GROUP RINGS WITH FINITE CENTRAL ENDOMORPHISM DIMENSION

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(Received 9 March, 1982)

1. Introduction. Let F be any field. Denote by $\mathfrak{X}_F$ the class of all groups G such that every irreducible FG-module has finite dimension over F and by $\mathfrak{Z}_F$ the class of all groups G such that every irreducible FG-module has finite dimension over its endomorphism ring. Clearly $\mathfrak{X}_F \subseteq \mathfrak{Z}_F$.

The study of the classes $\mathfrak{X}_F$ arose out of work of P. Hall and later Roseblade on residual finiteness of certain soluble groups. Recently [**2**, **5**, **7**, **8** and **9**] the soluble $\mathfrak{X}_F$-groups have been almost completely described. The classes $\mathfrak{Z}_F$ arise in connection with injective modules [**3**, Sections 3.2 and 12.4]. In some unpublished work [**1**] B. Hartley has effectively described all locally finite $\mathfrak{Z}_F$-groups and, coupled with Section 3 of [**7**], this also describes all locally finite $\mathfrak{X}_F$-groups. It seems likely that a successful assault on soluble $\mathfrak{Z}_F$-groups will require considerably more knowledge of soluble 'linear' groups over division algebras than the present author has. We therefore suggest the following intermediate class.

Let $\mathfrak{Y}_F$ be the class of all groups G such that every irreducible FG-module has finite dimension over the centre of its endomorphism ring. By a theorem of Kaplansky [**3**, 5.3.4 and 5.1.6] this is the class of all groups G such that every primitive image of FG satisfies a polynomial identity. Much of the work on $\mathfrak{X}_F$ goes through with suitable modifications for $\mathfrak{Y}_F$ and the object of this note is to indicate these modifications.

It is convenient in places to use the algebra of group classes. As usual $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{A}$, $\mathfrak{P}$ and $\mathfrak{S}$ denote respectively the classes of finite, finitely generated, abelian, polycyclic and soluble groups and S, Q, L and R the subgroup, quotient, local and residual operators. If p is a prime, $\mathfrak{P}_p$ is the class of all groups with a series of finite length whose factors are cyclic or Prüfer p^∞-groups. Throughout u will denote the characteristic of the field F (so that $u \geqslant 0$). If G is a group and p a prime then $O_0(G) = \langle 1 \rangle$ and $O_p(G)$ is the maximal normal p-subgroup of G. Also $\Lambda(G)$ is the subgroup of elements g of G such that for every finite subset X of G, the FC-centre of $\langle g, X\rangle$ contains g.

THEOREM 1. *Let F be any field. Then*

$$\mathfrak{G} \cap \mathfrak{S}\mathfrak{F} \cap \mathfrak{Y}_F = \begin{cases} \mathfrak{G} \cap \mathfrak{A}\mathfrak{F} & \text{if } F \text{ is not locally finite,} \\ \mathfrak{P}\mathfrak{F} & \text{if } F \text{ is locally finite.} \end{cases}$$

THEOREM 2. *If F is a field of characteristic $u \geqslant 0$ that is not locally finite, then $\mathfrak{S}\mathfrak{F} \cap \mathfrak{Y}_F$ is the class of all groups $\mathfrak{G}$ with*

$$G/O_u(G) \in \mathfrak{A}\mathfrak{F},\ O_u(G) \subseteq \Lambda(G) \quad \textit{and} \quad O_u(G) \in \mathfrak{S}.$$

Our information about soluble $\mathfrak{Y}_F$-groups for $\mathfrak{F}$ locally finite is even less complete than about soluble $\mathfrak{X}_F$-groups.

Glasgow Math. J. **24** (1983) 169–176.

THEOREM 3. *Let F be a locally finite field of characteristic u and let G be a soluble-by-finite $\mathfrak{Y}_F$-group. Then G has normal subgroups $O_u(G) \subseteq B \subseteq H \subseteq N \subseteq G$ satisfying:*

(a) *$O_u(G) \subseteq \Lambda(G)$ and is $O_u(G)$ soluble;*

(b) *$B/O_u(G)$ is periodic abelian, $G/C_G(B/O_u(G))$ is periodic, B is the maximal periodic normal subgroup of H and has finite index in the maximal periodic normal subgroup of G;*

(c) *H/B is a torsion-free by finite $\mathfrak{P}_u$-group;*

(d) *N/H is abelian with no elements of order u;*

(e) *G/N is finite.*

By way of comparison consider the following. Let F be a locally finite field and let $G = A \times H$ where A is abelian and H polycyclic. By [**7**, 2.2 and 3.1] and [**9**, Theorem 3] we have that $G \in \mathfrak{X}_F$ if and only if $A \in \mathfrak{X}_F$, and the latter implies that A has finite torsion-free rank.

PROPOSITION. *With F and G as above, $G \in \mathfrak{Y}_F$ if and only if either A has finite torsion-free rank or H is abelian-by-finite.*

Thus although Theorems 1 and 2 above are strikingly similar to their $\mathfrak{X}_F$ counterparts, it would seem that characterizations of $\mathfrak{S} \cap \mathfrak{X}_F$ and $\mathfrak{S} \cap \mathfrak{Y}_F$ for F locally finite will have to differ noticeably. A further difference is that if F and K are fields with $F \leqslant K$ then $\mathfrak{Y}_F \supseteq \mathfrak{Y}_K$ (an easy result—but see below), while this is not usually true for $\mathfrak{X}_F$ and $\mathfrak{X}_K$. Indeed there is a tendency for the reverse to be true. For example $\mathfrak{A} \cap \mathfrak{X}_F \subseteq \mathfrak{A} \cap \mathfrak{X}_K$ always, and thus $\mathfrak{S} \cap \mathfrak{X}_F \subseteq \mathfrak{S} \cap \mathfrak{X}_K$ whenever F is not locally finite.

2. Preliminary remarks.

2.1. *$\mathfrak{X}_F \subseteq \mathfrak{Y}_F \subseteq \mathfrak{Z}_F$ for any F.*

2.2. *$\mathfrak{A} \subseteq \mathfrak{Y}_F$ for any F.*

2.3. *$\mathfrak{Y}_F$ is $\langle$S, Q$\rangle$-closed for any F.*

The quotient closure of $\mathfrak{Y}_F$ is trivial. Let H be a subgroup of the $\mathfrak{Y}_F$-group G and let W be an irreducible FH-module. By Hall's lemma [**7**, 2.1] there is an irreducible FG-module V containing W as an FH-submodule. Now $FH/\mathrm{Ann}_{FH}W$ is an image of $FH/\mathrm{Ann}_{FH}V$, which is isomorphic to a subalgebra of $FG/\mathrm{Ann}_{FG}V$. By hypothesis this satisfies a standard polynomial identity; whence $FH/\mathrm{Ann}_{FH}W$ does too.

2.4. *$G \in \mathfrak{Y}_F$ if and only if $G/(G \cap (1 + J(FG))) \in \mathfrak{Y}_F$. In particular $G \in \mathfrak{Y}_F$ if $G/(O_u(G) \cap \Lambda(G)) \in \mathfrak{Y}_F$.*

Here $J(FG)$ is the Jacobson radical of FG. The first part is immediate and the second follows since $(O_u(G) \cap \Lambda(G)) - 1$ generates a nil ideal of FG. Also immediate from the definitions is the following.

2.5. *$G/(G \cap (1 + J(FG)))$ is residually an irreducible linear group (over various extension fields of F, namely the centres of the endomorphism rings of the irreducible FG-modules).*

2.6. *$G \cap (1 + J(FG)) \subseteq O_u(G)$ with equality if $G \in \mathfrak{Y}_F$.*

https://doi.org/10.1017/S0017089500005243 Published online by Cambridge University Press

The containment is well known and is recorded in [**7**, 2.5]. Suppose $G \in \mathfrak{Y}_F$. Let ρ be an irreducible representation of G, finite dimensional over some extension field of F. Then $O_u(G\rho)$, being unipotent and completely reducible [**6**, 9.1v and 1.8], is trivial. Thus $O_u(G) \subseteq \bigcap \ker \rho$, which is $G \cap (1+J(FG))$ by 2.5.

2.7. *If H is a subgroup of the group G of finite index then $H \in \mathfrak{Y}_F$ if and only if $G \in \mathfrak{Y}_F$.*

If $G \in \mathfrak{Y}_F$ then $H \in \mathfrak{Y}_F$ by 2.3. Suppose $H \in \mathfrak{Y}_F$. Again by 2.3 we may assume that H is normal in G. Let V be an irreducible FG-module, set $R = FG/\mathrm{Ann}_{FG} V$ and let S be the natural image of FH in R. By a version of Clifford's theorem [**3**, 7.2.16] V is a direct sum of a finite number of irreducible FH-modules. Since $H \in \mathfrak{Y}_F$ it follows that the centre Z of S is a direct sum of a finite number of fields and that S is finitely generated as Z-module. Clearly G normalizes Z and H centralizes Z. Thus the finite group G/H acts on Z. By a result from invariant theory (in fact an easy extension of a result from Galois theory) Z can be generated as $C_Z(G)$-module by $|G/H|$ elements. Since $C_Z(G)$ is central in R the result follows.

As a companion to 2.7 we have the following.

2.8. *Let F, K be fields with $F \leq K$. Then $\mathfrak{Y}_F \supseteq \mathfrak{Y}_K$ with equality if $(K:F)$ is finite.*

Let $G \in \mathfrak{Y}_K$ and suppose that V is an irreducible FG-module. Then $V \cong FG/A$ for some right ideal A. Now $AK \neq KG$ so that A lies in a maximal right ideal B of KG. Clearly $A = B \cap FG$ and V embeds into the irreducible KG-module $W = KG/B$. Since $KG/\mathrm{Ann}_{KG} W$ satisfies a standard polynomial identity, so does $FG/\mathrm{Ann}_{FG} W$ and hence $FG/\mathrm{Ann}_{FG} V$. Therefore $G \in \mathfrak{Y}_F$.

Now let $G \in \mathfrak{Y}_F$ where $(K:F)$ is finite. Let W be an irreducible KG-module. Then W is finitely FG-generated, so W contains a maximal FG-submodule V. Now $W\mathrm{Ann}_{FG}(W/V)$ is a KG-submodule of W in V and so is zero. Thus $\mathrm{Ann}_{FG}(W/V) = FG \cap \mathrm{Ann}_{KG} W$. By hypothesis $FG/\mathrm{Ann}_{FG}(W/V)$ satisfies a polynomial identity. Hence so does $KG/\mathrm{Ann}_{KG} W = K((FG + \mathrm{Ann}_{KG} W)/\mathrm{Ann}_{KG} W)$, for example by [**3**, 5.1.3].

2.9 (B. Hartley). *$L\mathfrak{F} \cap \mathfrak{Y}_F$ is the class of locally finite groups G with $G/O_u(G)$ abelian-by-finite.*

Hartley's theorem [**1**] is that $G/O_u(G)$ is abelian-by-finite if merely $G \in L\mathfrak{F} \cap \mathfrak{Z}_F$. The converse follows from 2.2, 2.7 and 2.4.

2.10. *The wreath product $G = (\mathbb{Z}/n\mathbb{Z})\mathrm{wr}\,\mathbb{Z}$ is not in $\mathfrak{Z}_F$ (and hence not in $\mathfrak{Y}_F$) for $n = 0$, 2, 3, ...*

By 2.3 we may assume that n is a prime power. Then by [**3**, 9.2.8] the group ring FG is primitive and not simple, but if $G \in \mathfrak{Z}_F$ then every primitive image of G is simple by the density theorem.

2.11. *If $G \in \mathfrak{Y}_F$ then $G/O_u(G) \in LR\mathfrak{F} \cap R(\mathfrak{A}\mathfrak{F}) \subseteq \mathfrak{A}\,.\,R\mathfrak{F}$.*

We may assume that $O_u(G) = \langle 1 \rangle$. Linear groups are in $LR\mathfrak{F}$ [**6**, 4.2] and G is residually linear by 2.5 and 2.6. Thus $G \in LR\mathfrak{F}$. Let ρ be a finite-dimensional irreducible representation of G over some extension field of F. Clearly G can contain no non-cyclic

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free subgroups. If $u=0$ then $G\rho$ is soluble-by-finite by Tits' theorem [**6**, 10.17]. If $u\neq 0$ Tits' theorem yields only that $G\rho\in\mathfrak{S}\mathrm{L}\mathfrak{F}$. But then 2.9 with 2.3 and [**6**, 3.8] implies that here also $G\rho$ is soluble-by-finite. Since ρ is irreducible, in both cases $G\rho$ is abelian-by-finite by Mal'cev's theorem [**6**, 3.5]. This proves that $G\in\mathrm{R}(\mathfrak{A}\mathfrak{F})$ and clearly $\mathrm{R}(\mathfrak{A}\mathfrak{F})\subseteq\mathfrak{A}\mathrm{R}\mathfrak{F}$.

3. Finitely generated groups. Write $\tau(G)$ for the maximal periodic normal subgroup of a group G.

3.1. *Let G be a soluble-by-finite $\mathfrak{Y}_F$-group, where F is not locally finite. Then $G/\tau(G)$ is abelian-by-finite.*

If G is countable we can repeat the proof of [**7**, 5.1] (except that the representations are now over extension fields of F). A standard argument reduces the general case to the countable case (cf. the proof of [**8**, Lemma 1]).

3.2. *Let G be a finitely generated $\mathfrak{Y}_F$-group. If A is a periodic abelian section of G (and u-free if $u>0$) then A has finite exponent.*

Suppose otherwise. By [**8**, Lemma 2] there is an infinite image of A of rank 1. Hence we may assume that A has rank 1. But then there is an irreducible FA-module that is faithful on A. Thus by Hall's lemma there is a finite-dimensional irreducible representation ρ of G over some extension field of F whose kernel avoids A. Tits's and Mal'cev's theorems yield that $G\rho$ is abelian-by-finite. Also G is finitely generated and A is periodic. Thus A, being isomorphic to a section of $G\rho$, is finite. This contradiction completes the proof.

3.3. *Let $x\in\mathbb{C}^*$ be such that, for some prime u, x is integral over $\mathbb{Z}[1/u]$ but not over $\mathbb{Z}$. Let $A=\mathbb{Z}[x,x^{-1}]\subseteq\mathbb{C}$. Multiplication by x is an automorphism of A; so we may let G be the split extension $\langle x\rangle[A$ of A by $\langle x\rangle$. If F is any field of characteristic u then G is not a $\mathfrak{Y}_F$ group.*

We already know [**7**, 4.5] that G is not an $\mathfrak{X}_F$-group. In the proof of that result we constructed a certain irreducible FG-module V where

$$V=\bigcup_{i\in\mathbb{Z}}U_i,\quad U_i\cong_{FA}FA/(A_i-1)FA\cong F(A/A_i)\quad\text{and}\quad A_i=\sum_{j\leq i}\mathbb{Z}x^j\subseteq A.$$

Let K be the kernel of the representation of G on V. Then $A\cap K\subseteq\bigcap_i A_i=\{0\}$. Thus $[A,K]=\langle 1\rangle$ and yet $\langle x\rangle$ acts faithfully on A. Therefore $K=\langle 1\rangle$. If $G\in Y_F$ then Mal'cev's theorem yields that $G\cong G/K$ has an abelian normal subgroup B of finite index m say. Then x^m acts trivially on $A\cap B$, which is impossible since x has infinite order, $A\cap B$ is nontrivial and A is a domain. Consequently $G\notin Y_F$.

We remark in passing that the above proof contains the following.

3.4. *If G and F are as in 3.3 then FG is primitive.*

3.5. *Proof of Theorem* 1. In view of 3.1 it suffices to prove that if $G\in\mathfrak{G}\cap\mathfrak{S}\mathfrak{F}\cap\mathfrak{Y}_F$ then G is polycyclic-by-finite. By the usual reductions (using passage to a subgroup of

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finite index, induction on derived length and the maximal condition on normal subgroups) we may assume that G has a non-trivial abelian normal subgroup A that is either of prime exponent or is torsion-free, and that every proper image of G is polycyclic.

Suppose A has prime exponent q. In view of 2.3 and 2.10 there is no subgroup of G isomorphic to $(\mathbb{Z}/q\mathbb{Z})\mathrm{wr}\,\mathbb{Z}$. Now A is finitely G-generated, G/A is polycyclic and every proper image of $\mathbb{F}_q[X, X^{-1}]$ is finite. It follows that A is finite, so that G is polycyclic in this case as required. (I have lifted this trick from [**2**].)

Now assume that A is torsion-free. Trivially $A \cap \tau(G) = \langle 1 \rangle$, so that if F is not locally finite then G is abelian-by-finite by 3.1 (applied twice). Suppose F is locally finite. By a lemma of P. Hall [**4**, 9.53 Corollary 1] A contains a free abelian subgroup A_0 such that A/A_0 is periodic with finite spectrum. If q is a prime not in the spectrum of A/A_0 then $A_0 \cap A^q = A_0^q$. Now G/A^q is polycyclic. Hence A_0/A_0^q is finite and A has a finite rank. Let $A_1/A_0 = O_{u'}(A/A_0)$, so A/A_1 is a u-group and A_1/A_0 a u'-group. A_1/A_0 has finite exponent by 3.2 and A_1 is torsion-free. Thus A_1 is free abelian of finite rank.

G, being soluble of finite rank, is nilpotent-by-abelian-by-finite [**4**, 3.25]. Repeating if necessary our initial reductions we may assume that G' centralizes A. Now we choose A of minimal rank. Thus $\mathbb{Q}\otimes_{\mathbb{Z}} A$ is irreducible as $\mathbb{Q}G$-module and by Schur's lemma there is a finite field extension K *of* $\mathbb{Q}$, an embedding $\lambda : A \to K$ and a homomorphism $\mu : G \to K^*$ such that the G-module structure of A is induced by multiplication of $A\lambda$ by $G\mu$ in K. Since A/A_1 is a u-group we have $A\lambda \subseteq A_1\lambda\mathbb{Z}[1/u]$ and so $G\mu$ is integral over $\mathbb{Z}[1/u]$.

Suppose $G\mu$ is integral over $\mathbb{Z}$. Then $\mathbb{Z}[G\mu]$ is finitely generated as $\mathbb{Z}$-module and consequently so is $A_1\lambda\mathbb{Z}[G\mu]$. Thus $\langle A_1^G \rangle$ is a finitely generated (abelian) group. Since $A_1 \neq \langle 1 \rangle$ the group $G/\langle A_1^G \rangle$ is polycyclic and therefore so is G. We are left with the case where G contains an element g such that $g\mu$ is not integral over $\mathbb{Z}$. Necessarily $g\mu$ has infinite order. Let $a \in A \setminus \langle 1 \rangle$. Then $\langle g, a \rangle \in \mathfrak{Y}_F$ by 2.3 and yet

$$\langle g, a \rangle = \langle g \rangle \langle a^{\langle g \rangle} \rangle \cong \langle g\mu \rangle [a\lambda\,\mathbb{Z}[g\mu, g^{-1}\mu] \cong \langle g\mu \rangle [\mathbb{Z}[g\mu, g^{-1}\mu].$$

This contradicts 3.3 and completes the proof of Theorem 1.

4. Soluble groups.

4.1. *Let F be a field (with* $\mathrm{char}F = u \geq 0$ *as always) and* $G = \langle x \rangle [A$ *be the split extension of its abelian normal subgroup A of finite torsion-free rank by the infinite cyclic group $\langle x \rangle$. Suppose that $A \setminus \langle 1 \rangle$ contains no elements of order u, and that if F is locally finite then A is periodic. If $G \in \mathfrak{Y}_F$ then $C_{\langle x \rangle}(A) \neq \langle 1 \rangle$.*

This generalizes [**8**, Lemma 4 and **9**, Lemma 1] and the proof of 4.1 is similar to these. Consider the proof of Lemma 4 of [**8**]. In the second and third paragraphs of that proof we construct certain direct sums of finitely many irreducible representations of G. In the context of 4.1 they are no longer finite dimensional over F but are finite-dimensional over suitable extension fields of F. The results of [**6**] still apply and the construction of X goes through as in [**8**].

In paragraphs 4, 5 and 6 of the proof we constructed an extension field K of F, an irreducible KG-module V and a maximal FG-submodule W of V. Recall that we were

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seeking a contradiction. Let σ be the representation of G on V and τ of G on V/W. Since $G \in \mathfrak{Y}_F$ the group $G\tau$ is an irreducible linear group over some extension of F. By [**8**, Lemma 3] there is a normal subgroup L of G of finite index and containing A . ker τ such that $L\tau$ is abelian. Now $V((\ker \tau)-1)$ is a KG-submodule of V in W and therefore is zero. Thus $\ker \sigma = \ker \tau$. Let $m = (G:L)$. Then $[x^m, A] \subseteq \ker \sigma$, which implies (see the construction of V) that $a\phi = a^{x^m}\phi$ for every $a \in A$. This is false by construction.

4.2. *Let G be a $\mathfrak{Y}_F$-group with $O_u(G) = \langle 1 \rangle$. Then G has a normal subgroup K with G/K u-free-abelian by finite and $\tau(K)$ abelian and of finite index in $\tau(G)$.*

By Hartley's theorem (2.9) and [**3**, 12.1.2] there exists an abelian characteristic subgroup T of $\tau(G)$ of finite index. Let Q be a maximal u-subgroup of $\tau(G)$. Since $O_u(G) = \langle 1 \rangle$ the subgroup T is a u'-group and Q is finite. By 2.11 there exists K_1 normal in G with G/K_1 abelian-by-finite and $Q \cap K_1 = \langle 1 \rangle$. Also by 2.11 there exists K_2 normal in G with G/K_2 abelian-by-finite, $T \subseteq K_2$ and $(\tau(G) \cap K_2)/T = O_u(G/T)$. Set $K_0 = K_1 \cap K_2$. Clearly G/K_0 is abelian-by-finite, and $\tau(K_0) = K_0 \cap \tau(G) = K_0 \cap T$ by elementary Sylow theory. If X is any irreducible linear group over an extension field of F then $O_u(X) = \langle 1 \rangle$. Thus the proof of 2.11 shows that we can choose K_0 with $O_u(G/K_0) = \langle 1 \rangle$. Finally set $K = TK_0$.

4.3. *Let G be a soluble-by-finite $\mathfrak{Y}_F$-group where F is not locally finite. Then $G/O_u(G)$ is abelian-by-finite.*

This can be proved along the lines of the proof of [**8**, Theorem 2]. However we indicate a better approach using ideas from both [**5**] and [**8**].

By [**8**, Lemma 1] we may assume that G is countable. Also we may assume that $O_u(G) = \langle 1 \rangle$. Choose K as in 4.2. Also $G/\tau(G)$ is abelian-by-finite by 3.1. Hence G has a normal subgroup H of finite index with $H' \subseteq K \cap \tau(G)$. Thus H' is periodic abelian. Also $O_u(H) = \langle 1 \rangle$.

Let $x_1, x_2, \ldots$ be a transversal of H' to H. Suppose we have constructed $r_1, \ldots, r_{i-1} > 0$ such that $A_i = \langle x_j^{r_j} : j < i \rangle H'$ is abelian. Since A_i is normal in H we have $O_u(A_i) = \langle 1 \rangle$ and 4.1 yields the existence of $r_i > 0$ such that $[A_i, x_i^{r_i}] = \langle 1 \rangle$ (if $|x_i| < \infty$ set $r_i = |x_i|$). Then $A_{i+1} = A_i \langle x_i^{r_i} \rangle$ is abelian. By induction we construct an abelian normal subgroup $A = \bigcup_{i \geq 1} A_i$ containing H' with H/A periodic.

Let $a \in A$. We claim that $[a, H]$ has finite exponent $e(a)$ say. For, if not, by [**8**, Lemma 2] there is an infinite-rank-1 image of $[a, H]$; $[a, H]$ is contained in H' and is periodic. By Hall's lemma there is an irreducible representation ρ of H over some extension of F such that $[a, H]\rho$ is infinite. But $(H\rho : C_{H\rho}(A\rho))$ is finite, e.g. by [**8**, Lemma 3], so $[a, H]\rho = \langle a^{-1}a^H \rangle \rho$ is finitely generated, abelian and periodic. This contradiction confirms the existence of finite $e(a)$.

Let Q be a maximal torsion-free subgroup of A and set $B = \langle a^{e(a)} : a \in Q \rangle$. Now $a \in A$ stabilizes the series $H \supseteq [a, H] \supseteq \langle 1 \rangle$ since $A \supseteq [a, H]$ is abelian. Thus $a^{e(a)}$ centralizes H and in particular B is normal in H. Also H/B is periodic, so by 2.9 again H/B has a normal subgroup N/B of finite index such that $N'B/B$ is a u-group. But $N' \subseteq H'$ is a u'-group, so that $N' \subseteq B \cap H' = \langle 1 \rangle$ since $B \subseteq Q$ is torsion-free. The proof of 4.3 is complete.

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4.4. *Proof of Theorem* 2. Let $G \in \mathfrak{S}\mathfrak{F} \cap \mathfrak{Y}_F$. Then $G/O_u(G) \in \mathfrak{A}\mathfrak{F}$ by 4.3, $O_u(G) \subseteq \Lambda(G)$ by 2.6 and [**7**, 2.9], and $O_u(G)$ is clearly soluble.

Conversely suppose G is a group with $G/O_u(G) \in \mathfrak{A}\mathfrak{F}$ and $O_u(G) \subseteq \Lambda(G)$. Then $G/O_u(G) \in \mathfrak{Y}_F$ by 2.2 and 2.7, so that $G \in \mathfrak{Y}_F$ by 2.4. If also $O_u(G)$ is soluble then $G \in \mathfrak{S}\mathfrak{F}$.

4.5. *Proof of Theorem* 3. By 2.6 and [**7**, 2.9] we have $O_u(G) \subseteq \Lambda(G)$ and $O_u(G)$ clearly is soluble. From now on assume that $O_u(G) = \langle 1 \rangle$. By 4.2 there exist normal subgroups $B \subseteq H_1 \subseteq N_1$ of G with $(G:N_1)$ finite, N_1/H_1 abelian and u-free, $B = \tau(H_1)$ an abelian u'-group and $(\tau(G):B)$ finite. $G/C_G(B)$ is periodic by 4.1. Since $G \in \mathfrak{S}\mathfrak{F}$ we may choose N_1 to be soluble.

Let X be a free abelian section of N_1/B of infinite rank. There exists a purely transcendental extension K of F for which there is a homomorphism of FX onto K that is one-to-one on X. Hence by Hall's lemma there is an irreducible representation ρ of G/B over some extension of F such that $\ker \rho$ avoids X. Also $G\rho$ is abelian-by-finite with $O_u(G\rho) = \langle 1 \rangle$. Apply this to a free abelian subgroup of maximal rank in each factor with infinite torsion-free rank of the derived series of N_1/B. It follows that G contains normal subgroups $N_2 \subseteq N_1$ and $H_2 \subseteq H_1 \cap N_2$ with $(G:N_2)$ finite, N_2/H_2 abelian and u-free, $B \subseteq H_2$ and H_2/B is poly-(abelian with finite torsion-free rank). By a theorem of Mal'cev [**4**, 9.34 and 9.39.3] and the finiteness of $(\tau(G):B)$, the section H_2/B is torsion-free by finite, and soluble of finite rank.

Now let X be a periodic abelian u'-section of H_2/B of rank 1. Then X can be embedded into the multiplicative group of the algebraic closure of F so that by Hall's lemma applied to the extension of F generated by this image of X there exists an irreducible representation ρ of G over some extension of F such that $\ker \rho$ avoids X. Again $G\rho$ is abelian-by-finite with $O_u(G\rho) = \langle 1 \rangle$. Hence we can find normal subgroups $N \subseteq N_2$ and $H \subseteq H_2 \cap N$ with $(G:N)$ finite, N/H abelian and u-free, $B \subseteq H$ and $H/B \in \mathfrak{P}_u$. The proof is complete.

4.6. *Proof of Proposition.* Let $G \in \mathfrak{Y}_F$ and suppose that A has a free abelian subgroup X of infinite rank. If x is an indeterminate there is a homomorphism of X onto $F(x)^*$, which can be extended to a homomorphism ϕ of A into $\overline{F(x)}^*$ by injectivity, the bar here denoting the algebraic closure. Then $K = F[A\phi]$ is a field that is not locally finite. Let V be any irreducible KH-module. Then ϕ extends to a homomorphism of FG onto KH and V becomes an irreducible FG-module with $\mathrm{End}_{FG} V = \mathrm{End}_{KH} V$. Since $G \in \mathfrak{Y}_F$ these endomorphism rings are finite dimensional over their centres and we have $H \in \mathfrak{P} \cap \mathfrak{Y}_K$. Thus H is abelian-by-finite by Theorem 1.

Conversely if H is abelian-by-finite, so is G, and $G \in \mathfrak{Y}_F$ by 2.2 and 2.7. Now suppose that A has finite torsion-free rank. Let ρ be an irreducible representation of FG and set $J = (FA)\rho$. As in the proof of [**9**, Theorem 3], the ring J is a field. If X is a free abelian subgroup of A of maximal rank, then J is integral over $(FX)\rho$, so that the latter too is a field. But X is finitely generated and thus $(FX)\rho$ is a finite extension of F by the Nullstellensatz. Consequently J is locally finite. Now $(FG)\rho$ is a homomorphic image of JH and by Roseblade's theorem [**3**, 12.3.7] $JH \in \mathfrak{X}_F$. Therefore $(FG)\rho$ is finite dimensional over its central subfield J and we have $G \in \mathfrak{Y}_F$ as required.

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