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On product-preserving Kan extensions

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In this article we examine the problem of when a left Kan extension of a finite-product-preserving functor is finite-product preserving. This extension property is of significance in the development of finitary universal algebra in a closed category, details of which will appear elsewhere. We give a list of closed categories with the required extension property.

Introduction

The aim of this article is to introduce and discuss a colimit-limit commutativity property called axiom π . If V is a symmetric monoidal closed category, then a V-category C is said to satisfy axiom π (relative to V) if the left Kan V-extension of any finite-V-product-preserving functor into C again preserves finite V-products. One basic use of this extension property is in the construction of free-algebra functors and, more generally, left adjoints to algebraic functors in finitary universal algebra; details of this will appear elsewhere (see Borceux and Day [1]).

In Section 1 we discuss various equivalent forms of axiom π with a view to using it in finitary universal algebra. In Section 2 we describe some basic constructions which inherit axiom π . In Section 3 we show that cartesian closed categories satisfy axiom π , as do closed categories which are finitarily algebraic over a cartesian closed category. We also see that certain closed functor categories satisfy axiom π .

Throughout the article the symbol V stands for a symmetric monoidal closed category $V = (V, I, \otimes, \ldots)$. Most of the other notations are

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standard (see Mac Lane [8], and Eilenberg and Kelly [5]), or are explained in the text.

1. V-cartesian products and Kan extensions

A V-category is said to have finite V-products if it has finite products and they are preserved by the V-representable functors (see Day and Kelly [4], §2).

Let A and B be two V-categories. Their product in the category of V-categories is denoted $A \times B$. It is defined by:

(1) $\mathrm{obj}(A \times B) = \mathrm{obj}(A) \times \mathrm{obj}(B)$;

(2) $(A \times B)\big((A, B), (A', B')\big) = A(A, A') \times B(B, B')$;

(3) $j_{(A,B)} = \big(j_A, j_B\big)$;

(4) $M^{(A',B')}_{(A,B)(A'',B'')} = \left\{M^{B'}_{BB''} \times M^{A'}_{AA''}\right\} \cdot \big(p_2 \otimes p_2, p_1 \otimes p_1\big)$, where p_i denotes the i-th projection of a product.

PROPOSITION 1.1. *Let A be a V-category. The diagonal functor $\Delta : A \to A \times A$ is a V-functor.*

PROPOSITION 1.2. *Let A be a V-category with finite V-products. The cartesian product $\times : A \times A \to A$ is a V-functor.*

The verifications are straightforward. //

We now have the following result dealing with mean tensor products in the sense of Borceux and Kelly [2]; we frequently use the contraction notation $HA \circ GA$ in place of $H * G$.

THEOREM 1.3. *Let A be a V-category with finite V-products, $H, H' : A^{\mathrm{op}} \to V$ be two V-functors, and $G : A \to C$ be a V-functor. Then the following isomorphism holds as soon as mean tensor products exist:*

$$\big(H(-) \times H'(-)\big) * G(-) \cong \big(H(-) \times H'(=)\big) * G(- \times =) .$$

In alternative notation:

$$(HA \times H'A) \circ GA \cong (HA' \times H'A'') \circ G(A' \times A'') .$$

Proof. We have the following situation:

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$$H(-) \times H'(=) : A^{op} \times A^{op} \xrightarrow{H\times H'} V \times V \xrightarrow{\times} V ,$$

$$H(-) \times H'(-) : A^{op} \xrightarrow{(H,H')} V \times V \xrightarrow{\times} V ,$$

$$G(- \times =) : A \times A \xrightarrow{\times} A \xrightarrow{G} C .$$

For brevity we write $T_1 = (HA \times H'A) \circ GA$ and $T_2 = (HA' \times H'A'') \circ G(A' \times A'')$. Now T_1 and T_2 are defined by the fact that there exist V-natural transformations:

$$\lambda_A : HA \times H'A \to C(GA, T_1) ,$$

$$\rho_{A'A''} : HA' \times H'A'' \to C(G(A' \times A''), T_2)$$

generating V-natural isomorphisms:

$$[X, C(T_1, C)] \cong \int_A [HA \times H'A, [X, C(GA, C)]] ,$$

$$[X, C(T_2, C)] \cong \int_{A'A''} [HA' \times HA'', [X, C(G(A' \times A''), C)]] ,$$

for all $X \in V$ and $C \in C$. But T_1 and T_2 are isomorphic as soon as the two *sets* of V-natural transformations are isomorphic. This last correspondence between

$$\alpha_A : HA \times H'A \to [X, C(GA, C)]$$

and

$$\beta_{A'A''} : HA' \times H'A'' \to [X, C(G(A \times A'), C)]$$

is given by $\beta_{A'A''} = \alpha_{A'\times A''} . (Hp_1 \times H'p_2)$ and $\alpha_A = [1, C(G.\Delta, 1)] . \beta_{AA}$ where p_i denotes projection from a product. //

DEFINITION 1.4. Let C be a V-category with finite V-products. Consider the following situation in V-cat : functors $H, H' : A^{op} \to V$ and $G : A \to C$ where A is small and has finite V-products preserved by G . The category C is said to satisfy *axiom* π , or to be $\pi(V)$, if, in any such situation, $HA \circ GA$ and $H'A' \circ GA'$ exist and the canonical transformation:

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$$(HA \times H'A') \circ (GA \times GA') \to (HA \circ GA) \times (H'A' \circ GA')$$

is an isomorphism.

We note that this canonical transformation is obtained in the following way. Consider the V-natural transformation

$$HA \times H'A' \xrightarrow{P_1} HA \xrightarrow{\alpha_A} C(GA,\, HA \circ GA) \xrightarrow{C(P_1,1)} C(GA \times GA',\, HA \circ GA) \ ,$$

where α_A is the canonical transformation defining $HA \circ GA$. This V-natural transformation gives rise to the factorisation

$$(HA \times H'A') \circ (GA \times GA') \to HA \circ GA \ ,$$

which is the first component in the transformation we are looking for.

THEOREM 1.5. *Let* C *be a* V*-category with finite* V*-products and small* V*-colimits. The following conditions are equivalent:*

(i) C *is* $\pi(V)$ *;*

(ii) in the situation of Definition 1.4 *the canonical transformation*

$$(HA \times H'A) \circ GA \to (HA' \circ GA') \times (H'A'' \circ GA'')$$

is an isomorphism;

(iii) for any V*-category* B *, any small* V*-category* A *with finite* V*-products, any* V*-functor* $M : A \to B$ *, and any finite-*V*-product preserving* V*-functor* $G : A \to C$ *, the left Kan* V*-extension of* G *along* M *exists pointwise and preserves finite* V*-products.*

Proof. The equivalence of *(i)* and *(ii)* follows from Theorem 1.3. Also *(i)* implies *(iii)* because, by Theorem 1.3, if $B \times B'$ is a V-product in B then

$$\begin{aligned} \operatorname{lan} G(B \times B') &\cong B(MA,\, B \times B') \circ GA \\ &\cong \big(B(MA,\, B) \times B(MA,\, B')\big) \circ GA \\ &\cong \big(B(MA',\, B) \times B(MA'',\, B')\big) \circ G(A' \times A'') \\ &\cong \big(B(MA',\, B) \times B(MA'',\, B')\big) \circ (GA' \times GA'') \ , \end{aligned}$$

while

$$\operatorname{lan} G(B) \times \operatorname{lan} G(B') \cong \big(B(MA',\, B) \circ GA'\big) \times \big(B(MA'',\, B') \circ GA''\big) \ .$$

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Finally *(iii)* implies *(ii)* on taking $M : A \to B$ to be the Yoneda embedding $Y : A \to [A^{op}, V]$. //

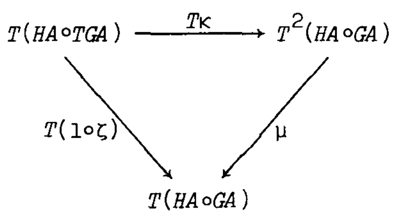
2. Hereditary properties of axiom π

PROPOSITION 2.1. *If* C *is a* V*-category with finite* V*-products and small* V*-colimits then, for any small* V*-category* A , $[A, C]$ *is* $\pi(V)$ *if* C *is* $\pi(V)$. //

Similarly, any product of $\pi(V)$ categories is $\pi(V)$.

PROPOSITION 2.2. *If* C *is a* V*-category with finite* V*-products and small* V*-colimits and* $\mathsf{T} = (T, \mu, \eta)$ *is a* V*-monad on* C *which preserves* V*-coequalisers of reflective pairs of morphisms and finite* V*-products, then* C^{T} *is* $\pi(V)$ *if* C *is* $\pi(V)$.

Proof. By a computation analogous to that for ordinary colimits (see Linton [7]), the mean tensor product $HA \circ GA$ in C^{T} is computed as the V-coequaliser in C^{T} of the reflective pair



where $\kappa : HA \circ TGA \to T(HA \circ GA)$ is the canonical comparison transformation for mean tensor products in C ; we omit the underlying-object functor $C^{\mathsf{T}} \to C$ from the notation. The result now follows from examination of the diagram:

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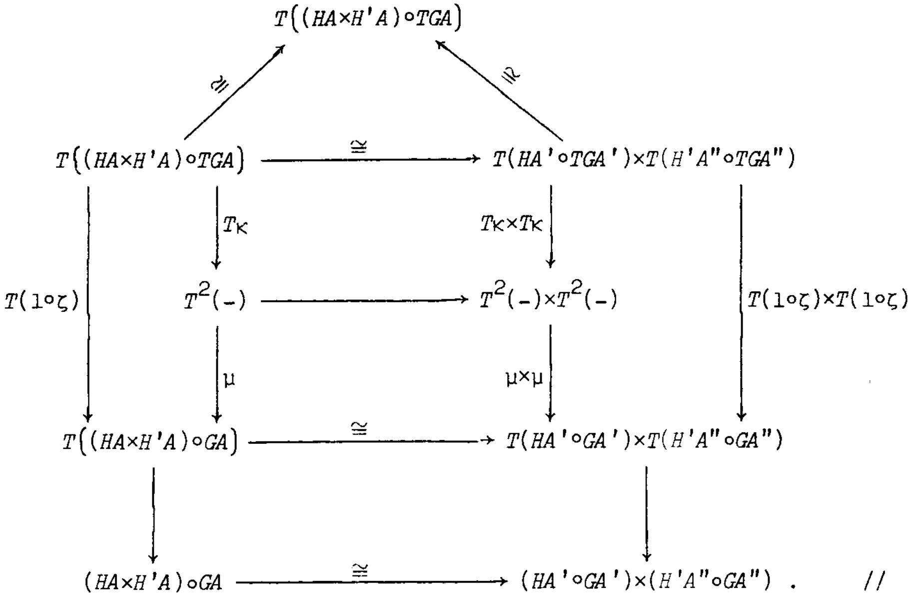

3. Examples

EXAMPLE 3.1. If V is a cartesian closed category, then it is $\pi(V)$ because the cartesian product preserves mean tensor products:

$$\begin{aligned}(HA \circ GA) \times (H'A' \circ GA') &\cong HA \circ \big(GA \times (H'A' \circ GA')\big)\\ &\cong HA \circ \big(H'A' \circ (GA \times GA')\big)\\ &\cong (HA \times H'A') \circ (GA \times GA') .\end{aligned}$$

EXAMPLE 3.2. If V is cartesian closed and has small limits and colimits, and if T is a finitary commutative V-theory (see Day [3], Example 4.3), then the monoidal closed category $W = \mathrm{T}^b$ of T-algebras in V is $\pi(W)$. In fact we shall establish a stronger result.

We first suppose that V is a given symmetric monoidal closed "base" category and that all categorical algebra is *relative* to this V . Let W and W' be symmetric monoidal closed categories and let $U : W \to W'$ be a symmetric monoidal closed functor such that $\hat{U} : U_*W \to W'$ has a left W'-adjoint F ; thus U_*W is W'-tensored by Kelly [6], 5.1. **Consider** W-functors $H : \mathrm{A}^{op} \to W$ and $G : \mathrm{A} \to W$. These give W'-functors $U_*H : U_*\mathrm{A}^{op} \to U_*W$ and $U_*G : U_*\mathrm{A} \to U_*W$. We then have

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$$UGA \circ HA = \int^{U_*A} UGA \circ HA = \int^{U_*A} FUGA \otimes HA$$

in U_*W .

LEMMA 3.2.1. *Suppose* $U : W \to W'$ *is a faithful symmetric monoidal closed functor. Let*

$$S' : U_*A^{op} \otimes U_*A \xrightarrow{\tilde{U}_*} U_*(A^{op} \otimes A) \xrightarrow{U_*S} U_*W .$$

Then $\int^{A} S(AA) \cong \int^{U_*A} S'(AA)$ *, one coend existing if and only if the other does.* //

Now consider the composite

$$\int^{U_*A} HA \otimes FUGA \xrightarrow{\int 1\otimes\varepsilon} \int^{U_*A} HA \otimes GA \xrightarrow{\kappa} \int^{A} HA \otimes GA$$

in the original situation.

PROPOSITION 3.2.2. *If* U *preserves* $\int^{U_*A} UGA \circ HA$ *and* U *reflects isomorphisms, then* $1 \otimes \varepsilon$ *and* κ *are isomorphisms.*

Proof. The map κ is an isomorphism by Lemma 3.2.1 and faithfulness of U . Moreover $HA \cong \int^{A} A(AB) \otimes HB \cong \int^{U_*A} A(AB) \otimes HB$ by the W-representation theorem. So it suffices to consider H representable. But $U\left(\int^{U_*A} GUA \circ A(AB)\right) \cong \int^{U_*A} UGA \otimes UA(AB) \cong UGB$ by the W'-representation theorem, as required. //

COROLLARY 3.2.3. *If* W' *is* $\pi(W')$ *and* U *reflects isomorphisms and preserves* $\int^{U_*A} UGA \circ HA$ *whenever* G *preserves finite* V*-products, then* W *is* $\pi(W)$. //

In order to establish our original assertion regarding $W = T^b$ we let V be $\pi(V)$ and let P be a small V-category together with a selected set Λ of finite V-products. Suppose $W = [P, V]_\Lambda$ has a symmetric monoidal closed structure, denoting the basic functor by $U : W \to V$.

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Consider W-functors $H : A^{op} \to W$ and $G : A \to W$ where A has finite W-products and they are preserved by G. We form $\int^{U_*A} HA(B) \otimes UGA$ in V for each B P and obtain a functor of B; because V is $\pi(V)$, this functor lies in W, by Theorem 1.5 *(ii)*. It is clearly $UGA \circ HA = \int^{U_*A} UGA \circ HA$ in U_*W. It is also $\int^{U_*A} HA \otimes FUGA$ in U_*W. But, by construction, it is $UGA \circ HA$ in $[P, V]$ and so is preserved by U if U has a right V-adjoint. Thus, if U restricted to W reflects isomorphisms then W is $\pi(W)$ by the preceding corollary.

EXAMPLE 3.3. The preceding example raises the problem of when a closed functor category of the form $W = [P, V]$ is $\pi(W)$. The authors have not yet obtained a general solution to this problem although there are simple cases of interest.

PROPOSITION 3.3.1. *If V is $\pi(V)$ and X is a discrete set then $W = V^X$ is $\pi(W)$.*

Proof. Each W-category A gives rise to a family $\{A_x ; x \in X\}$ of V-categories with $\text{obj } A_x = \text{obj } A$ and $A_x(AA') = A(AA')_x$. Similarly, each W-functor $H : A \to B$ yields a family of V-functors $H_x : A_x \to B_x$. Moreover,

$$\int^{A} HA \otimes GA = \left(\int^{A_x} H_x A \otimes G_x A\right)_{x \in X} ,$$

from which it follows that W is $\pi(W)$ if V is $\pi(V)$. //

Another case which admits a simple solution is that in which P is comonoidal (see Day [3]) and $J \cong P(I, -)$. If the ground functor $U : [P, V] \to V$ is V-faithful, then $W = [P, V]$ is $\pi(W)$ if V is $\pi(V)$.

EXAMPLE 3.4. The closed category W of Banach spaces with the greatest cross-norm tensor product is $\pi(W)$. The proof of this fact will appear elsewhere.

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