THE CALCULATION OF $\pi(N)$

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(received 20 August 1962, revised 20 December 1962)

The aim of this paper is to derive two formulae for $\pi(N)$ that need involve only a few of the smallest primes. The first is

$$\pi(N) = m + b_1 P_1 + b_2 P_2 + b_{11} P_{11} + b_3 P_3 + b_{21} P_{21} + \cdots \tag{1}$$

Here m is a small integer, the b's are integers that will be found later, and $P_{ij\cdots k}$ denotes the number of products $f^i g^j \cdots h^k \leqq N$, in which $f, g, \cdots, h$ are unequal integers greater than 1 and prime to the first m primes. The suffixes run through all partitions of all integers.

It will be proved that

$$b_{(n)} = (1/n) \sum (-)^{n/d-1} \mu(d) C(n/d) \quad (d|(n)), \tag{2}$$

where (n) denotes a partition $ij \cdots k$ of n, d runs through the integers that divide all of $i, j, \cdots, k$, $\mu(d)$ is the Möbius function, and $C(n/d)$ denotes the multinomial coefficient

$$\frac{(n/d)!}{(i/d)!(j/d)! \cdots (k/d)!} \tag{3}$$

associated with the partition $(n)/d$. When $d = 1$ only, (2) is simply

$$b_{(n)} = \frac{(-)^{n-1}(n-1)!}{i!j! \cdots k!} \quad ((i, j, \cdots, k) = 1).$$

It will also be proved that when the partition is a single integer,

$$b_n = 0 \ (n \geqq 3). \tag{4}$$

A modification of (1) was suggested by Dr J. C. Butcher. Let

$$(n) = 1^\alpha 2^\beta \cdots \nu^\gamma.$$

Then $P_{(n)}$ as defined above denotes the number of products

$$f_1 f_2 \cdots f_\alpha (g_1 g_2 \cdots g_\beta)^2 \cdots (h_1 h_2 \cdots h_\gamma)^\nu \leqq N$$

of integers greater than 1, prime to the first m primes, and all different, i.e.,

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$$f_i \neq f_j, \quad g_i \neq g_j, \cdots,$$

(5) $$f_i \neq g_j, \quad f_i \neq h_j, \quad g_i \neq h_j, \cdots.$$

Let $Q_{(n)}$ denote the number of products as just defined except that they need not satisfy (5). The second formula is

(6) $$\pi(N) = m + c_1 Q_1 + c_2 Q_2 + c_{11} Q_{11} + \cdots,$$

where

(7) $$c_{(n)} = (1/n) \sum (-)^{n/d-1} \mu(d)(n/d)! \lambda_{(n)/d} \quad (d|(n)),$$

(8) $$\lambda_{ij\cdots k} = \lambda_i \lambda_j \cdots \lambda_k,$$

and $\lambda_1, \lambda_2, \cdots$ are defined by

(9) $$e^x = (1+\lambda_1 x)(1+\lambda_2 x^2)(1+\lambda_3 x^3) \cdots \text{to } \infty.$$

If $d = 1$ only with $(n) = ij \cdots k$, (7) becomes

$$c_{(n)} = (-)^{n-1}(n-1)! \lambda_{(n)} \quad ((i, j, \cdots, k) = 1).$$

Formulae (1) and (6) are believed to have the advantages that a computer program giving the P's or Q's for $\pi(N)$ can be devised so as to give them for $\pi(N/l)$ also, where l runs through any desired set of integers, and that the same P's and Q's can be used in formulae similar to (1) and (6) for the numbers of integers with prime factorizations pq, p^2q, pqr, etc. (These formulae have yet to be worked out.) It may also be possible to find the number of primes in each of a set of residue classes, e.g. $+1$ and -1 mod 4.

Proof of (2)

$N_{st\cdots u}$ will denote the number of integers in a given set whose prime factorizations are of the form $p^s q^t \cdots r^u$. The set can be any that does not include 1, and for the present purpose it consists of the integers greater than 1 but not greater than N that are prime to the first m primes. Such a set will be referred to as the set N.

The N's are connected by the relation

$$N_1 + N_2 + N_{11} + \cdots = P_1.$$

Further relations can be obtained from the number of ways in which an integer I belonging to the set can be expressed as a product $f^i g^j \cdots h^k$ enumerated by $P_{ij\cdots k}$. The number of ways depends only on the exponents in the prime factorization, $p^s q^t \cdots r^u$ say, of I, so it can be denoted by $c_{ij\cdots k}^{st\cdots u}$, and we have

(10) $$c_{ij\cdots k}^{1} N_1 + c_{ij\cdots k}^{2} N_2 + c_{ij\cdots k}^{11} N_{11} + \cdots = P_{ij\cdots k},$$

where $ij \cdots k$ can be any partition of any integer.

The first few coefficients in the first few relations (10) are tabulated below. Eliminating all N's but the first gives

$$N_1 = b_1 P_1 + b_2 P_2 + b_{11} P_{11} + \cdots, \text{ say,} \tag{11}$$

which is true of any set. In the case of the set N

$$N_1 = \pi(N) - m,$$

and, once the coefficients in (11) are determined, we have (1).

In general $f, g, \cdots, h$ and $p, q, \cdots, r$, unlike the product I, need not belong to the set. But they do belong to the set N, i.e., they too are prime to the first m primes.

Suffix in (10) and (11)	Superfix in (10) = 1	2	11	3	21	111	4	31	22	211	1111	Coeff. in (11)
1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	.	.	.	.	1	.	1	.	.	−1
11			1	1	2	3	1	3	3	5	7	−1
3				1	.	.	.	.	.	.	.	.
21					1	.	1	1	2	1	.	1
111						1	.	1	1	3	6	2
4							1	.	.	.	.	.
31								1	.	.	.	−1
22									1	.	.	−1
211										1	.	−3
1111					Coefficients in (10)						1	−6

To find all elements in the matrix of (10) and then find the first row of its reciprocal seems hopeless. A different approach is adopted.

The number of ways of expressing an integer whose prime factorization is $p^s q^t \cdots r^u$ as a product of a factors that need not be unequal, unity being an admissible factor and permutations of factors being counted separately, depends only on a and $s, t, \cdots, u$, so it can be denoted by $d_a^{st\cdots u}$. Similarly to (10) there is a set of relations

$$d_a^1 N_1 + d_a^2 N_2 + d_a^{11} N_{11} + \cdots = U_a \qquad (a = 1(1)n), \tag{12}$$

where U_a denotes the number of products of a factors as just defined that belong to the set N, and n is made so large that $N_{(s)}$ is zero if $s > n$. We shall derive (2) from the solution of (12), the coefficients in which are easily found, while the U's are simple combinations of the P's.

The coefficients are multiplicative, for $d_a^s, d_a^t, \cdots, d_a^u$ are just the respective numbers of ways of putting s things p, t things $q, \cdots, u$ things r into a numbered boxes, whence

$$d_a^{st\cdots u} = d_a^s d_a^t \cdots d_a^u.$$

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The formula

(13) $d_a^s = a(a+1)\cdots(a+s-1)/s! = C(a+s-1, s) = C(a+s-1, a-1)$

is true for any superfix when the suffix is 1 (only one box). So it will be assumed true for any superfix with suffixes $1(1)a$ and proved by induction. On this assumption d_{a+1}^s enumerates distributions of which

$C(a+s-1, a-1)$ have 0 things in the first box,
$C(a+s-2, a-1)$ have $1, \cdots,$
$C(a-1, a-1)$ have s.

The sum of the binomial coefficients is the coefficient of x^{a-1} in

$$(1+x)^{a+s-1}+(1+x)^{a+s-2}+\cdots+(1+x)^{a-1}.$$

This is the coefficient of x^a in

$$(1+x)^{a+s}-(1+x)^{a-1},$$

and so, as required for the induction,

$$d_{a+1}^s = C(a+s, a).$$

The n equations (12) cannot be solved for the individual N's, but only for n linear combinations $v, v_2, \cdots, v_n$ of them. One possible set of combinations is obtained by using (13) to rearrange the equations as polynomials in a:

(14) $$av+a^2v_2+\cdots+a^nv_n = U_a \qquad (a = 1(1)n).$$

It will be seen later that only v need be investigated. By (14)

(15) $$v = |A|^{-1}\{M_1U_1-M_2U_2+\cdots+(-)^{n-1}M_nU_n\},$$

where $|A|$ is the $n\times n$ alternant $|i^j|$, and M_i is the minor of $|A|$ obtained by deleting its ith row and first column. By easy algebra

$$|A| = 1!2!\cdots n!, \quad |A|^{-1}M_i = C(n, i)/i,$$

and substituting in (15) gives

(16) $$v = \sum(-)^{i-1}C(n, i)U_i/i \qquad (i = 1(1)n).$$

The U_i are now replaced by numbers V_i, defined as for U_i except that unity is not an admissible factor. The factors in each product enumerated by U_i are those in one enumerated by V_j $(j = 1(1)i)$, in the same order but distributed in j positions out of i, the vacant positions being filled by 1's. The number of ways of choosing the j positions is $C(i, j)$, whence

$$U_i = \sum C(i, j)V_j \qquad (j = 1(1)i),$$

and (16) becomes

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$$v = \sum_{i=1}^{n} \sum_{j=1}^{i} (-)^{i-1} C(n, i) C(i, j) V_j / i.$$

Since

$$\frac{C(i, j)}{i} = \frac{(i-1)!}{j!(i-j)!} = \frac{C(i-1, i-j)}{j},$$

we have

(17) $$v = \sum_{i=1}^{n} \sum_{j=1}^{i} (-)^{i-1} C(n, i) C(i-1, i-j) V_j / j.$$

The cofactor of $(-)^{j-1} V_j / j$ is

$$\sum C(n, i) \cdot (-)^{i-j} C(i-1, i-j) \qquad (i = j(1)n),$$

which is the coefficient of x^i / x^{i-j} or x^j in

$$(1+x)^n (1+1/x)^{-j}, = x^j (1+x)^{n-j}.$$

The coefficient is 1, so (17) becomes

(18) $$v = \sum_{j=1}^{n} (-)^{j-1} V_j / j = \sum_{j=1}^{\infty} (-)^{j-1} V_j / j.$$

The limit n, which can be as large as we please, is replaced by ∞.

We now extract the value of N_1 from (18). By (13) and the multiplicative property d_a^{st} and all more complex forms have the factor a^2, and d_a^s has the factor a but not a^2. Therefore N_{st} and all more complex forms are absent from v, and the coefficient in v of N_s is that of a in (13). This is $1/s$, whence

(19) $$v = \sum N_d / d \qquad (d = 1(1)\infty).$$

Now the number N_d of dth prime-powers in the set N is the number of primes in the set $N^{1/d}$, i.e. $(N^{1/d})_1$. Hence (19) and similar formulae for $v(N^{1/x})$ can be written

$$v(N) = \sum (N^{1/d})_1 / d, \ v(N^{1/x})/x = \sum (N^{1/dx})_1 / dx \qquad (d = 1(1)\infty),$$

a relation between two functions of x. Inverting, making $x = 1$, and using (18), we get in turn

$$(N^{1/x})_1 / x = \sum \mu(d) v(N^{1/dx}) / dx \qquad (d = 1(1)\infty),$$

(20) $$N_1 = \sum_{d=1}^{\infty} \mu(d) v(N^{1/d}) / d = \sum_{d=1}^{\infty} \sum_{j=1}^{\infty} (-)^{j-1} \mu(d) V_j(N^{1/d}) / dj.$$

The next step is to substitute

(21) $$V_j(N^{1/d}) = \sum C(j) P_{(j)}(N^{1/d}) = \sum C(j) P_{d(j)}(N) = \sum C(j) P_{d(j)},$$

where the summations are over all partitions (j), $C(j)$ denotes a multinomial

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coefficient as in (3), and $d(j)$ denotes the partition of dj obtained by multiplying each element of (j) by d. The second member includes $C(j)$ because permutations of factors are counted separately in V_j but not in $P_{(j)}$, and the second step follows from the definition of the P's (just as $(N^{1/d})_1 = N_d$). Substituting in (20) gives

$$N_1 = \sum_{d=1}^{\infty} \sum_{j=1}^{\infty} \sum_{(j)} (-)^{j-1}\mu(d)C(j)P_{d(j)}/dj. \tag{22}$$

Finally, the coefficient of $P_{(n)}$ is obtained from the terms in (22) with (j) equal to $(n)/d$. It is

$$(1/n)\sum (-)^{n/d-1}\mu(d)C(n/d) \qquad (d|(n)),$$

as announced in (2). And when $(n) = n$, the coefficient is

$$(1/n)\sum (-)^{n/d-1}\mu(d) \qquad (d|n),$$

which vanishes if n is odd and greater than 1, for the signs affecting the Möbius functions are then all the same. If n is even and greater than 2, let $n = 2^y z$ where z is odd and either y or z is greater than 1. Then the $\sum$ can be split up into

$$\sum (-)^{n/d-1}\mu(d) + \sum (-)^{n/2d-1}\mu(2d) + \cdots \qquad (d|z),$$

the unwritten sums vanishing because $\mu(4) = 0$. The first two sums cancel if $z = 1$ (whence $4|n$), and vanish separately if $z > 1$. Therefore

$$\sum (-)^{n/d-1}\mu(d) = 0 \qquad (n \geqq 3;\, d|n), \tag{23}$$

and (4) follows.

Although $1/n$ appears in (2), $b_{(n)}$ is integral. For the matrix of (10) has unit determinant, as will be seen from the table.

If the maximum value of n is set at ν, the formula is valid if

$$N < p_{m+1}^{\nu} p_{m+2}.$$

For instance if $m = 3$ and $\nu = 13$, it is valid if

$$N < 7^{13} \cdot 11 \approx 1.06 \times 10^{12}.$$

The calculation of the $P_{(n)}$'s for the larger values of n consists in the elaborate computation of many small numbers. This can be avoided by calculating V's instead. Replacing dj in (20) by n, we get

$$N_1 = \sum_{n=1}^{\infty} \sum_{d|n} (-)^{n/d-1}\mu(d)V_{n/d}(N^{1/d})/n. \tag{24}$$

For values of n up to a suitable intermediate value $n = i$, making $dj \leqq i$ in (20) leads via (22) to (1) as far as $b_{(i)}P_{(i)}$. For $n = i+1(1)\nu$, (24) can be used. Thus for $N = 10^{12}$ with $m = 3$ the numerous $b_{(13)} P_{(13)}$'s can be replaced by

$$\{V_{13}(10^{12}) - V_1(10^{12/13})\}/13.$$

Every product contributing to the V_{13} contains at least eight 7's since $7^7 \cdot 11^6 > 10^{12}$, so only $V_5(10^{12}/7^8)$ need be found. This and indeed the few non-zero $P_{(13)}$'s can easily be calculated by hand.

One can similarly use (24) to shorten (6).

Proof of (7)

With $j = n, d = 1$, (21) becomes

$$V_n = \sum_{(n)} C(n) P_{(n)},$$

(25) $$\frac{V_n}{n!} = \frac{P_n}{n!} + \frac{P_{n-1,1}}{(n-1)!1!} + \cdots.$$

Now expanding (9) and using (8) gives

$$e^x = 1 + \lambda_1 x + \lambda_2 x^2 + (\lambda_3 + \lambda_{12}) x^3 + \cdots + \Lambda_i x^i + \cdots,$$

Λ_i denoting the sum of all λ's whose suffixes are partitions of i into unequal integers. Hence

(26) $$1/i! = \Lambda_i,$$

and substituting in (25) gives

(27) $$V_n/n! = \sum_{(n)} \Lambda_{(n)} P_{(n)},$$

where

$$\Lambda_{ij\cdots k} = \Lambda_i \Lambda_j \cdots \Lambda_k.$$

It will now be shown how $V_n/n!$ is expressed in terms of Q's.

A Q can be expressed in terms of P's. For instance a product fg^3h^3, contributing 1 to $P_{(n)}$ with $(n) = 133$, can be dissected into fgg^2h^3 and fhh^2g^3, contributing 2 to $Q_{(\nu)}$ with $(\nu) = 1123$; neither of these is obtained by dissecting any other product that contributes to any $P_{(7)}$; and every contribution to Q_{1123} comes thus from *some* $P_{(7)}$. Therefore

$$Q_{1123} = 2P_{133} + \cdots,$$

the 2 reflecting the fact that either of the 3's in 133 can be partitioned into 12 to give 1123. In general, the contribution of $P_{(n)}$ to $Q_{(\nu)}$ is equal to $P_{(n)}$ multiplied by the number of ways in which elements of (n) can be partitioned, each element into unequal integers, so as to give (ν). Every such

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partition of any element j of (n) is the suffix of a λ in the Λ_j forming part of the coefficient of $P_{(n)}$, as in the first line of

$$\begin{aligned}\Lambda_{133}P_{133} &= \Lambda_1\Lambda_3\Lambda_3P_{133} = \lambda_1(\lambda_3+\lambda_{12})(\lambda_3+\lambda_{12})P_{133}\\ &= (\lambda_1\lambda_3\lambda_3+2\lambda_1\lambda_{12}\lambda_3+\lambda_1\lambda_{12}\lambda_{12})P_{133} = (\lambda_{133}+2\lambda_{1123}+\lambda_{11122})P_{133}\\ &= \lambda_{1123}\cdot 2P_{133}+\cdots.\end{aligned}$$

The number of partitions of (n) into (ν) is the number of ways in which $\lambda_{(\nu)}$ can be formed from products of individual λ's in the first line, and this is the coefficient of $\lambda_{(\nu)}$ in the expansion of $\Lambda_{(n)}$, as in the second line. Therefore the contribution of $P_{(n)}$ to $Q_{(\nu)}$ is the cofactor of $\lambda_{(\nu)}$ in the expansion of $\Lambda_{(n)}P_{(n)}$, as in the third line. This applies to every $P_{(n)}$ that contributes to $Q_{(\nu)}$, and every $P_{(n)}$ appears in (27). Hence the sum of all the cofactors of $\lambda_{(\nu)}$ in the expansion of (27) is equal to $Q_{(\nu)}$. This applies to every $Q_{(\nu)}$, and so

$$V_n/n! = \sum_{(\nu)}\lambda_{(\nu)}Q_{(\nu)} = \sum_{(n)}\lambda_{(n)}Q_{(n)}, \tag{28}$$

the sets of partitions (ν) and (n) being the same.

By (28), and similarly to (21),

$$V_j(N^{1/d}) = j!\sum\lambda_{(j)}Q_{(j)}(N^{1/d}) = j!\sum\lambda_{(j)}Q_{d(j)}(N) = j!\sum\lambda_{(j)}Q_{d(j)},$$

with summations over all partitions (j). Substituting in (20) gives

$$N_1 = \sum_{d=1}^{\infty}\sum_{j=1}^{\infty}\sum_{(j)}(-)^{j-1}\mu(d)j!\lambda_{(j)}Q_{d(j)}/dj.$$

The coefficient of $Q_{(n)}$, obtained from the terms with (j) equal to $(n)/d$, is as in (7). This completes the proof.

The property corresponding to (4) is

$$\sum_{\neq} c_{(n)} = 0 \qquad (n \geqq 3),$$

the sign $\neq$ indicating summation over partitions of n into unequal integers. For by (7)

$$n\sum_{\neq} c_{(n)} = \sum_{\neq}\sum_{d|(n)}(-)^{n/d-1}\mu(d)(n/d)!\lambda_{(n)/d}.$$

The cofactor of $(-)^{n/d-1}\mu(d)(n/d)!$ is the sum of the λ's whose suffixes are partitions of n/d into unequal integers. Therefore, using (26) in the third step and (23) in the last, we have

$$\begin{aligned}n\sum_{\neq} c_{(n)} &= \sum_{d|n}(-)^{n/d-1}\mu(d)(n/d)!\sum_{\neq}\lambda_{(n/d)}\\ &= \sum_{d|n}(-)^{n/d-1}\mu(d)(n/d)!\Lambda_{n/d} = \sum_{d|n}(-)^{n/d-1}\mu(d) = 0.\end{aligned}$$

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Although all λ's but λ_1 are fractional, as will soon be seen, $c_{(n)}$ is integral. For the matrix of the equations between N's and Q's, like that of (10), has unit determinant.

Calculation of the coefficients. The λ's are given by $\lambda_1 = 1$ and

$$(29) \qquad n\lambda_n = (-1)^n + \sum d(-\lambda_d)^{n/d} \qquad (d|n;\ 1 < d < n),$$

obtained by taking logarithms of (9) and equating coefficients. To justify this, the first n factors are multiplied out:

$$\begin{aligned}(1+\lambda_1 x)\cdots(1+\lambda_n x^n) &= 1+\lambda_1 x+\cdots \\ &= 1+x+\cdots+x^n/n!+a_{n+1}x^{n+1}+\cdots+a_w x^w, \text{ say } \left(w = \tfrac{1}{2}n(n+1)\right), \\ &= e^x+\{a_{n+1}-1/(n-1)!\}x^{n+1}+\cdots = e^x(1+x^{n+1}X),\end{aligned}$$

where X is a convergent power series in x. If x is so small that

$$|\lambda_i x^i| < 1 \quad (i = 1(1)n) \quad \text{and} \quad |x^{n+1}X| < 1,$$

we can use the formula for log $(1+y)$ with $-1 < y \leqq 1$, getting

$$\sum_{d=1}^{n} \sum_{j=1}^{\infty} (-)^{j-1}\lambda_d^j x^{dj}/j = x+x^{n+1}X+\cdots.$$

The terms with $dj = 1, n$ show that $\lambda_1 = 1$ and

$$\sum (-)^{n/d-1} d\lambda_d^{n/d} = 0 \qquad (n > 1;\ d|n).$$

Here the terms with $d = 1, n$ are $(-1)^{n-1}$, $n\lambda_n$. Writing them separately, we get (29).

It can be proved that the $\sum$ in (29) is $O(n^{-1})$, so that (9) converges if $-1 < x \leqq 1$.

Values of some λ's follow.

$n =$	2	3	4	5	6	7	8	9	10	11	12
$(-)^n\lambda_n =$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{1}{5}$	$\frac{13}{72}$	$\frac{1}{7}$	$\frac{27}{128}$	$\frac{8}{81}$	$\frac{91}{800}$	$\frac{1}{11}$	$\frac{1213}{13824}$

From these were calculated the values of $c_{(n)}$ below. Coefficients with

$$(n) = 2^i 1^{n-2i} \qquad (n \geqq 3;\ n-2i \geqq 1)$$

are omitted to save space. They are

$$b_{(n)} = c_{(n)} = (-)^{n-1}(n-1)!/2^i.$$

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(n)	$b_{(n)}$	$c_{(n)}$	(n)	$b_{(n)}$	$c_{(n)}$	(n)	$b_{(n)}$	$c_{(n)}$	(n)	$b_{(n)}$	$c_{(n)}$
1	1	1	51	−1	24	421	15	135	51^3	−42	1008
2	−1	−1	42	−3	−23	41^3	30	270	4^2	−8	−708
1^2	−1	−1	41^2	−5	−45	3^21	20	80	431	−35	630
3	0	−1	3^2	−3	−13	32^2	30	−60	42^2	−51	−471
4	0	−2	321	−10	20	321^2	60	−120	421^2	−105	−945
31	−1	2	31^3	−20	40	31^4	120	−240	41^4	−210	−1890
2^2	−1	−1	2^3	−16	−16	8	0	−1062	3^22	−70	−280
5	0	−5	7	0	−103	71	−1	720	3^21^2	−140	−560
41	1	9	61	1	130	62	−3	−456	32^21	−210	420
32	2	−4	52	3	−72	61^2	−7	−910	321^3	−420	840
31^2	4	−8	51^2	6	−144	53	−7	−336	31^5	−840	1680
6	0	−21	43	5	−90	521	−21	504	2^4	−312	−312

I thank the referee for a helpful report.

References

The earliest practical formula for $\pi(N)$ is Meissel's of 1870, described in Uspensky and Heaslet's *Elementary Number Theory*, pp. 120−2. More recent is D. H. Lehmer's formula described in his paper "On the exact number of primes less than a given limit", *Illinois J. Math.*, **3** (1959) 381−8.

The expansion similar to (9) of e^{-x} forms the subject of contributions (in English) to *Nordisk Matematisk Tidskrift* by O. Kolberg, L. Carlitz, and F. Herzog (**8** (1960) 33−4, **9** (1961) 117−22, and **10** (1962) 78−9 respectively).

Patent Office, Canberra.