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STRONGLY GORENSTEIN FLAT MODULES

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Dedicated to Professor Boxun Zhou on the occasion of his 90th birthday
and Professor Wenting Tong on his 70th birthday

Abstract

In this paper, strongly Gorenstein flat modules are introduced and investigated. An R-module M is called strongly Gorenstein flat if there is an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective R-modules with $M = \ker(P^0 \to P^1)$ such that $\mathrm{Hom}(-, F)$ leaves the sequence exact whenever F is a flat R-module. Several well-known classes of rings are characterized in terms of strongly Gorenstein flat modules. Some examples are given to show that strongly Gorenstein flat modules over coherent rings lie strictly between projective modules and Gorenstein flat modules. The strongly Gorenstein flat dimension and the existence of strongly Gorenstein flat precovers and pre-envelopes are also studied.

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1. Introduction

Let R be a ring. A right R-module N is called *Gorenstein flat* [15] if there is an exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ of flat right R-modules with $N = \ker(F^0 \to F^1)$ such that $- \otimes E$ leaves the sequence exact whenever E is an injective left R-module. Gorenstein flat modules have been studied extensively by many authors (see, for example, [5, 7, 8, 10, 12–16, 19, 20, 23]). These modules have nice properties when the ring in question is n-Gorenstein (a ring R is called n-*Gorenstein* if R is a left and right Noetherian ring with self-injective dimension at most n for an integer $n \geq 0$ on either side).

Following [22], a left R-module M is called FP-*injective* if $\mathrm{Ext}^1(N, M) = 0$ for all finitely presented left R-modules N. The FP-*injective dimension of* M,

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denoted by FP-id(M), is defined to be the smallest nonnegative integer n such that $\mathrm{Ext}^{n+1}(F, M) = 0$ for every finitely presented left R-module F. If no such n exists, set FP-id$(M) = \infty$. A ring R is said to be an *n-FC ring* [10] if R is a left and right coherent ring with FP-id$({}_RR) \leq n$ and FP-id$(R_R) \leq n$ for an integer $n \geq 0$. A ring R is called an *FC ring* if R is 0-FC.

It is clear that every n-Gorenstein ring is n-FC, but the converse is not true in general. For example, let $R = \Pi F$, an infinite product of a field F; then R is a commutative von Neumann regular ring. Clearly, R is n-FC but not n-Gorenstein for any $n \geq 0$. Furthermore, if $S = R[X_1, X_2, \ldots, X_m]$, the ring of polynomials in m indeterminates over R, then S is n-FC but not n-Gorenstein for all $n \geq m$ (see [18, Theorem 7.3.1]).

The main purpose of this paper is to introduce and study strongly Gorenstein flat modules over n-FC rings. In Section 2, the definition and some general results are given. A right R-module M is called *strongly Gorenstein flat* if there is an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective right R-modules with $M = \ker(P^0 \to P^1)$ such that $\mathrm{Hom}(-, \mathcal{F}lat)$ leaves the sequence exact, where $\mathcal{F}lat$ stands for the class of all flat right R-modules. It is proven that a right R-module M over a left coherent ring R is strongly Gorenstein flat if and only if M has an exact right $\mathcal{F}lat$-resolution and $\mathrm{Ext}^i(M, F) = 0$ for all flat right R-modules F and all $i \geq 1$. For an n-FC ring R, we prove that $wD(R) \leq n$ if and only if every strongly Gorenstein flat right R-module is projective (flat). Several well-known classes of rings are characterized in terms of strongly Gorenstein flat modules. For instance, R is a right perfect ring if and only if every flat right R-module is strongly Gorenstein flat; R is a QF ring if and only if every right R-module is strongly Gorenstein flat. Some examples are given to show that strongly Gorenstein flat modules over coherent rings lie strictly between projective modules and Gorenstein flat modules.

In Section 3, we define and investigate the strongly Gorenstein flat dimension for modules and rings. Given a right R-module M, the strongly Gorenstein flat dimension $SGfd(M)$ of M is defined to be the infimum of the set of n such that there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ of right R-modules, where each G_i is strongly Gorenstein flat. If no such n exists, set $SGfd(M) = \infty$. The right strongly Gorenstein flat dimension $rSGFD(R)$ of a ring R is defined to be the supremum of the set of $SGfd(M)$ such that M is any right R-module. It is shown that $rFID(R) \leq rSGFD(R) \leq rD(R)$ for any ring R and the equalities hold if $wD(R) < \infty$, where $rFID(R)$ is defined to be the supremum of the set of $\mathrm{id}(M)$ such that M is any flat right R-module. For a two-sided coherent ring R and an integer $n \geq 1$, we get that R is an n-FC ring if and only if $SGfd(M) \leq n$ for any (right and left) finitely presented R-module M if and only if the nth $\mathcal{F}lat$-cosyzygy of every (right and left) finitely presented R-module is strongly Gorenstein flat.

Section 4 is devoted to the existence of precovers and pre-envelopes by strongly Gorenstein flat modules. Let $\mathcal{SGF}$ be the class of all strongly Gorenstein flat right R-modules and $\mathcal{I}_n$ the class of all right R-modules of injective dimension at most n, where n is a fixed nonnegative integer. We show that, for a left coherent ring R

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with $rSGFD(R) \leq n$, every right R-module M has an $\mathcal{SGF}$-precover. Let R be a left coherent ring with $FP\text{-id}({}_RR) \leq n$. It is shown that $rSGFD(R) \leq n$ if and only if $rFID(R) \leq n$ if and only if $(\mathcal{SGF}, \mathcal{I}_n)$ is a cotorsion theory. Finally, for an n-FC ring R, we prove that every finitely presented right R-module has an $\mathcal{SGF}$-pre-envelope.

Next we recall some known notions and facts required in the paper. Let $\mathcal{C}$ be a class of R-modules and M an R-module. Following [11], we say that a homomorphism $\phi : M \to C$ is a *$\mathcal{C}$-pre-envelope* if $C \in \mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}(\phi, C') : \mathrm{Hom}(C, C') \to \mathrm{Hom}(M, C')$ is surjective for each $C' \in \mathcal{C}$. Dually we have the definition of a *$\mathcal{C}$-precover*.

Given a class $\mathcal{L}$ of R-modules, write $\mathcal{L}^\perp = \{C \mid \mathrm{Ext}^1(L, C) = 0 \text{ for all } L \in \mathcal{L}\}$ and ${}^\perp\mathcal{L} = \{C \mid \mathrm{Ext}^1(C, L) = 0 \text{ for all } L \in \mathcal{L}\}$. A pair $(\mathcal{F}, \mathcal{C})$ of classes of R-modules is called a *cotorsion theory* [13] if $\mathcal{F}^\perp = \mathcal{C}$ and $\mathcal{F} = {}^\perp\mathcal{C}$.

It is known that a left coherent ring is characterized by the condition that every right R-module has a $\mathcal{F}lat$-pre-envelope (see [11]). So, for a left coherent ring R, every right R-module M has a *right $\mathcal{F}lat$-resolution*, that is, there is a $\mathrm{Hom}(-, \mathcal{F}lat)$ exact complex $0 \to M \to F^0 \to F^1 \to \cdots$ (not necessarily exact) with each F^i flat. Let $L^0 = M$, $L^1 = \mathrm{coker}(M \to F^0)$, $L^i = \mathrm{coker}(F^{i-2} \to F^{i-1})$ for $i \geq 2$. The nth cokernel L^n $(n \geq 0)$ is called the nth *$\mathcal{F}lat$-cosyzygy* of M. Note that we can choose each F^i to be finitely generated projective if M is finitely presented by [13, Example 8.3.10]. So by the nth $\mathcal{F}lat$-cosyzygy of a finitely presented right R-module, we will mean the nth cokernel in such a right $\mathcal{F}lat$-resolution.

On the other hand, it has been proven that every right R-module over any ring R has a $\mathcal{F}lat$-precover (see [6]), and hence every right R-module N has a *left $\mathcal{F}lat$-resolution*, that is, there is a $\mathrm{Hom}(\mathcal{F}lat, -)$ exact complex $\cdots \to F_1 \to F_0 \to N \to 0$ with each F_i flat. Thus, for a left coherent ring R, $\mathrm{Hom}(-, -)$ is left balanced on $\mathcal{M}_\mathcal{R} \times \mathcal{M}_\mathcal{R}$ by $\mathcal{F}lat \times \mathcal{F}lat$ (see [13, Definition 8.2.13]), where $\mathcal{M}_\mathcal{R}$ denotes the category of right R-modules. So the *nth left derived functor*, which is denoted by $\mathrm{Ext}_n(-, -)$, can be computed using a right $\mathcal{F}lat$-resolution of the first variable or a left $\mathcal{F}lat$-resolution of the second variable.

Throughout this paper, R is an associative ring with identity and all modules are unitary. Then $rD(R)$ $(wD(R))$ stands for the right (the weak) global dimension of R. We denote by M_R $({}_RM)$ a right (left) R-module. The character module $\mathrm{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+; $\mathrm{fd}(M)$, $\mathrm{id}(M)$ and $\mathrm{pd}(M)$ stand for the flat, injective and projective dimensions of M, respectively. Let M and N be two R-modules. Then $\mathrm{Hom}(M, N)$ $(\mathrm{Ext}^n(M, N))$ means $\mathrm{Hom}_R(M, N)$ $(\mathrm{Ext}^n_R(M, N))$, and similarly $M \otimes N$ $(\mathrm{Tor}_n(M, N))$ denotes $M \otimes_R N$ $(\mathrm{Tor}^R_n(M, N))$ for an integer $n \geq 1$. For unexplained definitions and terminology, we refer the reader to [1, 13, 21, 23].

2. Strongly Gorenstein flat modules

We start with the following result.

DEFINITION 2.1. A right R-module M is called *strongly Gorenstein flat* if there is an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective right R-modules with $M = \ker(P^0 \to P^1)$ such that $\mathrm{Hom}(-, \mathcal{F}lat)$ leaves the sequence exact.

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REMARK 2.2.

(1) It is clear that every projective module is strongly Gorenstein flat.
(2) The class of strongly Gorenstein flat right R-modules is closed under direct sums by definition.
(3) If $\mathcal{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ is a $\mathrm{Hom}(-, \mathcal{F}lat)$ exact exact sequence of projective right R-modules, then by symmetry, all the images, the kernels and the cokernels of $\mathcal{P}$ are strongly Gorenstein flat.

The next proposition shows that strongly Gorenstein flat modules over coherent rings are indeed stronger than Gorenstein flat modules.

PROPOSITION 2.3. *Let R be a left coherent ring. Then every strongly Gorenstein flat right R-module is Gorenstein flat.*

PROOF. Let M be a strongly Gorenstein flat right R-module. Then there is an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective right R-modules with $M = \ker(P^0 \to P^1)$ such that $\mathrm{Hom}(-, \mathcal{F}lat)$ leaves the sequence exact. Let E be an injective left R-module. Then E^+ is flat by [17, Theorem 2.2]. So we get the exact sequence

$$\cdots \to \mathrm{Hom}(P^1, E^+) \to \mathrm{Hom}(P^0, E^+) \to \mathrm{Hom}(P_0, E^+) \to \mathrm{Hom}(P_1, E^+) \to \cdots$$

which gives the exactness of

$$\cdots \to (P^1 \otimes E)^+ \to (P^0 \otimes E)^+ \to (P_0 \otimes E)^+ \to (P_1 \otimes E)^+ \to \cdots .$$

Thus we have the exact sequence

$$\cdots \to P_1 \otimes E \to P_0 \otimes E \to P^0 \otimes E \to P^1 \otimes E \to \cdots .$$

Hence M is Gorenstein flat. □

LEMMA 2.4. *Let M be a strongly Gorenstein flat right R-module. Then:*

(1) $\mathrm{Ext}^i(M, G) = 0$ *for all right R-modules G with* $\mathrm{fd}(G) < \infty$ *and all* $i \geq 1$;
(2) *either M is projective or* $\mathrm{fd}(M) = \infty$.

PROOF. (1) By hypothesis, there is a $\mathrm{Hom}(-, \mathcal{F}lat)$ exact exact sequence $\cdots \to P_1 \to P_0 \to M \to 0$ with each P_i projective. Thus $\mathrm{Ext}^i(M, F) = 0$ for all flat right R-modules F and all $i \geq 1$. By dimension shifting, $\mathrm{Ext}^i(M, G) = 0$ for all right R-modules G with $\mathrm{fd}(G) < \infty$ and all $i \geq 1$.

(2) Suppose $\mathrm{fd}(M) < \infty$. We may assume that $\mathrm{fd}(M) \leq n$ with $1 \leq n < \infty$. Then there is a $\mathrm{Hom}(-, \mathcal{F}lat)$ exact exact sequence $0 \to F_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ with F_n flat and each P_i projective. It is easy to see that M is isomorphic to a direct summand of P_0, and hence M is projective. □

COROLLARY 2.5. *A right R-module M is projective if and only if M is flat and strongly Gorenstein flat.*

Now we give some characterizations of strongly Gorenstein flat modules.

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THEOREM 2.6. *Let R be a left coherent ring. Then the following statements are equivalent for a right R-module M.*

(1) *M is strongly Gorenstein flat.*

(2) *M has an exact right $\mathcal{F}lat$-resolution and $\mathrm{Ext}^i(M, F) = 0$ for all right R-modules F with $\mathrm{fd}(F) < \infty$ and all $i \geq 1$.*

(3) *M has an exact right $\mathcal{F}lat$-resolution and $\mathrm{Ext}^i(M, F) = 0$ for all flat right R-modules F and all $i \geq 1$.*

Moreover, if FP-id$({}_RR) < \infty$, then the above conditions are equivalent to

(4) $\mathrm{Ext}^i(M, F) = 0$ *for all flat right R-modules F and all $i \geq 1$.*

PROOF. That (1) implies (2) follows from Lemma 2.4(1).

That (2) implies (3) is trivial.

To show that (3) implies (1), let $f : M \to F^0$ be a $\mathcal{F}lat$-pre-envelope of M. Consider the short exact sequence $0 \to K^0 \to P^0 \xrightarrow{\pi} F^0 \to 0$ with P^0 projective. There exists $g : M \to P^0$ such that $\pi g = f$ since K^0 is flat and $\mathrm{Ext}^1(M, K^0) = 0$. It is easy to verify that $g : M \to P^0$ is a $\mathcal{F}lat$-pre-envelope. Thus, for any flat right R-module F, there is the exact sequence

$$\mathrm{Hom}(P^0, F) \to \mathrm{Hom}(\mathrm{im}(g), F) \to 0.$$

In addition, the exactness of $0 \to \mathrm{im}(g) \to P^0 \to \mathrm{coker}(g) \to 0$ yields the exact sequence

$$\mathrm{Hom}(P^0, F) \to \mathrm{Hom}(\mathrm{im}(g), F) \to \mathrm{Ext}^1(\mathrm{coker}(g), F) \to 0.$$

Hence $\mathrm{Ext}^1(\mathrm{coker}(g), F) = 0$. So $\mathrm{coker}(g)$ has a $\mathcal{F}lat$-pre-envelope $\mathrm{coker}(g) \to P^1$ with P^1 projective by the proof above. Continuing this process, we can get a $\mathrm{Hom}(-, \mathcal{F}lat)$ exact complex $0 \to M \to P^0 \to P^1 \to \cdots$ with each P^i projective. Note that $\mathrm{Ext}_i(M, ({}_RR)^+) = 0$ for all $i \geq 1$ and $\mathrm{Ext}_0(M, ({}_RR)^+) \cong M^+$ since M has an exact right $\mathcal{F}lat$-resolution and $({}_RR)^+$ is injective. So the complex $0 \to M \to P^0 \to P^1 \to \cdots$ is exact since $({}_RR)^+$ is a cogenerator. On the other hand, there is a $\mathrm{Hom}(-, \mathcal{F}lat)$ exact exact sequence $\cdots \to P_1 \to P_0 \to M \to 0$ with each P_i projective since $\mathrm{Ext}^i(M, F) = 0$ for all flat right R-modules F and all $i \geq 1$. Now we get a $\mathrm{Hom}(-, \mathcal{F}lat)$ exact exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective right R-modules with $M \cong \mathrm{coker}(P_1 \to P_0)$. So M is strongly Gorenstein flat.

That (3) implies (4) is trivial.

Finally, we prove that (4) implies (1). By the proof that (3) implies (1), we obtain a $\mathrm{Hom}(-, \mathcal{F}lat)$ exact complex

$$\mathcal{E} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective right R-modules such that $M \cong \mathrm{coker}(P_1 \to P_0)$. Next we will show that $\mathrm{Hom}(\mathcal{E}, N)$ is exact for any right R-module N with $\mathrm{fd}(N) = n < \infty$. We proceed

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by induction on n. The case $n = 0$ is clear. Let $n \geq 1$. There is an exact sequence $0 \to K \to P \to N \to 0$ with P projective, which induces an exact sequence

$$0 \to \mathrm{Hom}(\mathcal{E}, K) \to \mathrm{Hom}(\mathcal{E}, P) \to \mathrm{Hom}(\mathcal{E}, N) \to 0$$

of complexes. Note that $\mathrm{fd}(K) = n - 1$, so $\mathrm{Hom}(\mathcal{E}, K)$ is exact by induction. Thus $\mathrm{Hom}(\mathcal{E}, N)$ is exact by [21, Theorem 6.3]. In particular, $\mathrm{Hom}(\mathcal{E}, ({}_RR)^+)$ is exact since $\mathrm{fd}(({}_RR)^+) = FP\text{-id}({}_RR) < \infty$ by [17, Theorem 2.2]. Therefore $\mathcal{E}$ is an exact sequence. So M is strongly Gorenstein flat. □

REMARK 2.7. By the proof that (3) implies (1) in Theorem 2.6, any strongly Gorenstein flat right R-module M over a left coherent ring R has a $\mathcal{F}lat$-pre-envelope $\alpha : M \to F$ with F projective. Moreover, $\mathrm{coker}(\alpha)$ is still strongly Gorenstein flat.

Recall that a right R-module M is called *Gorenstein projective* [12] if there is an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective right R-modules with $M = \ker(P^0 \to P^1)$ and such that $\mathrm{Hom}(-, P)$ leaves the sequence exact whenever P is a projective right R-module.

COROLLARY 2.8. *Let R be an n-Gorenstein ring. Then the class of strongly Gorenstein flat right R-modules coincides with the class of Gorenstein projective right R-modules.*

PROOF. By definition, every strongly Gorenstein flat right R-module is Gorenstein projective.

Conversely, let M be a Gorenstein projective right R-module. Then, by [13, Remark 10.2.2], $\mathrm{Ext}^i(M, L) = 0$ for all $i \geq 1$ and all right R-modules L of finite projective dimension. For any flat right R-module N, we have $\mathrm{pd}(N) < \infty$ by [13, Proposition 9.1.2]. So $\mathrm{Ext}^i(M, N) = 0$ for all $i \geq 1$. Thus M is strongly Gorenstein flat by Theorem 2.6. □

COROLLARY 2.9. *Let n be a fixed nonnegative integer. The following statements are equivalent for a left coherent ring R.*

(1) *Every nth $\mathcal{F}lat$-cosyzygy of any right R-module is strongly Gorenstein flat.*
(2) $FP\text{-id}({}_RR) \leq n$ *and* $\mathrm{Ext}^i(L^n, F) = 0$ *for all nth $\mathcal{F}lat$-cosyzygies L^n of right R-modules, all flat right R-modules F and all $i \geq 1$.*

PROOF. To prove that (1) implies (2), it is enough to show that $FP\text{-id}({}_RR) \leq n$. There is a $\mathrm{Hom}(\mathcal{F}lat, -)$ exact exact sequence $0 \to K_n \stackrel{\alpha}{\to} F_{n-1} \stackrel{\varphi}{\to} F_{n-2} \to \cdots \to F_1 \to F_0 \to ({}_RR)^+ \to 0$ with each F_i flat. Let $\beta : F_n \to K_n$ be a $\mathcal{F}lat$-precover of K_n. Since the nth $\mathcal{F}lat$-cosyzygy of K_n is strongly Gorenstein projective, $\mathrm{Ext}_{n-1}(K_n, ({}_RR)^+) = 0$. So

$$\mathrm{Hom}(K_n, F_n) \stackrel{(\alpha\beta)_*}{\to} \mathrm{Hom}(K_n, F_{n-1}) \stackrel{\varphi_*}{\to} \mathrm{Hom}(K_n, F_{n-2})$$

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is exact. Since $\alpha \in \ker(\varphi_*) = \operatorname{im}(\alpha\beta)_*$, there exists $\gamma \in \operatorname{Hom}(K_n, F_n)$ such that $\alpha = \alpha\beta\gamma$. Thus $\beta\gamma = 1$, and hence K_n is flat. So $\operatorname{fd}(({}_RR)^+) \leq n$. It follows that $FP\text{-id}({}_RR) = \operatorname{fd}(({}_RR)^+) \leq n$ by [17, Theorem 2.2].

The reverse implication follows from Theorem 2.6. □

PROPOSITION 2.10. *Let R be a left coherent ring and $0 \to A \to B \to C \to 0$ an exact sequence of right R-modules.*

(1) *If A and C are strongly Gorenstein flat, then so is B.*
(2) *If B and C are strongly Gorenstein flat, then so is A.*
(3) *If A and B are strongly Gorenstein flat, then C is strongly Gorenstein flat if and only if $\operatorname{Ext}^1(C, F) = 0$ for all flat right R-modules F.*

Thus the class of strongly Gorenstein flat right R-modules is closed under direct summands.

PROOF. If C is strongly Gorenstein flat, then $\operatorname{Ext}^1(C, F) = 0$ for all flat right R-modules F by Lemma 2.4. But the fact that $\operatorname{Ext}^1(C, F) = 0$ for all flat right R-modules F implies that $0 \to A \to B \to C \to 0$ is $\operatorname{Hom}(-, \mathcal{F}lat)$ exact. Thus, by [13, Theorem 8.2.5(2)], we obtain the long exact sequence

$$\begin{aligned}\cdots &\to \operatorname{Ext}_n(A, ({}_RR)^+) \to \operatorname{Ext}_{n-1}(C, ({}_RR)^+) \to \operatorname{Ext}_{n-1}(B, ({}_RR)^+)\\ &\to \operatorname{Ext}_{n-1}(A, ({}_RR)^+) \to \cdots \to \operatorname{Ext}_1(A, ({}_RR)^+) \to \operatorname{Ext}_0(C, ({}_RR)^+)\\ &\to \operatorname{Ext}_0(B, ({}_RR)^+) \to \operatorname{Ext}_0(A, ({}_RR)^+) \to 0.\end{aligned}$$

So (1), (2) and (3) follow from Theorem 2.6.

The last statement holds by (1), (2), Remark 2.2(2) and [20, Proposition 1.4]. □

REMARK 2.11. We note that [13, Theorem 11.5.6] is an immediate consequence of Proposition 2.10 by Corollary 2.8.

Recall that R is a *right perfect ring* [4] if every right R-module has a projective cover. It is well known that R is right perfect if and only if every flat right R-module is projective.

COROLLARY 2.12. *Let R be a left coherent and right perfect ring, n a fixed nonnegative integer and $FP\text{-id}({}_RR) \leq n$. The following statements are equivalent for a right R-module M.*

(1) *M is strongly Gorenstein flat.*
(2) *M is Gorenstein projective.*
(3) *M is Gorenstein flat.*

PROOF. That (1) holds if and only if (2) holds is clear since R is right perfect.

That (1) implies (3) follows from Proposition 2.3.

To prove that (3) implies (1), let F be a flat right R-module and $i \geq 1$. Then there is a pure exact sequence $0 \to F \to F^{++} \to L \to 0$. Note that F^+ is injective,

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and hence F^{++} is flat since R is left coherent. Thus L is flat and so projective since R is right perfect. It follows that F is isomorphic to a direct summand of F^{++}. Note that $\mathrm{Ext}^i(M, F^{++}) \cong \mathrm{Tor}_i(M, F^+)^+$ and $\mathrm{Tor}_i(M, F^+) = 0$ by (3). Thus $\mathrm{Ext}^i(M, F^{++}) = 0$, and so $\mathrm{Ext}^i(M, F) = 0$. Hence M is strongly Gorenstein flat by Theorem 2.6. □

LEMMA 2.13. *Let R be an n-FC ring. Consider the following conditions for a right R-module M.*

(1) *M is strongly Gorenstein flat.*
(2) *M is Gorenstein projective.*
(3) *M is Gorenstein flat.*

Then (1) *implies* (2) *and (2) implies* (3). *If M is finitely presented, then* (3) *implies* (1).

PROOF. That (1) implies (2) is trivial. That (2) implies (3) follows from [10, Theorem 5].

To prove that (3) implies (1), let F be a flat right R-module and $i \geq 1$. Since R is right coherent and M is finitely presented, there is a standard isomorphism $\mathrm{Ext}^i(M, F)^+ \cong \mathrm{Tor}_i(M, F^+)$ by [21, Theorem 9.51] and the remark following it. But $\mathrm{Tor}_i(M, F^+) = 0$ by (3). Thus $\mathrm{Ext}^i(M, F)^+ = 0$ and so $\mathrm{Ext}^i(M, F) = 0$. Hence M is strongly Gorenstein flat by Theorem 2.6. □

PROPOSITION 2.14. *The following statements are equivalent for an n-FC ring R.*

(1) *$wD(R) \leq n$.*
(2) *Every Gorenstein flat right R-module is flat.*
(3) *Every strongly Gorenstein flat right R-module is projective.*
(4) *Every strongly Gorenstein flat right R-module is flat.*

PROOF. That (1) holds if and only (2) holds follows from [10, Theorem 13].

To prove that (2) implies (3), let M be a strongly Gorenstein flat right R-module. Then M is Gorenstein flat by Proposition 2.3. So M is flat by (2), and hence M is projective by Corollary 2.5.

That (3) implies (4) is trivial.

To prove that (4) implies (2), let M be a Gorenstein flat right R-module. Then, by [10, Theorem 5], $M \cong \varinjlim M_i$ for some inductive system $((M_i), (f_{ij}))$, where each M_i is a finitely presented Gorenstein projective right R-module. By Lemma 2.13, each M_i is strongly Gorenstein flat, and so flat by (4). Thus M is flat. □

PROPOSITION 2.15. *The following statements are equivalent for a ring R.*

(1) *R is right perfect.*
(2) *Every flat right R-module is strongly Gorenstein flat.*

In particular, if the class of strongly Gorenstein flat right R-modules is closed under direct limits, then R is right perfect.

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PROOF. That (1) implies (2) is clear.

As to the converse, let M be a flat right R-module. Then M is projective by (2) and Corollary 2.5, and so R is right perfect.

If the class of strongly Gorenstein flat right R-modules is closed under direct limits, then any direct limit of projective right R-modules is both strongly Gorenstein flat and flat, and hence it is also projective by Corollary 2.5. Thus R is right perfect by [4, Theorem P]. □

PROPOSITION 2.16. *The following statements are equivalent for a ring R.*

(1) *R is a QF ring (that is, R is a 0-Gorenstein ring).*

(2) *Every right R-module is strongly Gorenstein flat.*

PROOF. This follows from the fact that R is a QF ring if and only if injective right R-modules coincide with projective right R-modules. □

PROPOSITION 2.17. *The following statements are equivalent for a ring R.*

(1) *R is an FC ring.*

(2) *Every (right and left) finitely presented R-module is strongly Gorenstein flat.*

PROOF. To prove that (1) implies (2), let M be a finitely presented right R-module and F a flat right R-module. Then F is FP-injective by [22, Lemma 4.1]. So $\mathrm{Ext}^i(M, F) = 0$ for all $i \geq 1$. Thus M is strongly Gorenstein flat by Theorem 2.6.

As to the converse, for any finitely presented right R-module N, $\mathrm{Ext}^1(N, R_R) = 0$ since N is strongly Gorenstein projective. So R_R is FP-injective. Similarly, ${}_RR$ is FP-injective. In addition, every finitely presented R-module has a $\mathcal{F}lat$-pre-envelope. So R is left and right coherent by [2, Proposition 2]. □

We conclude this section with some examples which show that strongly Gorenstein flat modules over coherent rings lie strictly between projective modules and Gorenstein flat modules.

EXAMPLE 2.18. Let $R = \mathbb{Z}/4\mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers. Then R is a commutative QF ring. It is clear that $2R$ is a strongly Gorenstein flat R-module, but $2R$ is not projective.

EXAMPLE 2.19. Let R be a commutative coherent domain which is not a field, and let Q denote the field of fractions of R. Then Q is a (Gorenstein) flat R-module which is not contained in any free R-module. So Q is not a strongly Gorenstein flat R-module by definition.

EXAMPLE 2.20. Let R be a trivial extension of $\mathbb{Z}$ by $\mathbb{Q}/\mathbb{Z}$, that is,

$$R = \left\{ \begin{pmatrix} m & x \\ 0 & m \end{pmatrix} \middle| \, m \in \mathbb{Z}, x \in \mathbb{Q}/\mathbb{Z} \right\}$$

with addition and multiplication as in ordinary matrices. Then R is a commutative FC ring (see [9, Example 1]). But $R/\mathrm{Rad}(R) = R/(0, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}$, where $\mathrm{Rad}(R)$ is the Jacobson radical of R. So R is neither a von Neumann regular ring nor a perfect

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ring. Thus there exists a strongly Gorenstein flat R-module which is not projective (flat) by Proposition 2.14, and there exists a (Gorenstein) flat R-module which is not strongly Gorenstein flat by Proposition 2.15.

3. Strongly Gorenstein flat dimension

In this section, we define and investigate the strongly Gorenstein flat dimension for modules and rings.

DEFINITION 3.1. For a right R-module M, let $SGfd(M)$ denote the infimum of the set of n such that there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ of right R-modules, where each G_i is strongly Gorenstein flat and call $SGfd(M)$ the strongly Gorenstein flat dimension of M. If no such n exists, set $SGfd(M) = \infty$. Put $rSGFD(R)$ equal to the supremum of the set of $SGfd(M)$ such that M is any right R-module and $rFID(R)$ equal to the supremum of the set of $\mathrm{id}(M)$ such that M is any flat right R-module.

PROPOSITION 3.2. *Let R be a ring. Then $rFID(R) \le rSGFD(R) \le rD(R)$. The equalities hold if $wD(R) < \infty$.*

PROOF. It is clear that $rSGFD(R) \le rD(R)$. Let $rSGFD(R) = n < \infty$ and F be a flat right R-module. For any right R-module N, there is an exact sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to N \to 0$ with each P_i strongly Gorenstein flat by assumption. It follows that $\mathrm{Ext}^{n+i}(N, F) \cong \mathrm{Ext}^i(P_n, F) = 0$ for all $i \ge 1$ by Lemma 2.4(1). Thus $\mathrm{id}(F) \le n$, and so $rFID(R) \le rSGFD(R)$.

Let $wD(R) < \infty$. We only need to show that $rD(R) \le rFID(R)$. In fact, we may assume that $rFID(R) = m < \infty$. For any right R-module M, there exist a nonnegative integer k and an exact sequence $0 \to F_k \to F_{k-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ with each F_i flat since $\mathrm{fd}(M) < \infty$. Note that $\mathrm{id}(F_i) \le m$, and so $\mathrm{id}(M) \le m$. Thus $rD(R) \le rFID(R)$, and hence the equalities follow. □

REMARK 3.3. Clearly, $rSGFD(R) = 0$ if and only if R is a QF ring, which holds if and only if $rFID(R) = 0$. For example, $R = \mathbb{Z}/4\mathbb{Z}$ is a QF ring, and so $rFID(R) = rSGFD(R) = 0$. But $rD(R) = wD(R) = \infty$.

LEMMA 3.4. *Let R be a left coherent ring and n a fixed nonnegative integer. Consider the following conditions for a right R-module M.*

(1) $SGfd(M) \le n$.
(2) *For any exact sequence $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ with each P_i projective, K_n is strongly Gorenstein flat.*
(3) $\mathrm{Ext}^{n+i}(M, F) = 0$ *for any flat right R-module F and any $i \ge 1$.*

Then (1) and (2) are equivalent and imply (3). Moreover, (3) implies (2) if FP-$\mathrm{id}({}_RR) < \infty$.

PROOF. By (1), there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ of right R-modules with each G_i strongly Gorenstein flat. By Proposition 2.10 and [3, Lemma 3.12], K_n is strongly Gorenstein flat, thus (1) implies (2).

That (2) implies (1) is clear by definition.

For any flat right R-module F and any $i \geq 1$, we have $\mathrm{Ext}^{n+i}(M, F) \cong \mathrm{Ext}^i(K_n, F) = 0$ by (2) and Lemma 2.4(1), thus (2) implies (3).

That (3) implies (2) holds by Theorem 2.6. □

COROLLARY 3.5. *The following statements are true for a ring R.*

(1) *If R is a left coherent ring and FP-id$({}_RR) < \infty$, then $rSGFD(R) = rFID(R)$.*
(2) *If R is an n-Gorenstein ring, then $rSGFD(R) = rFID(R) \leq n$.*

PROOF. (1) holds by Lemma 3.4; (2) comes from (1) and [13, Proposition 9.1.2]. □

We are now in a position to give the main result of this section.

THEOREM 3.6. *The following statements are equivalent for a two-sided coherent ring R and a fixed integer $n \geq 1$.*

(1) *R is an n-FC ring.*
(2) *$SGfd(M) \leq n$ for any (right and left) finitely presented R-module M.*
(3) *The nth $\mathcal{F}lat$-cosyzygy of every (right and left) finitely presented R-module is strongly Gorenstein flat.*

PROOF. To show that (1) implies (2), let M be a finitely presented R-module and F be a flat R-module. Then FP-id$(F) \leq n$ by (1) and [13, Theorem 8.4.31], and so $\mathrm{Ext}^{n+i}(M, F) = 0$ for all $i \geq 1$. Thus $SGfd(M) \leq n$ by Lemma 3.4.

We now prove that (2) implies (1). For any finitely presented right R-module M, $\mathrm{Ext}^{n+1}(M, R_R) = 0$ by (2) and Lemma 3.4. So FP-id$(R_R) \leq n$. Similarly, it follows that FP-id$({}_RR) \leq n$.

To show that (1) implies (3), let M be a finitely presented R-module and $0 \to M \to P^0 \to P^1 \to \cdots$ be a right $\mathcal{F}lat$-resolution of M with each P_i finitely generated projective and L^n the nth $\mathcal{F}lat$-cosyzygy. Then the sequence is exact at P^k for $k \geq n-1$ by [13, Theorem 8.4.31]. So we get an exact sequence

$$0 \to L^n \to P^n \to \cdots \to P^{2n-1} \to L^{2n} \to 0.$$

Note that L^n and L^{2n} are finitely presented. Let F be a flat R-module. Evidently FP-id$(F) \leq n$ by [13, Theorem 8.4.31]. So $\mathrm{Ext}^i(L^n, F) \cong \mathrm{Ext}^{n+i}(L^{2n}, F) = 0$ for all $i \geq 1$. Thus L^n is strongly Gorenstein flat by Theorem 2.6.

Finally, to show that (3) implies (1), let N be a finitely presented right R-module. Then N admits a right $\mathcal{F}lat$-resolution

$$\mathcal{E} : 0 \to N \to P^0 \to P^1 \to \cdots .$$

Note that the nth $\mathcal{F}lat$-cosyzygy L^n is strongly Gorenstein flat by (3). So there is a $\mathrm{Hom}(-, \mathcal{F}lat)$ exact exact sequence $0 \to L^n \to F^1 \to F^2 \to \cdots$ with each F^i projective. Thus $\mathcal{E}$ is exact at P^k for $k \geq n-1$. Hence FP-id$({}_RR) \leq n$ by [13, Theorem 8.4.31]. Similarly, FP-id$(R_R) \leq n$. □

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4. Precovers and pre-envelopes by strongly Gorenstein flat modules

In this section, we study the existence of strongly Gorenstein flat precovers and pre-envelopes. Let $\mathcal{SGF}$ be the class of all strongly Gorenstein flat right R-modules.

THEOREM 4.1. *Let R be a left coherent ring and n be a fixed nonnegative integer with $rSGFD(R) \le n$. Then every right R-module M has an $\mathcal{SGF}$-precover $\alpha : N \to M$ such that if $0 \to K \to N \xrightarrow{\alpha} M \to 0$ is exact, then $\mathrm{pd}(K) \le n-1$ whenever $n \ge 1$. Moreover, if $\mathrm{pd}(M) < \infty$, then N is projective.*

PROOF. By hypothesis, there is an exact sequence $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ such that each P_i is projective and K_n is strongly Gorenstein flat by Lemma 3.4 since $SGfd(M) \le n$. Then there exists a $\mathrm{Hom}(-, \mathcal{F}lat)$ exact exact sequence $0 \to K_n \to P^0 \to \cdots \to P^{n-2} \to P^{n-1} \to L^n \to 0$ with each P^i projective. Note that L^n is strongly Gorenstein flat. So the following diagram commutes:

$$
\begin{array}{ccccccccccccccc}
0 & \longrightarrow & K_n & \longrightarrow & P^0 & \longrightarrow & \cdots & \longrightarrow & P^{n-2} & \longrightarrow & P^{n-1} & \longrightarrow & L^n & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle h_0} & & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle h_n} & & \\
0 & \longrightarrow & K_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

By [13, Proposition 1.4.14], we get an exact sequence

$$0 \to K_n \to P^0 \oplus K_n \to P^1 \oplus P_{n-1} \to \cdots \to P^{n-1} \oplus P_1 \to L^n \oplus P_0 \to M \to 0$$

which gives the exactness of the sequence

$$0 \to P^0 \to P^1 \oplus P_{n-1} \to \cdots \to P^{n-1} \oplus P_1 \to L^n \oplus P_0 \to M \to 0.$$

Let $N = L^n \oplus P_0$. Consider the short exact sequence $0 \to K \to N \to M \to 0$. It is clear that $\mathrm{pd}(K) \le n-1$ and N is strongly Gorenstein flat. So $\mathrm{Ext}^1(H, K) = 0$ for any strongly Gorenstein flat right R-module H by Lemma 2.4(1). Thus $N \to M$ is an $\mathcal{SGF}$-precover.

If $\mathrm{pd}(M) < \infty$, then $\mathrm{pd}(N) < \infty$ since $\mathrm{pd}(K) < \infty$. It follows that N is projective by Lemma 2.4(2). □

Let $\mathcal{GP}$ be the class of all Gorenstein projective right R-modules and $\mathcal{I}_n$ the class of all right R-modules of injective dimension at most n, where n is a fixed nonnegative integer. Enochs and Jenda proved that $(\mathcal{GP}, \mathcal{I}_n)$ is a cotorsion theory if R is an n-Gorenstein ring (see [13, Remark 11.5.10]). This result is a particular case of the following theorem by Corollaries 2.8 and 3.5(2).

THEOREM 4.2. *Let R be a left coherent ring, n be a fixed nonnegative integer and $FP\text{-id}({}_R R) \le n$. Then the following statements are equivalent.*

(1) *$rSGFD(R) \le n$.*
(2) *$rFID(R) \le n$.*
(3) *$(\mathcal{SGF}, \mathcal{I}_n)$ is a cotorsion theory.*

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PROOF. The case $n = 0$ is clear. Next we assume that $n \geq 1$.

That (1) holds if and only if (2) holds follows from Corollary 3.5(1).

To show that (2) implies (3), let E be an injective right R-module. Then there is a split exact sequence $0 \to E \to \Pi({}_R R)^+$ since $({}_R R)^+$ is a cogenerator. Since R is left coherent, $\mathrm{fd}(({}_R R)^+) = FP\text{-id}({}_R R) \leq n$, and so $\mathrm{fd}(\Pi({}_R R)^+) \leq n$. Hence $\mathrm{fd}(E) \leq n$. Thus $\mathrm{fd}(Q) < \infty$ for any $Q \in \mathcal{I}_n$. So $\mathcal{I}_n \subseteq \mathcal{SGF}^\perp$ and $\mathcal{SGF} \subseteq {}^\perp\mathcal{I}_n$ by Lemma 2.4(1).

Let M be a right R-module such that $M \in \mathcal{SGF}^\perp$, that is, $\mathrm{Ext}^1(N, M) = 0$ for all strongly Gorenstein flat right R-modules N. Then there is an exact sequence $0 \to M \to E \to L \to 0$ with E injective. Note that (2) is equivalent to (1), so there exists an exact sequence $0 \to K \to G \to L \to 0$ with G strongly Gorenstein flat and $\mathrm{pd}(K) \leq n - 1$ by Theorem 4.1. Then we obtain the following pullback diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & H & \to & G & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & E & \to & L & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since $\mathrm{Ext}^1(G, M) = 0$, the sequence $0 \to M \to H \to G \to 0$ is split. Note that $\mathrm{id}(H) \leq n$ since $\mathrm{id}(K) \leq n$ by (2). Thus $\mathrm{id}(M) \leq n$, and so $\mathcal{SGF}^\perp \subseteq \mathcal{I}_n$. Hence $\mathcal{SGF}^\perp = \mathcal{I}_n$.

Now let $N \in {}^\perp\mathcal{I}_n$, that is, $\mathrm{Ext}^1(N, F) = 0$ for all right R-modules F such that $\mathrm{id}(F) \leq n$. By Theorem 4.1, we get an exact sequence $0 \to A \to B \to N \to 0$ with B strongly Gorenstein projective and $\mathrm{pd}(A) \leq n - 1$. So the sequence is split since $\mathrm{id}(A) \leq n$ by (2), and hence N is strongly Gorenstein flat. Thus ${}^\perp\mathcal{I}_n \subseteq \mathcal{SGF}$, and hence ${}^\perp\mathcal{I}_n = \mathcal{SGF}$. Therefore, $(\mathcal{SGF}, \mathcal{I}_n)$ is a cotorsion theory.

Finally, to show that (3) implies (1), let M be any right R-module. There is an exact sequence $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ with each P_i projective. Since $\mathrm{Ext}^1(K_n, H) \cong \mathrm{Ext}^{n+1}(M, H) = 0$ for any R-module H with $\mathrm{id}(H) \leq n$, K_n is strongly Gorenstein flat by (3). Thus $SGfd(M) \leq n$ by Lemma 3.4, and so $rSGFD(R) \leq n$. □

THEOREM 4.3. *Let R be a left coherent ring and n a fixed nonnegative integer such that $SGFd(N) \leq n$ for any finitely presented right R-module N. Then every finitely presented right R-module has an $\mathcal{SGF}$-pre-envelope.*

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PROOF. Let M be a finitely presented right R-module. Then there is a $\mathrm{Hom}(-, \mathcal{F}lat)$ exact complex $0 \to M \to F^0 \to F^1 \to \cdots \to F^{n-1} \to L^n \to 0$ with each F_i finitely generated projective since R is left coherent. Note that there exists an exact sequence $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to L^n \to 0$ with each P_i projective. Thus K_n is strongly Gorenstein flat by Lemma 3.4 since $SGFd(L^n) \leq n$. For any strongly Gorenstein flat right R-module G and any homomorphism $f : M \to G$, there is a $\mathrm{Hom}(-, \mathcal{F}lat)$ exact exact sequence $0 \to G \to H^0 \to H^1 \to \cdots \to H^{n-1} \to Q^n \to 0$ with each H^i projective, and so we obtain the following commutative diagram:

$$\begin{array}{ccccccccccccc}
0 & \to & M & \to & F^0 & \to & F^1 & \to \cdots \to & F^{n-1} & \to & L^n & \to & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle g} & & \\
0 & \to & G & \to & H^0 & \to & H^1 & \to \cdots \to & H^{n-1} & \to & Q^n & \to & 0
\end{array}$$

But we also have the following commutative diagram:

$$\begin{array}{ccccccccccccc}
0 & \to & M & \to & F^0 & \to & F^1 & \to \cdots \to & F^{n-1} & \to & L^n & \to & 0 \\
 & & \downarrow{\scriptstyle h_n} & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & \to & K_n & \to & P_{n-1} & \to & P_{n-2} & \to \cdots \to & P_0 & \to & L^n & \to & 0 \\
 & & \downarrow{\scriptstyle g_n} & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle g} & & \\
0 & \to & G & \to & H^0 & \to & H^1 & \to \cdots \to & H^{n-1} & \to & Q^n & \to & 0
\end{array}$$

Then the usual homotopy argument shows that $f - g_n h_n$ factors through F^0. So $f : M \to G$ factors through the strongly Gorenstein flat right R-module $K_n \oplus F^0$. Thus $M \to K_n \oplus F^0$ is an $\mathcal{SGF}$-pre-envelope. □

Recall that a left R-module M is called *Gorenstein injective* [12] if there is an exact sequence $\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ of injective left R-modules with $M = \ker(E^0 \to E^1)$ such that $\mathrm{Hom}(E, -)$ leaves the sequence exact whenever E is an injective left R-module.

The following theorem is a generalization of [13, Corollary 11.8.3].

THEOREM 4.4. *Let R be an n-FC ring. Then every finitely presented right R-module M has an $\mathcal{SGF}$-pre-envelope $M \to P$ with P finitely presented.*

PROOF. The result can be obtained as a corollary of Theorems 3.6 and 4.3. Here we give another proof which may be of independent interest.

Let M be a finitely presented right R-module. By [13, Lemma 5.3.12], there is a cardinal number $\aleph_\alpha$ such that for any homomorphism $f : M \to L$ with L Gorenstein flat, there is a pure submodule Q of L such that $\mathrm{Card}(Q) \leq \aleph_\alpha$ and $f(M) \subseteq Q$. We claim that Q is Gorenstein flat. In fact, the pure exact sequence $0 \to Q \to L \to L/Q \to 0$ induces a split exact sequence $0 \to (L/Q)^+ \to L^+ \to Q^+ \to 0$. Since L^+ is Gorenstein injective by [20, Theorem 3.6], Q^+ is Gorenstein injective by

[20, Theorem 2.6]. Thus Q is Gorenstein flat by [20, Theorem 3.6] again. It follows that f has a factorization $M \to Q \to L$ with $\mathrm{Card}(Q) \le \aleph_\alpha$ and Q Gorenstein flat. Now let $(\varphi_i)_{i\in I}$ give all such homomorphisms $\varphi_i : M \to Q_i$. So any homomorphism $M \to H$ with H Gorenstein flat has a factorization $M \to Q_j \to H$ for some $j \in I$. Thus the homomorphism $\varphi : M \to \prod_{i\in I} Q_i$ induced by all φ_i is a Gorenstein $\mathcal{F}lat$-pre-envelope since $\prod_{i\in I} Q_i$ is Gorenstein flat by [10, Corollary 8]. But M is finitely presented, so φ factors through a finitely presented strongly Gorenstein flat right R-module P by Lemma 2.13 and [10, Theorem 5], that is, there exist homomorphisms $g : M \to P$ and $h : P \to \prod_{i\in I} Q_i$ such that $\varphi = hg$. It is easy to verify that g is an $\mathcal{SGF}$-pre-envelope of M. □

We round off the paper with the following remark which may be viewed as an illustration of the usefulness of strongly Gorenstein flat modules.

REMARK 4.5. (1) If R is a left coherent ring, then every projective module is strongly Gorenstein flat, and every strongly Gorenstein flat module is Gorenstein flat, but no two of these concepts are equivalent by Proposition 2.3 and Examples 2.18, 2.19 and 2.20.

(2) If R is an n-Gorenstein ring, then strongly Gorenstein flat modules are precisely Gorenstein projective modules by Corollary 2.8.

(3) By definition, every strongly Gorenstein flat module is Gorenstein projective, but we have not been able to find examples of Gorenstein projective modules which are not strongly Gorenstein flat. If one can find a Gorenstein projective module M over a coherent ring which is not Gorenstein flat, then M is not strongly Gorenstein flat by Proposition 2.3.

(4) Holm [19] asked whether every Gorenstein projective module is Gorenstein flat. This question remains open, but it is now known that if R is a left coherent ring with finite right finitistic projective dimension (or R is a commutative Noetherian ring with finite Krull dimension), then every Gorenstein projective right R-module is Gorenstein flat (see [19, Proposition 5.5] and [20, Proposition 3.4]).

(5) If the class of strongly Gorenstein flat modules happens to be the class of Gorenstein projective modules, then Proposition 2.3 gives an affirmative answer to Holm's question for any left coherent ring.

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