of ϑ^{2r} at $UT(\infty)$ is zero.

$\Gamma_Q(2)$ is a normal subgroup of Γ_ϑ, $U^2\in\Gamma_Q(2)$, $T^k\in\Gamma_Q(2)$ iff $2\mid k$. Moreover $v^4(U^2)=v^4(T^2)=1$. Hence $M_r(\Gamma_\vartheta, v^{2r})\subset M_r(\Gamma_Q(2), 1)$ for $2\mid r$. The group

$$\Gamma_Q^*(2)=\left\{L \mid L=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_Q,\ a\equiv d\equiv 1 \bmod(4),\ b\equiv c\equiv 0 \bmod(2)\right\}$$

is a normal subgroup of Γ_ϑ, and $\Gamma_Q(4)\subset\Gamma_Q^*(2)$. We have $U^2\in\Gamma_Q^*(2)$, $T^k\in\Gamma_Q^*(2)$ iff $4\mid k$. Moreover $v^2(U^2)=v^2(T^4)=1$. Hence $M_r(\Gamma_\vartheta, v^{2r})\subset M_r(\Gamma_Q^*(2), 1)\subset M_r(\Gamma_Q(4), 1)$.

LEMMA 3. *Suppose that $f\in M_r(\Gamma_Q(2), 1)$. Then*

$$f\in M_r(\Gamma_\vartheta, v^{2r})\Leftrightarrow f\mid T=(-1)^{r/2}f \quad \text{for } 2\mid r.$$

Suppose that $f\in M_r(\Gamma_Q(4), 1)$. Then

$$f\in M_r(\Gamma_\vartheta, v^{2r})\Leftrightarrow f\mid U^2=f, \qquad f\mid T=-i^r f \quad \text{for } 2\nmid r.$$

Proof. If $f\in M_r(\Gamma_Q(2), 1)$ or $f\in M_r(\Gamma_Q(4), 1)$, then the conditions (7), (9) of section 2 for elements of $M_r(\Gamma_\vartheta, v^{2r})$ are satisfied. Since Γ_ϑ is generated by U^2, T, we have that $f\in M_r(\Gamma_\vartheta, v^{2r})$ iff $f\mid U^2=f$, $f\mid T=(-i)^r f$. But $f\mid U^2=f$ for $f\in M_r(\Gamma_Q(2), 1)$.

From (18) we easily deduce that

$$UTUTUT=-E=T^2, \qquad UTU=T^3U^{-2}UT.$$

Suppose $f\in M_r(\Gamma_\vartheta, v^{2r})\subset M_r(\Gamma_Q(4), 1)$. Then (see Lemma 2)

$$(f(z)\mid UT)\mid U=f(z)\mid UTU=f(z)\mid T^3U^{-2}UT=(-i)^{3r}f(z)\mid UT.$$

This is impossible if the Fourier expansion of f at $1=UT(\infty)$ ((10) with $L=UT$, $\rho=4$, $\delta=1$, $\nu=m$) contains a non-vanishing term with $m\not\equiv r \bmod(4)$. Hence

$$f\in M_r(\Gamma_\vartheta, v^{2r})\Rightarrow a(m/4; f, UT)=0 \quad \text{for} \quad m\not\equiv r \bmod(4). \tag{24}$$

LEMMA 4. *Let r be a positive integer. Then*

$$\vartheta^{2r}=h_r+h_r^+, \qquad h_r\in M_r^E(\Gamma_\vartheta, v^{2r}), \qquad h_r^+\in M_r^+(\Gamma_\vartheta, v^{2r}). \tag{25}$$

If $2\mid r$, then h_r is, up to a constant factor, uniquely determined by

$$h_r\in M_r^E(\Gamma_Q(2), 1) \quad \text{and} \quad \begin{cases} h_r\mid T=-h_r & \text{for } 4\nmid r,\\ h_r\mid T=h_r,\ \ h_r \text{ vanishes at } UT(\infty) & \text{for } 4\mid r.\end{cases} \tag{26}$$

If $2\nmid r$, then h_r is, up to a constant factor, uniquely determined by

$$h_r\in M_r^E(\Gamma_Q(4), 1) \quad \text{and} \quad h_r\mid U^2=h_r, \qquad h_r\mid T=-i^r h_r. \tag{27}$$

Proof. Put $\rho=2$ for $2\mid r$, $\rho=4$ for $2\nmid r$. It is easily seen (compare Lemma 2) that $M_r^E(\Gamma_\vartheta, v^{2r})=M_r^E(\Gamma_Q(\rho), 1)\cap M_r(\Gamma_\vartheta, v^{2r})$. If $4\nmid r$, then, according to (24), every function from $M_r(\Gamma_\vartheta, v^{2r})$ vanishes at the cusp $1=UT(\infty)$, and $\dim_{\mathbb{C}} M_r^E(\Gamma_\vartheta, v^{2r})=1$. The conditions given for $4\nmid r$ are those of Lemma 3. If $4\mid r$, then $\dim_{\mathbb{C}} M_r^E(\Gamma_\vartheta, v^{2r})=2$, but ϑ^{2r} vanishes at $1=UT(\infty)$ (Lemma 2), hence h_r must vanish at $UT(\infty)$.

It is well-known and easy to show that

$$\dim_{\mathbb{C}} M_r^+(\Gamma_\vartheta, v^{2r}) = [\tfrac{1}{8}(2r-1)] = \begin{cases} 0 & \text{for } r \le 4, \\ 1 & \text{for } 5 \le r \le 8, \\ 2 & \text{for } 9 \le r \le 12. \end{cases}$$

Suppose we are given a basis $\hat{g}_r$ $(5 \le r \le 8)$, or $\hat{g}_r, \hat{f}_r$ $(9 \le r \le 12)$ of $M_r^+(\Gamma_\vartheta, v^{2r})$. In order to calculate the coefficients in

$$h_r^+ = b(\hat{g}_r)\hat{g}_r \quad (5 \le r \le 8) \quad \text{or} \quad h_r^+ = b(\hat{g}_r)\hat{g}_r + b(\hat{f}_r)\hat{f}_r \quad (9 \le r \le 12), \tag{28}$$

we need only know the coefficients for one $(5 \le r \le 8)$ or two $(9 \le r \le 12)$ suitably chosen exponents in the Fourier expansions of these functions at some cusp. We can, e.g., take

$$a(1/2; f, E) \quad (5 \le r \le 8), \qquad a(1/2; f, E) \text{ and } a(1; f, E) \quad (9 \le r \le 12). \tag{29}$$

Suppose further that we are given a function $\tilde{h}_r$, satisfying the conditions given in Lemma 4 for h_r. Then (provided $\tilde{h}_r \ne 0$),

$$h_r = b(\tilde{h}_r)\tilde{h}_r. \tag{30}$$

In order to calculate $b(\tilde{h}_r)$ and h_r^+ at the same time, we need the Fourier coefficients for one more exponent, in addition to (29). Since

$$a(m/4; \vartheta^{2r}, UT) = 0 \quad \text{for} \quad m \le 4, \qquad r \ge 5 \tag{31}$$

(see (23)), we take (compare (24))

$$a(m_0/4; f, UT) \quad \text{with} \quad m_0 \equiv r \bmod (4), \qquad 1 \le m_0 \le 4. \tag{32}$$

TABLE 1. FOURIER COEFFICIENTS (29), (32) FOR CERTAIN POWERS ϑ^{2r} OF ϑ^2.

f	ϑ^{12}	ϑ^{16}	ϑ^{18}	ϑ^{20}	ϑ^{24}	ϑ^{28}	ϑ^{32}
$a(1/2; f, E)$	$2^3.3$	2^5	$2^2.3^2$	$2^3.5$	$2^4.3$	$2^3.7$	2^6
$a(1; f, E)$	$2^3.3.11$	$2^5.3.5$	$2^2.3^2.17$	$2^3.5.19$	$2^4.3.23$	$2^3.3^3.7$	$2^6.31$
$a(m_0/4; f, UT)$	0	0	0	0	0	0	0

5. Eisenstein series for $\Gamma_{\mathbb{Q}}(2)$ for $2 \mid r$. We have $\dim_{\mathbb{C}} M_r^E(\Gamma_{\mathbb{Q}}(2), 1) = 3$ for $2 \mid r$, $r > 2$, and 3 Eisenstein series, corresponding to

$$(\rho_1, \rho_2) = (1, 0), (0, 1), (1, 1).$$

We have to look for eigenfunctions of T with eigenvalues ± 1 (see Lemma 4). Using the transformation formulae of Lemma 1 we see that, for eigenvalue 1, T has eigenfunctions

$$\tilde{G}_r(z; (2), 1, 0; \mathbb{Q}) + \tilde{G}_r(z; (2), 0, 1; \mathbb{Q}), \quad \tilde{G}_r(z; (2), 1, 1; \mathbb{Q}),$$

and, for eigenvalue -1, T has eigenfunction

$$\tilde{G}_r(z; (2), 1, 0; \mathbb{Q}) - \tilde{G}_r(z; (2), 0, 1; \mathbb{Q}).$$

Now, according to Theorem 1, $\tilde{G}_r(z; (2), 1, 1; \mathbb{Q})$ (for $r > 2$) vanishes at the cusp $\infty = E(\infty)$, and, of course, at $0 = T(\infty)$. Being an Eisenstein series $\ne 0$, it is not a cusp form and

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cannot, therefore, vanish at $1=UT(\infty)$, as is required in (26) of Lemma 4. Put

$$\tilde{h}_r(z)=\tilde{G}_r(z;(2),1,0;\mathbb{Q})+(-1)^{r/2}\tilde{G}_r(z;(2),0,1;\mathbb{Q}). \tag{33}$$

Then

$$\tilde{h}_r(z)\mid UT=\tilde{G}_r(z;(2),1,1,\mathbb{Q})+(-1)^{r/2}\tilde{G}_r(z;(2),1,0;\mathbb{Q}), \tag{34}$$

showing that $\tilde{h}_r$ vanishes at the cusp $1=UT(\infty)$. Hence

THEOREM 6. *Suppose that $r\in\mathbb{N}$, $2\mid r$. Define $\tilde{h}_r$ by* (33). *Then $\tilde{h}_r$ is a function satisfying the conditions* (26) *of Lemma* 4 *for h_r. Consequently, h_r is, up to a constant factor, equal to $\tilde{h}_r$.*

The coefficients of the Fourier expansion of $\tilde{h}_r$ at the cusp $\infty=E(\infty)$ can be calculated directly from Theorem 1. Since

$$(\kappa+1)\frac{r}{2}\equiv\frac{r}{2}\bmod(2)\quad\text{for}\quad 2\mid\kappa,\qquad(\kappa+1)\frac{r}{2}\equiv0,\qquad\kappa\equiv\frac{m}{\kappa}\bmod(2)\quad\text{for}\quad 2\nmid\kappa$$

we get (note that the second sum only occurs for $2\mid m$, i.e. for $(-1)^m=1$)

$$\begin{aligned}a(m/2;\tilde{h}_r,E)&=(-1)^m\left(\sum_{\substack{\kappa=1\\ \kappa\mid m,2\nmid\kappa}}^{\infty}(-1)^{(\kappa+1)r/2+m/\kappa}\left(\frac{m}{\kappa}\right)^{r-1}+\sum_{\substack{\kappa=1\\ \kappa\mid m,2\mid\kappa}}^{\infty}(-1)^{(\kappa+1)r/2+m/\kappa}\left(\frac{m}{\kappa}\right)^{r-1}\right)\\&=(-1)^m\sum_{d\mid m}(-1)^{d+(m/d+1)r/2}d^{r-1}.\end{aligned}$$

Thus

$$a(m/2;\tilde{h}_r,E)=\begin{cases}(-1)^m\sum\limits_{d\mid m}(-1)^d\,d^{r-1} & \text{for } 4\mid r,\\ (-1)^m\sum\limits_{d\mid m}(-1)^{d+m/d+1}d^{r-1} & \text{for } 4\nmid r.\end{cases} \tag{35}$$

Using $w(m/2;\rho_2;\mathbb{Q})=(-1)^{m\rho_2}$ (Theorem 2), from (34) and Theorem 2 we find that the coefficients of the Fourier expansion of $\tilde{h}_r$ at $1=UT(\infty)$ are given by

$$a(m/2;\tilde{h}_r,UT)=((-1)^m+(-1)^{r/2})c_r(m;2;\mathbb{Q}),$$

i.e.

$$a(m/2;\tilde{h}_r,UT)=\begin{cases}2(-1)^m c_r(m;2,\mathbb{Q}) & \text{for } m\equiv\frac{r}{2}\bmod(2),\\ 0 & \text{for } m\not\equiv\frac{r}{2}\bmod(2).\end{cases} \tag{36}$$

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TABLE 2. FOURIER COEFFICIENTS (29), (32) FOR $\tilde{h}_r$ ($m_0 \equiv r/2 \bmod(2)$, $1 \le m_0 \le 2$).

r	6	8	10	12	14	16
$a(1/2; \tilde{h}_r, E)$	1	1	1	1	1	1
$a(1; \tilde{h}_r, E)$	3.11	127	$3^3.19$	23.89	3.2731	7.31.151
$a(m_0/2; \tilde{h}_r, UT)$	−2	2^8	−2	2^{12}	−2	2^{16}

6. Eisenstein series for $\Gamma_{\mathbb{Q}}(4)$ for $2 \nmid r$. We have $\dim_{\mathbb{C}} M_r^E(\Gamma_{\mathbb{Q}}(4), 1) = 6$ for $r > 2$ and 6 Eisenstein series, corresponding to

$$(\rho_1, \rho_2) = (1, 0), (1, 1), (1, -1), (1, 2), (2, 1), (0, 1).$$

We have to look for simultaneous eigenfunctions of T with eigenvalues $\pm i$ and of U^2 with eigenvalue 1 (see Lemma 4). Using the transformation formulae of Lemma 1 we see that eigenfunctions of T with eigenvalue $\pm i$ are

$$\tilde{G}_r(z; (4), 1, 0; \mathbb{Q}) \pm i\tilde{G}_r(z; (4), 0, 1; \mathbb{Q}), \qquad \tilde{G}_r(z; (4), 1, 2; \mathbb{Q}) \pm i\tilde{G}_r(z; (4), 2, 1; \mathbb{Q}),$$

and

$$\tilde{G}_r(z; (4), 1, -1; \mathbb{Q}) \pm i\tilde{G}_r(z; (4), 1, 1; \mathbb{Q}).$$

Checking the action of the transformation U^2 on these functions, we have

THEOREM 7. *Suppose $r \in \mathbb{N}$, $2 \nmid r$. Then*

$$\tilde{h}_r(z) = \tilde{G}_r(z; (4), 1, 0; \mathbb{Q}) + \tilde{G}_r(z; (4), 1, 2; \mathbb{Q}) - i^r(\tilde{G}_r(z; (4), 0, 1; \mathbb{Q}) + \tilde{G}_r(z; (4), 2, 1; \mathbb{Q}))$$

is a function satisfying the conditions of (27) (*Lemma* 4) *for h_r. Consequently, h_r is, up to a constant factor, equal to $\tilde{h}_r$.*

In order to calculate the coefficients of the Fourier expansion of $\tilde{h}_r$ at $\infty = E(\infty)$, put $m = 2^t m_1$, $2 \nmid m_1$. From Theorem 1 we find that, for $2 \nmid r$,

$$a_r(m/4; (4), 2, 1; \mathbb{Q}) = \sum_{\substack{\kappa_1 \equiv 1 \bmod(4) \\ 2\kappa_1 | m, \kappa_1 \in \mathbb{Z}}} \operatorname{sgn} \kappa_1 \left(\frac{m}{2\kappa_1}\right)^{r-1} (i^{m/2\kappa_1} - i^{-m/2\kappa_1}),$$

i.e.,

$$a_r(m/4; (4), 2, 1; \mathbb{Q}) = \begin{cases} 2i^{m_1} c_r(m_1; 4; \mathbb{Q}) & \text{for} \quad t = 1, \\ 0 & \text{for} \quad t \neq 1, \end{cases} \tag{37}$$

and

$$a_r(m/4; (4), 0, 1; \mathbb{Q}) = \sum_{\substack{\kappa_1 \equiv 1 \bmod(4) \\ 2^{s+1}\kappa_1 | m, s \in \mathbb{N}, \kappa_1 \in \mathbb{Z}}} \operatorname{sgn} \kappa_1 \left(\frac{m}{2^{s+1}\kappa_1}\right)^{r-1} (i^{m/2^{s-1}\kappa_1} - i^{-m/2^{s-1}\kappa_1})$$

i.e.,

$$a_r(m/4;(4),0,1;\mathbb{Q})=\begin{cases}2i^{m_1}c_r(m_1;4;\mathbb{Q}) & \text{for} \quad t\geq 2,\\ 0 & \text{for} \quad t<2.\end{cases} \tag{38}$$

Suppose $2\mid m$. Put $m=2^t m_1$, $2\nmid m_1$, $t\geq 1$. Since (Theorem 2)

$$w(m/4;\rho_2;\mathbb{Q})=i^{m\rho_2}=1 \quad \text{for} \quad 2\mid m, \qquad 2\mid \rho_2,$$

and $2\mid(r+m_1)$ for $2\nmid m_1$, $2\nmid r$, from (37), (38) and Theorem 2, we deduce that

$$a(m/4;\tilde{h}_r,E)=c_r(m;4,\mathbb{Q})-2(-1)^{(r+m_1)/2}c_r(m_1;4,\mathbb{Q}) \quad \text{for} \quad 2\mid m. \tag{39}$$

Since $\tilde{h}_r\mid U^2=\tilde{h}_r$, we have

$$a(m/4;\tilde{h}_r,E)=0 \quad \text{for} \quad 2\nmid m. \tag{40}$$

The transformation formulae of Lemma 1 show that

$$\tilde{h}_r(z)\mid UT=\tilde{G}_r(z;(4),1,-1;\mathbb{Q})-\tilde{G}_r(z;(4),1,1;\mathbb{Q})$$
$$-i^r(\tilde{G}_r(z;(4),1,0;\mathbb{Q})-\tilde{G}_r(z;(4),1,2;\mathbb{Q})).$$

Using Theorem 2, we find that the coefficients of the Fourier expansion of $\tilde{h}_r$ at $1=UT(\infty)$ are given by

$$a(m/4;\tilde{h}_r,UT)=\begin{cases}-4i^rc_r(m;4;\mathbb{Q}) & \text{for} \quad m\equiv r \bmod(4)\\ 0 & \text{for} \quad m\not\equiv r \bmod(4).\end{cases} \tag{41}$$

TABLE 3. FOURIER COEFFICIENTS (29), (32) FOR $\tilde{h}_r$ ($m_0\equiv r \bmod(4)$, $1\leq m_0\leq 4$).

r	5	7	9	11	15
$a(1/2;\tilde{h}_r,E)$	2.17	2.3².7	2.257	2.3.11.31	2.3.43.127
$a(1;\tilde{h}_r,E)$	2.257	2.3².5.7.13	2.65537	2.3.5².11.31.41	2.3.5.29.43.113.127
$a(m_0/4;\tilde{h}_r,UT)$	$-4i$		$-4i$		

7. Eisenstein series for $\Gamma_K(2)$ for $K=\mathbb{Q}(\sqrt{2})$. In this section $K=\mathbb{Q}(\sqrt{2})$, $\rho=2$, $\delta=2\sqrt{2}$, $\delta^{(1)}>0$, $\delta^{(2)}<0$ (see section 2). The class number of K is 1, a fundamental unit is $\varepsilon_0=1+\sqrt{2}$, residues mod(2) are

$$0,1,\varepsilon_0,\sqrt{2}.$$

Every residue class prime to (2) contains a unit (this we need for Theorems 2 and 3). Since $\mathcal{N}(\varepsilon)=1$ for every unit $\varepsilon\equiv 1 \bmod(2)$, r can be any positive integer (see section 2). We have 6 Eisenstein series, corresponding to

$$(\rho_1,\rho_2)=(1,0),(0,1),(1,1) \quad \text{and} \quad (1,\varepsilon_0),(1,\sqrt{2}),(\sqrt{2},1).$$

The last three pairs have ρ_1,ρ_2 independent mod(2) over $\mathbb{Z}$; in these cases, according to Theorem 5, the functions $\mathcal{S}_1\tilde{G}_r(z;(2),\rho_1,\rho_2;K)$ for $r>1$ are cusp forms from $M_{2r}(\Gamma_{\mathbb{Q}}(2),1)$. From Lemma 1 and Theorem 5 we deduce that the functions

$$g_r(\tau;K)=\tilde{G}_r(\tau;(2),1,\varepsilon_0;K), \tag{42}$$

$$f_r(\tau;K)=\tilde{G}_r(\tau;(2),1,\sqrt{2};K)+(-1)^r\tilde{G}_r(\tau;(2),\sqrt{2},1;K) \tag{43}$$

and their restrictions to the diagonal ($\mathscr{S}_1 g_r$ and $\mathscr{S}_1 f_r$) are eigenfunctions of T with eigenvalue $(-1)^r$. Hence, according to Lemma 3, we have

$$\mathscr{S}_1 g_r(z;K), \qquad \mathscr{S}_1 f_r(z;K) \in M^+_{2r}(\Gamma_\vartheta, v^{4r}). \tag{44}$$

From Lemma 1 we get

$$g_r(\tau;K) \mid UT = \tilde{G}_r(\tau;(2),\sqrt{2},1;K),$$
$$f_r(\tau;K) \mid UT = (-1)^r \tilde{G}_r(\tau;(2),1,\varepsilon_0;K) + \tilde{G}_r(\tau;(2),1,\sqrt{2};K),$$

and, from Theorem 5,

$$(\mathscr{S}_1 g_r) \underset{2r}{|} UT = \mathscr{S}_1\Big(g_r(*;K) \underset{r}{|} UT\Big), \qquad (\mathscr{S}_1 f_r) \underset{2r}{|} UT = \mathscr{S}_1\Big(f_r(*;K) \underset{r}{|} UT\Big). \tag{45}$$

In order to calculate the Fourier coefficients, put $\nu = a + b\sqrt{2}$. Then, from Theorem 2, we have

$$w\left(\frac{a+b\sqrt{2}}{4\sqrt{2}};\rho_2;K\right) = \begin{cases}(-1)^a & \text{for} \quad \rho_2 = \sqrt{2}, \\ (-1)^{a+b} & \text{for} \quad \rho_2 = \varepsilon_0,\end{cases}$$

and, for $\sqrt{2} \mid \nu$, i.e. $2 \mid a$,

$$w_r\left(\frac{a+b\sqrt{2}}{4\sqrt{2}};\sqrt{2},1;K\right) = (-1)^r(-1)^{\mathscr{S}(\nu/4)} + (-1)^{\mathscr{S}(\nu\varepsilon_0/4)}$$
$$= \begin{cases}0 & \text{for} \quad 2 \nmid (b-r), \\ 2(-1)^{(2r+a)/2} & \text{for} \quad 2 \mid (b-r).\end{cases}$$

Theorem 2 (with $\gamma = \sqrt{2}$) now gives the following values for the coefficients of the Fourier expansions at $\infty = E(\infty)$ and $1 = UT(\infty)$:

$$a\left(\frac{a+b\sqrt{2}}{4\sqrt{2}};g_r,E\right) = (-1)^{a+b} c_r(a+b\sqrt{2};2;K),$$

$$a\left(\frac{a+b\sqrt{2}}{4\sqrt{2}};g_r,UT\right) = \begin{cases}0 & \text{for} \quad 2 \nmid a \text{ or } 2 \nmid (b-r), \\ 2(-1)^{(2r+a)/2} c_r((a+b\sqrt{2})/\sqrt{2};2;K) & \text{for} \quad 2 \mid a \text{ and } 2 \mid (b-r),\end{cases}$$

and

$$a\left(\frac{a+b\sqrt{2}}{4\sqrt{2}};f_r,E\right) = \begin{cases}(-1)^a c_r(a+b\sqrt{2};2;K) & \text{for} \quad 2 \nmid a \text{ or } 2 \nmid (b-r), \\ c_r(a+b\sqrt{2};2;K) + 2(-1)^{a/2} c_r((a+b\sqrt{2})/\sqrt{2};2;K) & \\ & \text{for} \quad 2 \mid a \text{ and } 2 \mid (b-r),\end{cases}$$

$$a\left(\frac{a+b\sqrt{2}}{4\sqrt{2}};f_r,UT\right) = \begin{cases}0 & \text{for} \quad 2 \nmid (b-r), \\ 2(-1)^a c_r(a+b\sqrt{2};2;K) & \text{for} \quad 2 \mid (b-r).\end{cases}$$

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If f is a modular form for $\Gamma_K(2)$, Theorem 4 gives the Fourier coefficients of $\mathscr{S}_1 f$ in terms of the Fourier coefficients of f. Put $\nu = a + b\sqrt{2}$. Then

$$\mathscr{S}(\nu/2\sqrt{2}) = b, \qquad \nu/2 \,.\, 2\sqrt{2} > 0 \Leftrightarrow a^2 < 2b^2, \qquad b > 0.$$

For the coefficients of $\mathscr{S}_1 g_r$ and $\mathscr{S}_1 f_r$ at the cusp $\infty = E(\infty)$ we find

$$a\left(\frac{b}{2}; \mathscr{S}_1 g_r, E\right) = (-1)^b \left(\sum_{\substack{a^2<2b^2 \\ 2|a, a\in\mathbf{Z}}} c_r(a+b\sqrt{2}; 2; K) - \sum_{\substack{a^2<2b^2 \\ 2\nmid a, a\in\mathbf{Z}}} c_r(a+b\sqrt{2}; 2; K) \right),$$

$$a\left(\frac{b}{2}; \mathscr{S}_1 f_r, E\right) = \begin{cases} \displaystyle\sum_{\substack{a^2<2b^2 \\ 2|a, a\in\mathbf{Z}}} c_r(a+b\sqrt{2}; 2; K) - \sum_{\substack{a^2<2b^2 \\ 2\nmid a, a\in\mathbf{Z}}} c_r(a+b\sqrt{2}; 2; K) & \text{for } 2\nmid(b-r), \\ \displaystyle\sum_{\substack{a^2<2b^2 \\ a\in\mathbf{Z}}} (-1)^a c_r(a+b\sqrt{2}; 2; K) + 2 \sum_{\substack{a^2<2b^2 \\ 2|a, a\in\mathbf{Z}}} (-1)^{a/2} c_r\left(\frac{a+b\sqrt{2}}{\sqrt{2}}; 2; K\right) & \text{for } 2\,|\,(b-r). \end{cases}$$

From (44), and (24) of section 4, we have

$$a(b/2; \mathscr{S}_1 g_r, UT) = a(b/2; \mathscr{S}_1 f_r, UT) = 0 \quad \text{for} \quad 2\nmid(b-r).$$

From (45) and Theorem 5 we deduce that, for $2\,|\,(b-r)$,

$$a\left(\frac{b}{2}; \mathscr{S}_1 g_r, UT\right) = 2(-1)^r \left(\sum_{\substack{a^2<2b^2 \\ 4|a, a\in\mathbf{Z}}} c_r\left(\frac{a+b\sqrt{2}}{\sqrt{2}}; 2; K\right) - \sum_{\substack{a^2<2b^2 \\ 2|a, 4\nmid a, a\in\mathbf{Z}}} c_r\left(\frac{a+b\sqrt{2}}{\sqrt{2}}; 2; K\right) \right),$$

$$a\left(\frac{b}{2}; \mathscr{S}_1 f_r, UT\right) = 2\left(\sum_{\substack{a^2<2b^2 \\ 2|a, a\in\mathbf{Z}}} c_r(a+b\sqrt{2}; 2; K) - \sum_{\substack{a^2<2b^2 \\ 2\nmid a, a\in\mathbf{Z}}} c_r(a+b\sqrt{2}; 2; K) \right).$$

In order to calculate the coefficients (29), (32) of section 4 (with $2r$ instead of r because of (44)) for these functions, let γ_7 stand for an integer from K with $|\mathscr{N}(\gamma_7)| = 7$. For $2\nmid r$ we find that

$$a(1/2); \mathscr{S}_1 g_r, E) = 2c_r(1; 2; K) - c_r(\sqrt{2}; 2; K), \qquad a(1/2; \mathscr{S}_1 f_r, E) = c_r(\sqrt{2}; 2; K),$$
$$a(1; \mathscr{S}_1 g_r, E) = a(1; \mathscr{S}_1 f_r, E) = c_r(2\sqrt{2}; 2; K) + 2c_r(2; 2; K) - 2c_r(\gamma_7; 2; K),$$
$$a(1/2; \mathscr{S}_1 g_r, UT) = -2c_r(1; 2; K), \qquad a(1/2; \mathscr{S}_1 f_r, UT) = 2c_r(\sqrt{2}; 2; K) - 4c_r(1; 2; K).$$

For $2\,|\,r$ we get that

$$a(1/2; \mathscr{S}_1 g_r, E) = -a(1/2; \mathscr{S}_1 f_r, E) = 2c_r(1; 2; K) - c_r(\sqrt{2}; 2; K),$$
$$a(1; \mathscr{S}_1 g_r, E) = c_r(2\sqrt{2}; 2; K) + 2c_r(2; 2; K) - 2c_r(\gamma_7; 2; K),$$
$$a(1; \mathscr{S}_1 f_r, E) = a(1; \mathscr{S}_1 g_r, E) + 2c_r(2; 2; K) - 4c_r(\sqrt{2}; 2; K),$$
$$a(1; \mathscr{S}_1 g_r, UT) = 2c_r(2; 2; K) - 4c_r(\sqrt{2}; 2; K),$$
$$a(1; \mathscr{S}_1 f_r, UT) = 2c_r(2\sqrt{2}; 2; K) + 4c_r(2; 2; K) - 4c_r(\gamma_7; 2; K).$$

Inserting the values $c_r(\nu; 2; K)$ from Theorem 3 we obtain

$$a(1/2; \mathcal{S}_1 g_r, UT) = -2 \quad \text{for} \quad 2 \nmid r, \qquad a(1/2; \mathcal{S}_1 g_r, UT) = 2 - 2^{r-1} \neq 0 \quad \text{for} \quad 2 \mid r, \qquad r > 2.$$

If $2 \nmid r$, $r \geq 5$, we find that

$$\begin{pmatrix} a(1/2; \mathcal{S}_1 g_r, E) & a(1/2; \mathcal{S}_1 f_r, E) \\ a(1/2; \mathcal{S}_1 g_r, UT) & a(1/2; \mathcal{S}_1 f_r, UT) \end{pmatrix} \equiv \begin{pmatrix} 2 & 0 \\ -2 & 4 \end{pmatrix} \mathrm{mod}(2^4).$$

If $2 \mid r$, $r \geq 6$, we have

$$\begin{pmatrix} a(1/2; \mathcal{S}_1 g_r, E) & a(1/2; \mathcal{S}_1 f_r, E) \\ a(1; \mathcal{S}_1 g_r, UT) & a(1; \mathcal{S}_1 f_r, UT) \end{pmatrix} \equiv \begin{pmatrix} 2 - 2^{r-1} & -2 + 2^{r-1} \\ 0 & -(r-1)2^5 \end{pmatrix} \mathrm{mod}(2^6).$$

Taking into account (44) and the dimension of $M_{2r}(\Gamma_\vartheta, v^{4r})$ (see section 4) we get

THEOREM 8. *Define* $g_r(\tau; \mathbb{Q}(\sqrt{2}))$ *by* (42), $f_r(\tau; \mathbb{Q}(\sqrt{2}))$ *by* (43). *Then*

(*i*) $\mathcal{S}_1 g_r(*; \mathbb{Q}(\sqrt{2}))$, $\mathcal{S}_1 f_r(*; \mathbb{Q}(\sqrt{2})) \in M_{2r}^+(\Gamma_\vartheta, v^{4r})$ *for* $r > 1$,

(*ii*) $\mathcal{S}_1 g_r(*; \mathbb{Q}(\sqrt{2})) \not\equiv 0$ *for* $r \neq 2$, $r > 1$,

(*iii*) $\mathcal{S}_1 g_r(*; \mathbb{Q}(\sqrt{2}))$, $\mathcal{S}_1 f_r(*; \mathbb{Q}(\sqrt{2}))$ *are linearly independent for* $r \geq 5$.

Further, $M_{2r}^+(\Gamma_\vartheta, v^{4r})$ *is generated by* $\mathcal{S}_1 g_r(*; \mathbb{Q}(\sqrt{2}))$ *for* $r = 3, 4$, *and by* $\mathcal{S}_1 g_r(*; \mathbb{Q}(\sqrt{2}))$ *and* $\mathcal{S}_1 f_r(*; \mathbb{Q}(\sqrt{2}))$ *for* $r = 5, 6$.

TABLE 4. FOURIER COEFFICIENTS (29), (32) ($m_0 \equiv r \bmod(2)$, $1 \leq m_0 \leq 2$).

r	3	4	5	6	7	8
$a(1/2; \mathcal{S}_1 g_r(*; \mathbb{Q}(\sqrt{2})), E)$	-2	-2.3	-2.7	$-2.3.5$	-2.31	$-2.3^2.7$
$a(1; \mathcal{S}_1 g_r(*; \mathbb{Q}(\sqrt{2})), E)$	0	$-2^4.3$	$-2^6.3$	$2^4.3.5^2$	$2^5.3.5.73$	$2^4.3^2.7.479$
$a(m_0/2; \mathcal{S}_1 g_r(*; \mathbb{Q}(\sqrt{2})), UT)$	-2	$2^5.3$	-2	$2^7.3.5$	-2	$2^9.3^2.7$
$a(1/2; \mathcal{S}_1 f_r(*; \mathbb{Q}(\sqrt{2})), E)$	2^2	2.3	2^4	$2.3.5$	2^6	$2.3^2.7$
$a(1; \mathcal{S}_1 f_r(*; \mathbb{Q}(\sqrt{2})), E)$	0	$2^4.3$	$-2^6.3$	$2^4.3.5.13$	$2^5.3.5.73$	$2^4.3^2.7^2.73$
$a(m_0/2; \mathcal{S}_1 f_r(*; \mathbb{Q}(\sqrt{2})), UT)$	2^2	$-2^5.3$	$2^2.7$	$2^5.3.5^2$	$2^2.31$	$2^5.3^2.7.479$

8. The Eisenstein series for $\Gamma_K(4)$ for $[K:\mathbb{Q}] = 3$ and $d_K = 7^2$. In this section K is the totally real cubic number field of smallest discriminant, namely the subfield $K = \mathbb{Q}(\zeta_7 + \zeta_7^{-1}) \subset \mathbb{Q}(\zeta_7)$ of the field of 7-th roots of unity. We have $d_K = 7^2$. Since we shall be interested only in odd values of r (especially $r = 3$), we assume that $2 \nmid r$. For $k = 1, 2, 3$ we define

$$\xi_k = \zeta_7^k + \zeta_7^{-k}, \qquad \eta_k = 2 - \xi_k, \tag{46}$$

$$\xi_j = \xi_k, \qquad \eta_j = \eta_k \quad \text{for} \quad j \in \mathbb{Z}, \qquad j \equiv k \bmod(3). \tag{47}$$

We shall need the following facts about K.

LEMMA 5. *For* $j \in \mathbb{Z}$, *let* ξ_j, η_j *be defined by* (47). *Then* $K = \mathbb{Q}(\xi_j)$. *The automorphism group of* K *is generated by* $\xi_j \mapsto \xi_{j+1}$. *Also,* ξ_j *is a unit, the characteristic polynomial of* ξ_j *is* $x^3 + x^2 - 2x - 1$, *and hence*

$$\mathcal{S}(\xi_j) = -1, \qquad \mathcal{S}(\xi_j \xi_{j+1}) = \mathcal{S}(\xi_j^{-1}) = -2, \qquad \mathcal{N}(\xi_j) = 1.$$

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An integral basis of K is $1, \eta_2, \eta_3$. *The rational prime* 7 *is completely ramified in K,* $(7)=(\eta_j)^3$, *and* $(\eta_1)=(\eta_2)=(\eta_3)$. *Put* $\delta=\eta_2\eta_3$. *Then δ is totally positive and (δ) is the different of K. We have*

$$\nu = a + b\eta_2 + c\eta_3, \qquad a, b, c \in \mathbb{Z} \Rightarrow \mathcal{S}(\nu/\delta) = a + 2(b+c).$$

Proof. Since the automorphism group of $\mathbb{Q}(\zeta_7)$ is generated by $\zeta_7 \mapsto \zeta_7^5$, we see that the automorphism group of K is generated by $\xi_j \mapsto \xi_{j+1}$. It is easily verified that ξ_1 is a root of x^3+x^2-2x-1. Since $\zeta_7, \zeta_7^2, \ldots, \zeta_7^6$ is an integral basis of $\mathbb{Q}(\zeta_7)$, ξ_1, ξ_2, ξ_3 is an integral basis of K which is easily modified to $1, \eta_2, \eta_3$. We find $\mathcal{N}(\eta_j)=7$. Also, η_1, η_2, η_3 are conjugate to each other and, of course (from (46)), totally positive. Since K is a cyclic field, $d_K=7^2$, the prime 7 is completely ramified and hence $(\eta_1)=(\eta_2)=(\eta_3)$. The remaining statements of Lemma 5 are obvious.

Lemma 6. *For $j\in\mathbb{Z}$, let ξ_j be defined by* (47). *The multiplicative group of residue classes prime to* (4) *in K is the direct product of a cyclic group of order* 14 *and two groups of order* 2. *A set of generators is given by*

$$\xi_j \textit{ of order } 14,\ \xi_{j+1}^7 \textit{ and } -1 \textit{ of order } 2 \bmod(4).$$

Every residue class prime to (4) *contains a unit.*

Proof. The characteristic polynomial of ξ_j (Lemma 5) is irreducible over $\mathbb{Z}/(2)$. Hence ξ_j generates the field of residue classes mod(2) in K. The residue classes mod(2) are given by $0, 1, \xi_j, \ldots, \xi_j^6$. To calculate ξ_j^7 mod(4) we use a multiple of the characteristic polynomial of ξ_j,

$$(x^4+x^3+x^2-2x+1)(x^3+x^2-2x-1) \equiv x^7-2x^6-1 \bmod(4),$$

and deduce that

$$\xi_j^7 \equiv 1+2\xi_j^6 \bmod(4).$$

From (46) we get $\xi_1^2=2+\xi_2$, hence

$$\xi_j^2 = 2+\xi_{j+1}, \qquad \xi_j^2 \equiv \xi_{j+1} \bmod(2). \tag{48}$$

Consequently,

$$\xi_{j+1}^7 \equiv 1+2\xi_{j+1}^6 \equiv 1+2\xi_j^5 \not\equiv \xi_j^7 \bmod(4).$$

Since

$$\alpha \equiv \beta \bmod(4) \Rightarrow \mathcal{N}(\alpha) \equiv \mathcal{N}(\beta) \bmod(4), \tag{49}$$

there are two types of residue classes prime to (4), those with $\mathcal{N}(\alpha)\equiv 1$, and those with $\mathcal{N}(\alpha)\equiv -1 \bmod(4)$. We have that ξ_j, ξ_{j+1}^7 generate classes with $\mathcal{N}(\alpha)\equiv 1 \bmod(4)$, $\mathcal{N}(-1)\equiv -1 \bmod(4)$. The rest is obvious.

By (49), $\mathcal{N}(\varepsilon)=1$ for every unit $\varepsilon\equiv 1 \bmod(4)$. Hence, for $\rho=4$, $2\nmid r$ is admissible (see section 2). From now on, we shall assume that $\rho=4$ and r is a positive integer with $2\nmid r$.

We have 72 Eisenstein series for $\Gamma_K(4)$ (corresponding to the 72 cusps). Of these, 54 correspond to pairs $(1, \rho_2)$ with ρ_2 a unit and $1, \rho_2$ linearly independent mod(4) over $\mathbb{Z}$. According to Theorem 5, the restriction to the diagonal of such an Eisenstein series yields a cusp form from $M_{3r}(\Gamma_{\mathbb{Q}}(4), 1)$, if $r > 1$. We are interested in simultaneous eigenfunctions of U^2, with eigenvalue 1, and T, with eigenvalue $\pm i$ (compare Lemma 3). Put

$$\tilde{G}_r(1, \rho_2) = \tilde{G}_r(*; (4), 1, \rho_2; K).$$

From Lemma 1, for a unit ρ_2, we deduce that (for $2 \nmid r$)

$$\tilde{G}_r(1, \rho_2) \mid T = \tilde{G}_r(\rho_2, -1) = \operatorname{sgn} \mathcal{N}(\rho_2) \tilde{G}_r(1, -\hat{\rho}_2) \qquad \rho_1 \hat{\rho}_2 \equiv 1 \bmod(4). \tag{50}$$

Eigenfunctions of T with eigenvalue $\pm i$ are

$$\begin{aligned} &\mp \tilde{G}_r(1, \xi_1) + i\tilde{G}_r(1, -\xi_1^{13}), &&\pm \tilde{G}_r(1, -\xi_1 \xi_2^7) + i\tilde{G}_r(1, \xi_1^{13} \xi_2^7), \\ &\mp \tilde{G}_r(1, \xi_1^8) + i\tilde{G}_r(1, -\xi_1^6), &&\pm \tilde{G}_r(1, -\xi_1^8 \xi_2^7) + i\tilde{G}_r(1, \xi_1^6 \xi_2^7). \end{aligned} \tag{51}$$

The result (50) shows that

$$\tilde{G}_r(1, -\rho_2) \mid T = -\operatorname{sgn} \mathcal{N}(\rho_2) \tilde{G}_r(1, \hat{\rho}_2) \qquad (\text{for } 2 \nmid r).$$

From the eigenfunctions of T with eigenvalue $\pm i$, given in (51), by $\rho_2 \mapsto -\rho_2$ we therefore obtain eigenfunctions with eigenvalue $\mp i$. From each of these eigenfunctions of T, by applying the two non-trivial automorphisms of K to the values ρ_2, we get two other eigenfunctions which, however, according to (15) of Theorem 5, have the same restrictions to the diagonal as the original functions. At this point we have used up 48 of the above-mentioned 54 Eisenstein series. Simultaneous eigenfunctions of U^2 with eigenvalue 1 and T with eigenvalue $-i$ are (i) the sum of the 4 eigenfunctions of T with eigenvalue $-i$, given in (51), viz.

$$\begin{aligned} f_r(\tau; K) = {} & \tilde{G}_r(1, \xi_1) + \tilde{G}_r(1, \xi_1^8) - \tilde{G}_r(1, -\xi_1 \xi_2^7) - \tilde{G}_r(1, -\xi_1^8 \xi_2^7) \\ & + i(\tilde{G}_r(1, -\xi_1^{13}) + \tilde{G}_r(1, -\xi_1^6) + \tilde{G}_r(1, \xi_1^{13} \xi_2^7) + \tilde{G}_r(1, \xi_1^6 \xi_2^7)), \end{aligned} \tag{52}$$

(ii) the sum of the 4 eigenfunctions of T with eigenvalue i, given in (51), modified by substituting $-\rho_2$ for ρ_2 in each of the Eisenstein series, viz.

$$\begin{aligned} g_r(\tau; K) = {} & -\tilde{G}_r(1, -\xi_1) - \tilde{G}_r(1, -\xi_1^8) + \tilde{G}_r(1, \xi_1 \xi_2^7) + \tilde{G}_r(1, \xi_1^8 \xi_2^7) \\ & + i(\tilde{G}_r(1, \xi_1^{13}) + \tilde{G}_r(1, \xi_1^6) + \tilde{G}_r(1, -\xi_1^{13} \xi_2^7) + \tilde{G}_r(1, -\xi_1^6 \xi_2^7)), \end{aligned} \tag{53}$$

and (iii) the 4 functions, obtained from $f_r(\tau; K)$, $g_r(\tau; K)$ by applying to the values of ρ_2 the two non-trivial automorphisms of K (as already mentioned, these functions have the same restrictions to the diagonal as $f_r(\tau; K)$ and $g_r(\tau; K)$). The simultaneous eigenfunctions of U^2 with eigenvalue 1 and T with eigenvalue i are obtained from the eigenfunctions of U^2 with eigenvalue 1 and T with eigenvalue $-i$ by substituting $-\rho_2$ for ρ_2 in the Eisenstein series.

The functions $\tilde{G}_r(1, \xi_1^7)$, $\tilde{G}_r(1, -\xi_1^7)$ (as well as the "conjugates" $\tilde{G}_r(1, \xi_2^7)$, $\tilde{G}_r(1, -\xi_2^7)$ and $\tilde{G}_r(1, \xi_3^7)$, $\tilde{G}_r(1, -\xi_3^7)$ by the non-trivial automorphisms of K) generate a simultaneous eigenspace of U^2 and T, containing no simultaneous eigenfunction $\not\equiv 0$ of U^2, with eigenvalue 1, and T, with eigenvalue i or $-i$. This accounts for the 6 remaining Eisenstein series.

In order to calculate the Fourier coefficients at $\infty = E(\infty)$, we first observe that we need only calculate the coefficients of f_r, since Theorem 1 and the definitions of f_r and g_r show that

$$a(\nu/4\delta; g_r, E) = -\overline{a(\nu/4\delta; f_r, E)}. \tag{54}$$

From

$$e^{2\pi i\mathcal{S}(\nu/4\delta)(\tau+2)} = e^{\pi i\mathcal{S}(\nu/\delta)}e^{2\pi i\mathcal{S}(\nu/4\delta)\tau}$$

and $f_r \mid U^2 = f_r$, we deduce that

$$a(\nu/4\delta; f_r, E) = 0 \quad \text{for} \quad \mathcal{S}(\nu/\delta) \not\equiv 0 \bmod(2). \tag{55}$$

From Lemma 5 ($\mathcal{S}(\xi_j) = -1$, $\mathcal{S}(\xi_j^{-1}) = -2$) and (48) we find that

$$\mathcal{S}(\nu/\delta) \equiv 0 \bmod(2) \Leftrightarrow \nu/\delta \equiv 0,\ \xi_1^3,\ \xi_1^5,\ \xi_1^6 \bmod(2). \tag{56}$$

Put (see Theorem 2)

$$w(\nu/4\delta; f_r, E) = \sum_{\rho_2} (\text{factor of } \tilde{G}_r(1, \rho_2) \text{ in } (52))\ w(\nu/4\delta; \rho_2; K).$$

Then

$$a(\nu/4\delta; f_r, E) = w(\nu/4\delta; f_r, E)c_r(\nu; 4; K). \tag{57}$$

Now $w(\nu/4\delta; f_r, E)$ depends on $\nu \bmod(4)$ and has to be calculated from Theorem 2 and (52) for $\nu/\delta \equiv 0, \xi_1^3, \xi_1^5, \xi_1^6 \bmod(2)$ (see (55), (56)). We find that

$$w(\nu/4\delta; f_r, E) = \begin{cases} 8 & \text{for } \nu/\delta \equiv -\xi_1^5\xi_2^7, -\xi_1^{12}\xi_2^7, \\ -8 & \text{for } \nu/\delta \equiv -\xi_1^5, -\xi_1^{12}, \\ 4i & \text{for } \nu/\delta \equiv -\xi_1^6, -\xi_1^{13}, -\xi_1^6\xi_2^7, -\xi_1^{13}\xi_2^7, 0, 2, 2\xi_1^4, 2\xi_1^6, \\ -4i & \text{for } \nu/\delta \equiv \xi_1^6, \xi_1^3, \xi_1^6\xi_2^7, \xi_1^{13}\xi_2^7, 2\xi_1, 2\xi_1^2, 2\xi_1^3, 2\xi_1^5, \\ 0 & \text{for all other values of } \nu/\delta \bmod(4). \end{cases}$$

In order to calculate the Fourier coefficients at $1 = UT(\infty)$, we first note that

$$\begin{aligned} f_r(\tau; K) \mid UT = &-\tilde{G}_r(1, \xi_1^2\xi_2^7) + \tilde{G}_r(1, -\xi_1^9) - \tilde{G}_r(1, -\xi_1^9\xi_2^7) + \tilde{G}_r(1, \xi_1^2) \\ &+ i(\tilde{G}_r(1, -\xi_1^{10}) - \tilde{G}_r(1, \xi_1^{10}\xi_2^7) + \tilde{G}_r(1, -\xi_1^3\xi_2^7) - \tilde{G}_r(1, \xi_1^3)). \end{aligned} \tag{58}$$

We then observe that

$$g_r(\tau; K) \mid UT = \overline{f_r(-\bar{\tau}; K) \mid UT},$$

hence

$$a(\nu/4\delta; g_r, UT) = \overline{a(\nu/4\delta; f_r, UT)}. \tag{59}$$

Again, we need only calculate the coefficients of $f_r(\tau; K)$. From $UTU = T^3U^{-2}UT$ we find (compare section 4)

$$(f_r(\tau; K)\,|UT)\,|\,U = (-i)^3 f_r(\tau; K)\,|\,UT.$$

Consequently (compare (55))

$$a(\nu/4\delta; f_r, UT) = 0 \quad \text{for} \quad \mathcal{S}(\nu/\delta) \not\equiv 1 \bmod(4). \tag{60}$$

Put

$$w(\nu/4\delta; f_r, UT) = \sum_{\rho_2} (\text{factor of } \tilde{G}_r(1, \rho_2) \text{ in } (58))\ w(\nu/4\delta; \rho_2; K).$$

Then, by Theorem 2, we have

$$a(\nu/4\delta; f_r, UT) = w(\nu/4\delta; f_r, UT)c_r(\nu; 4; K). \tag{61}$$

Since $w(\nu/4\delta; f_r, UT)$ depends on $\nu \bmod(4)$, it has to be calculated from Theorem 2 and (58) for ν with $\mathcal{S}\nu/\delta \equiv 1 \bmod(4)$ (see (60); for those ν, from (56), we have $\nu/\delta \equiv 1$, $\xi_1, \xi_1^2, \xi_1^4 \bmod(2)$). We find that

$$w(\nu/4\delta; f_r, UT) = \begin{cases} -8 & \text{for } \nu/\delta \equiv \xi_1^4, \xi_1^{11}, \\ 8 & \text{for } \nu/\delta \equiv -\xi_1^4\xi_2^7, -\xi_1^{11}\xi_2^7, \\ -8i & \text{for } \nu/\delta \equiv -1, -\xi_1^7, \\ 8i & \text{for } \nu/\delta \equiv \xi_2^7, -\xi_1^7\xi_2^7, \\ 0 & \text{for all other values of } \nu/\delta \bmod(4). \end{cases}$$

By Theorem 4, the Fourier coefficients of $\mathcal{S}_1 f_r$ at $\infty = E(\infty)$ can be calculated from the coefficients in (57) ($2\,|\,m$ because of (55)), and the coefficients at $1 = UT(\infty)$ from the coefficients in (61) ($m \equiv 1 \bmod(4)$ because of (60)), as follows:

$$a(m/4; \mathcal{S}_1 f_r, E) = \sum_{\nu>0, \mathcal{S}\nu/\delta = m} w(\nu/4\delta; f_r, E)c_r(\nu; 4; K) \quad \text{for} \quad 2\,|\,m, \tag{62}$$

$$a(m/4; \mathcal{S}_1 f_r, UT) = \sum_{\nu>0, \mathcal{S}\nu/\delta = m} w(\nu/4\delta; f_r, UT)c_r(\nu; 4; K) \quad \text{for} \quad 4\,|\,(m-1). \tag{63}$$

Now, for given $m \in \mathbb{N}$, we have to find the totally positive integers ν from K with $\mathcal{S}\nu/\delta = m$. We fix the conjugates $\nu^{(1)}$, $\nu^{(2)}$, $\nu^{(3)}$ of

$$\nu = a + b\eta_2 + c\eta_3$$

by putting (see (46))

$$\eta_k = 2 - \xi_k, \qquad \xi_k = \xi_1^{(k)} = 2\cos 2\pi k/7 \qquad (k = 1, 2, 3).$$

As coordinates in the plane $m = \mathcal{S}\nu/\delta = a + 2(b + c)$ (see Lemma 5), we use b, c. The region $\nu^{(1)}>0$, $\nu^{(2)}>0$, $\nu^{(3)}>0$ of this plane is the interior of the triangle the vertices of which are

$$(b_j, c_j)=m((\eta_{j+1}/\eta_j)-1, (\eta_{j-1}/\eta_j)-1) \qquad (j=1,2,3).$$

The latter are given, approximately, by

$$(b_1, c_1)=m(2{\cdot}25, 4{\cdot}05), \qquad (b_2, c_2)=m(0{\cdot}55, -0{\cdot}69), \qquad (b_3, c_3)=m(-0{\cdot}80, -0{\cdot}36).$$

We find that

$$\nu>0, \qquad m=1 \Leftrightarrow \nu=\begin{cases} 1 & \mathcal{N}(\nu)=1, \quad \nu/\delta\equiv\xi_1^8 \bmod(4),\\ -3+\eta_2+\eta_3, & \mathcal{N}(\nu)=1, \quad \nu/\delta\equiv\xi_1^2 \bmod(4),\\ -5+\eta_2+2\eta_3, & \mathcal{N}(\nu)=1, \quad \nu/\delta\equiv\xi_1^4 \bmod(4),\end{cases}$$

and we have the following table.

TABLE 5. THE TOTALLY POSITIVE INTEGERS $\nu=a+b\eta_2+c\eta_3$ WITH $\mathcal{S}\nu/\delta=2$, ν/δ mod(4).

a	4	4	2	−6	−12	−20	2	0	0	−2	−4	−6	−8	−10	−14
b	−1	0	1	1	3	4	0	0	1	1	1	2	2	2	3
c	0	−1	−1	3	4	7	0	1	0	1	2	2	3	4	5
$\mathcal{N}(\nu)$	1	1	1	1	1	1	8	7	7	13	13	8	13	8	7
$\nu/\delta\equiv$	ξ_1^{10}	ξ_1^{12}	ξ_1^6	ξ_1^6	ξ_1^{10}	ξ_1^{12}	$2\xi_1^8$	$-\xi_1^{12}$	$-\xi_1^{10}$	ξ_1^{13}	$\xi_1^3\xi_2^7$	ξ_1^2	$\xi_1^{12}\xi_2^7$	ξ_1^4	$-\xi_1^6$

Let γ_7 stand for an integer of K with $\mathcal{N}(\gamma_7)=7$, and γ_{13} for an integer with $\mathcal{N}(\gamma_{13})=13$. Then, for $m=1$, from (63) and (59) we find that

$$a(1/4; \mathcal{S}_1 f_r, UT)=-8c_r(1;4;K)=-8,$$
$$a(1/4; \mathcal{S}_1 g_r, UT)=\overline{a(1/4; \mathcal{S}_1 f_r, UT)}=-8.$$

From (62), (54) and Table 5 we get (writing $c_r(\mu)=c_r(\mu;4;K)$)

$$a(2/4; \mathcal{S}_1 f_r, E)=-8c_r(\gamma_7)-4i(c_r(\gamma_{13})-c_r(\gamma_7)+c_r(2)+2c_r(1)),$$
$$a(2/4; \mathcal{S}_1 g_r, E)=-\overline{a(2/4; \mathcal{S}_1 f_r, E)}.$$

For $r>1$, we have (see Theorem 3) that

$$c_r(\gamma_7)>0, \qquad c_r(\gamma_{13})-c_r(\gamma_7)+c_r(2)+2c_r(1)>0.$$

Hence, $\mathcal{S}_1 f_r$, $\mathcal{S}_1 g_r$ are linearly independent for $r>1$. Thus we have

THEOREM 9. *Suppose that $r\in\mathbb{N}$ with $2\nmid r$, and define $f_r(\tau;K)$ by (52) and $g_r(\tau;K)$ by (53). For $r>1$, $\mathcal{S}_1 f_r$ and $\mathcal{S}_1 g_r$ are linearly independent cusp forms from $M_{3r}(\Gamma_\vartheta, \tilde{v})$ with $\tilde{v}(U^2)=1$, $\tilde{v}(T)=-i$. Further, $\mathcal{S}_1 f_3$, $\mathcal{S}_1 g_3$ are a basis of $M_9^+(\Gamma_\vartheta, v^{18})$.*

By Theorem 9, h_9^+ from (25) of Lemma 4 is a linear combination of $\mathcal{S}_1 f_3(*;K)$ and $\mathcal{S}_1 g_3(*,K)$. The coefficients $b(\mathcal{S}_1 f_3(*;K))$ and $b(\mathcal{S}_1 g_3(*;K))$ (see (28), section 4) can easily be calculated if h_9 is known. In order to calculate h_9 and h_9^+ at the same time, we need one more coefficient of the Fourier expansions, namely $a(1;*,E)$. I shall not reproduce a table of the 54 totally positive integers ν with $\mathscr{S}\nu/\delta=4$. One finds the following values (the values of the already calculated coefficients for $r=3$ included):

$$a(1/4;\mathscr{S}_1 f_3, UT)=-8, \qquad a(1/4;\mathscr{S}_1 g_3, UT)=-8,$$
$$a(1/2;\mathscr{S}_1 f_3, E)=-2^7.3-2^4.47i, \qquad a(1/4;\mathscr{S}_1 g_3, E)=2^7.3-2^4.47i,$$
$$a(1;\mathscr{S}_1 f_3, E)=-2^9.3^2-2^6.149i, \qquad a(1;\mathscr{S}_1 g_3, E)=2^9.3^2-2^6.149i.$$

9. Identities for modular forms and representation numbers. As a typical example, let us calculate the number of representations of a positive integer as a sum of 18 squares. By (25) of Lemma 4 we have that

$$\vartheta^{18}=h_9+h_9^+, \qquad (h_9^+\in M_9(\Gamma_\vartheta^+, v^{18})).$$

Theorem 7 states that

$$h_9=b(\tilde{h}_9)\tilde{h}_9$$

for some constant $b(\tilde{h}_9)$. From Theorem 9 we get that

$$h_9^+=b(\mathscr{S}_1 f_3)\mathscr{S}_1 f_3(*;K)+b(\mathscr{S}_1 g_3)\mathscr{S}_1 g_3(*;K),$$

where K is the totally real cubic number field of discriminant 7^2. Hence

$$\vartheta^{18}=b(\tilde{h}_9)\tilde{h}_9+b(\mathscr{S}_1 f_3)\mathscr{S}_1 f_3(*;K)+b(\mathscr{S}_1 g_3)\mathscr{S}_1 g_3(*;K). \tag{64}$$

Using the values of the Fourier coefficients $a(1/2;*,E)$, $a(1;*,E)$, $a(1/4;*,UT)$, given for ϑ^{18} in Table 1, for $\tilde{h}_9$ in Table 3, and for $\mathscr{S}_1 f_3(*;K)$ and $\mathscr{S}_1 g_3(*;K)$ at the end of section 8, we get 3 equations for the 3 coefficients in (64). We find that

$$2^4.5.277\vartheta^{18}=2^5\tilde{h}_9-1033(\mathscr{S}_1 f_3(*;K)-\mathscr{S}_1 g_3(*;K))-i2^3(\mathscr{S}_1 f_3(*;K)+\mathscr{S}_1 g_3(*;K)). \tag{65}$$

The values of the Fourier coefficients at the cusp $\infty=E(\infty)$ are given in (39) of section 6, and (62), (54) of section 8. Suppose that $2\mid m$, $m=2^t m_1$, $2\nmid m_1$. Then, from (65), we obtain that

$$\begin{aligned}5.277A_{18}(m/2)&=2^2c_9(m;4;\mathbb{Q})-2^2(-1)^{(m_1+1)/2}c_9(m_1;4;\mathbb{Q})\\ &\quad-1033\sum_{\substack{\nu>0,\mathscr{S}\nu/\delta=m\\ w(\nu/4\delta;f_3,E)=\pm 8}}\frac{1}{8}w(\nu/4\delta;f_3,E)c_3(\nu;4;K)\\ &\quad-i\sum_{\substack{\nu>0,\mathscr{S}\nu/\delta=m\\ w(\nu/4\delta;f_3,E)=\pm 4i}}w(\nu/4\delta;f_3,E)c_3(\nu;4;K).\end{aligned} \tag{66}$$

Here $w(\nu/4\delta;f_3,E)$ depends on ν mod(4) (a table for the values is given is section 8). For the divisor function $c_r(*;*;*)$, see Theorem 3.

Using Theorems 6 and 8 we find ϑ^{2r} $(r=6,8,10,12)$ to be as follows:

$$\vartheta^{12}=2^3\tilde{h}_6-2^3\mathscr{S}_1g_3(*;\mathbb{Q}(\sqrt{2})), \tag{67}$$

$$3\,.\,17\vartheta^{16}=2^5\,.\,3\tilde{h}_8+2^8\mathscr{S}_1g_4(*;\mathbb{Q}(\sqrt{2})), \tag{68}$$

$$3^2\,.\,31\vartheta^{20}=2^3\,.\,3^2\tilde{h}_{10}-2^3\,.\,107\mathscr{S}_1g_5(*;\mathbb{Q}(\sqrt{2}))-2^3\,.\,7\mathscr{S}_1f_5(*;\mathbb{Q}(\sqrt{2})), \tag{69}$$

$$3^3\,.\,5\,.\,691\vartheta^{24}=2^4\,.\,3^3\,.\,5\tilde{h}_{12}-2^6\,.\,3917\mathscr{S}_1g_6(*;\mathbb{Q}(\sqrt{2}))+2^8\,.\,769\mathscr{S}_1f_6(*;\mathbb{Q}(\sqrt{2})). \tag{70}$$

From these results, formulae for the representation numbers can easily be obtained; e.g. from (67) we have that

$$A_{12}(m)=2^3(-1)^m\sum_{d|m}(-1)^{d+m/d+1}d^5-2^3(-1)^m\sum_{a^2<2m^2,a\in\mathbb{Z}}(-1)^a c_3(a+m\sqrt{2};2;\mathbb{Q}(\sqrt{2})), \tag{71}$$

and, from (68),

$$3\,.\,17A_{16}(m)=2^5\,.\,3\,.\,(-1)^m\sum_{d|m}(-1)^d d^7+2^8(-1)^m\sum_{a^2<2m^2,a\in\mathbb{Z}}(-1)^a c_4(a+m\sqrt{2};2;\mathbb{Q}(\sqrt{2})). \tag{72}$$

Rankin's functions Ψ_1, Ψ_1^* (see [**12**]) can easily be represented as linear combinations of $\mathscr{S}_1g_5(*;\mathbb{Q}(\sqrt{2}))$ and $\mathscr{S}_1f_5(*;\mathbb{Q}(\sqrt{2}))$ (Theorem 8). By comparing the coefficients $a(1/2;*,E)$, $a(1;*,E)$ we find that

$$-2\,.\,3\,.\,5\Psi_1=\mathscr{S}_1g_5(*;\mathbb{Q}(\sqrt{2}))-\mathscr{S}_1f_5(*;\mathbb{Q}(\sqrt{2})), \tag{73}$$

$$2^2\,.\,3\Psi_1^*=-2\mathscr{S}_1g_5(*;\mathbb{Q}(\sqrt{2}))-\mathscr{S}_1f_5(*;\mathbb{Q}(\sqrt{2})). \tag{74}$$

For Rankin's function Ψ_0 we need an eigenfunction of T with eigenvalue 1. We find that (compare (43), section 7)

$$2^3\,.\,3\Psi_0=\mathscr{S}_1\tilde{G}_5(*;(2),1,\sqrt{2};\mathbb{Q}(\sqrt{2}))+\mathscr{S}_1\tilde{G}_5(*;(2),\sqrt{2},1;\mathbb{Q}(\sqrt{2})). \tag{75}$$

For Dedekind's function $\eta(z)$ we find that

$$2\eta^{12}=-\mathscr{S}_1g_3(*;\mathbb{Q}(\sqrt{2})), \tag{76}$$

$$2^5\,.\,3^3\,.\,5\eta^{24}=\mathscr{S}_1g_6(*;\mathbb{Q}(\sqrt{2}))+\mathscr{S}_1f_6(*;\mathbb{Q}(\sqrt{2})). \tag{77}$$

In order to represent ϑ^{28} and ϑ^{32} we need 3 linearly independent cusp forms, since

$$\dim_{\mathbb{C}} M_{14}^+(\Gamma_\vartheta,v^{28})=\dim_{\mathbb{C}} M_{16}^+(\Gamma_\vartheta,v^{32})=3.$$

From Theorem 8 of section 7 we have 2 independent cusp forms,

$$\mathscr{S}_1g_r(*;\mathbb{Q}(\sqrt{2})),\ \mathscr{S}_1f_r(*;\mathbb{Q}(\sqrt{2}))\qquad(r=7,8).$$

Another cusp form can be obtained by the method of section 7, if we use $\mathbb{Q}(\sqrt{3})$ instead of $\mathbb{Q}(\sqrt{2})$. Define (compare (43), section 7)

$$f_r(\tau;\mathbb{Q}(\sqrt{3}))=\tilde{G}_r(\tau;(2),1,1+\sqrt{3};\mathbb{Q}(\sqrt{3}))+(-1)^r\tilde{G}_r(\tau;(2),1+\sqrt{3},1;\mathbb{Q}(\sqrt{3})).$$

Then $\mathscr{S}_1 f_r(*; \mathbb{Q}(\sqrt{3}))$ is a cusp form from $M_{2r}(\Gamma_\vartheta, v^{4r})$ for $r>1$. Comparing the coefficients given below and the coefficients for $\mathscr{S}_1 g_r(*; \mathbb{Q}(\sqrt{2}))$ and $\mathscr{S}_1 f_r(*; \mathbb{Q}(\sqrt{2}))$ given in Table 4 of section 7, for $r=7, 8$, one easily finds that these functions are linearly independent for $r=7$ and for $r=8$. Consequently, for $r=7, 8$, ϑ^{4r} can be represented as a linear combination of $\tilde{h}_{2r}$ and these 3 functions. Further, we have that

$a(1/2; \mathscr{S}_1 f_7(*; \mathbb{Q}(\sqrt{3}), E)) = -2^3 . 3 . 5^2$ $\qquad$ $a(1/2; \mathscr{S}_1 f_8(*; \mathbb{Q}(\sqrt{3}), E)) = -2^2 . 3 . 7 . 23,$
$a(1; \mathscr{S}_1 f_7(*; \mathbb{Q}(\sqrt{3}), E)) = -2^9 . 3^2 . 5^2,$ $\qquad$ $a(1; \mathscr{S}_1 f_8(*; \mathbb{Q}(\sqrt{3}), E)) = 2^5 . 3 . 7^2 . 11^2,$
$a(1/2; \mathscr{S}_1 f_7(*; \mathbb{Q}(\sqrt{3}), UT)) = 2^4 . 3 . 5^2,$ $\qquad$ $a(1; \mathscr{S}_1 f_8(*; \mathbb{Q}(\sqrt{3}), UT)) = 2^6 . 3 . 7 . 1583.$

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