ON THE UNICELLULARITY OF WEIGHTED SHIFTS

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(Received 10 July 1969)

Communicated by J. B. Miller

1. Introduction

An operator acting on a Banach space is said to be unicellular if its lattice of invariant subspaces is totally ordered by inclusion. Each weighted shift on a sequence space has a natural totally ordered set of invariant subspaces, and is unicellular if these are its only invariant subspaces.

Let l^p, $1 \leqq p < \infty$ be the space of all complex sequences $f = \{\phi_n\}_{n=0}^{\infty}$ for which $\sum_{n=0}^{\infty} |\phi_n|^p$ converges. For such a sequence, define $||f||_p = (\sum_{n=0}^{\infty} |\phi_n|^p)^{1/p}$. Let l^∞ be the space of all bounded complex sequences, and C^0 the subspace of l^∞ consisting of all null sequences. For a bounded sequence $f = \{\phi_n\}_{n=0}^{\infty}$, define $||f||_\infty = \sup_n |\phi_n|$. Each l^p, $1 \leqq p < \infty$ is a separable Banach space with norm $|| \, ||_p$. With $|| \, ||_\infty$ as norm, l^∞ is a non-separable Banach space and C^0 is a separable Banach subspace.

For each non-negative integer n, let e_n be the sequence having 1 in the n'th place and 0 elsewhere. Clearly each e_n is in each of the spaces l^∞, C^0, and l^p, and in the separable spaces C^0 and l^p, the e_n form a Schauder basis.

To each complex sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ in l^∞ there corresponds a linear operator A which maps each of the spaces l^∞, C^0 and l^p into itself and is defined by the equations:

$$Ae_n = \alpha_n e_{n+1}, \qquad n \geqq 0.$$

The boundedness of the sequence α ensures that A is bounded. Such an operator A is called a weighted shift and the sequence α is called the weight sequence.

Let $\mathscr{B}$ be any one of the sequence spaces l^∞, C^0 or l^p. For each non-negative integer k, let M_k be the subspace of $\mathscr{B}$ spanned by $\{e_k, e_{k+1}, e_{k+2}, \cdots\}$. Clearly the M_k are invariant under the weighted shift A (acting on $\mathscr{B}$) and are totally ordered by inclusion: $\mathscr{B} = M_0 \supset M_1 \supset M_2 \supset \cdots \supset \{0\}$. For each k let P_k be the orthogonal projection onto M_k. Since $P_k - P_{k+1}$ has rank 1, A is unicellular only if its only invariant subspaces are $\{0\}$ and the M_k.

A basic problem of weighted shifts is to find a necessary and sufficient condition, in terms of the weighted sequence α (or some related sequence), for unicellularity of the corresponding weighted shift. The first examples of a unicellular and a non-unicellular weighted shift were given by Donoghue [3] and Beurling [1] respectively. In each case the underlying space is the Hilbert space l^2. In Donoghue's example the weights are $\alpha_n = 2^{-n}$, and in Beurling's they are $\alpha_n = 1$. Nikolskii [7] and [8] has extended Donoghue's work and has obtained the following sufficient condition for unicellularity of a weighted shift on any of the spaces l^p, $1 < p < \infty$.

THEOREM (Nikolskii). *If the weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ can be decomposed into arithmetic subsequences $\mathscr{S}_i$, $i = 0, 1, 2, \cdots, r-1$, each of the form $\mathscr{S}_i = \{\alpha_{i+kr}\}_{k=0}^{\infty}$ such that*

(i) *Each subsequence $\mathscr{S}_i$ decreases monotonically to zero, and*

(ii) *The sequence $\{\prod_{i=0}^{r-1} \alpha_{i+kr}\}_{k=0}^{\infty}$ is in l^p for some $p \geqq 1$, then the corresponding weighted shift, acting on any of the spaces l^q, $1 < q < \infty$, is unicellular.*

Further, if the weighted shift acts on l^1, then it is unicellular if it satisfies condition (i) only. Gellar [4] has studied weighted shifts on certain types of Banach spaces and has obtained sufficient conditions for unicellularity of these. In this paper we find reasonably simple sufficient conditions for unicellularity of weighted shifts acting on the spaces C^0 and l^p. We find that because l^∞ is non-separable, there are no unicellular weighted shifts acting on that space.

2. Preliminaries

Let $\mathscr{B}$ be any one of the spaces l^∞, C^0 or l^p, and suppose that A is a weighted shift operator acting on $\mathscr{B}$. For each f in $\mathscr{B}$, let $\mathscr{C}(f)$ be the least subspace of $\mathscr{B}$ which contains f and is invariant under A. It is easily seen that

$$\mathscr{C}(f) = \operatorname{span} \{A^n f\}_{n=0}^{\infty}.$$

(The span of a set means the smallest closed linear subspace containing that set.) To each non-zero $f = \{f_n\}_{n=0}^{\infty}$ in $\mathscr{B}$ there corresponds a unique non-negative integer $k(f)$ with the properties: $P_{k(f)} f = f$ and $P_{k(f)+1} f \neq f$; $k(f)$ is simply the least integer k for which f_k is non-zero. Clearly, for each non-zero f, $\mathscr{C}(f) \subset M_{k(f)}$ and $\mathscr{C}(f) \not\subset M_{k(f)+1}$ and we have

LEMMA 2.1. *A necessary and sufficient condition that the weighted shift A be unicellular is that for each non-zero f in $\mathscr{B}$, we have $\mathscr{C}(f) = M_{k(f)}$.*

This lemma shows that there are no unicellular weighted shifts on l^∞. For if $\mathscr{B} = l^\infty$ then each M_k is non-separable. Since each of the cyclic subspaces $\mathscr{C}(f)$ is separable, $\mathscr{C}(f) = M_{k(f)}$ is impossible.

For a given weighted shift A, let

$$\Lambda = \{r : k(f) = r \quad \text{implies} \quad \mathscr{C}(f) = M_r\}.$$

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In terms of Λ, lemma 2.1 is:

The weighted shift A is unicellular if and only if Λ is the set of all non-negative integers.

This condition can be altered slightly to the following:

LEMMA 2.2. *A necessary and sufficient condition that the weighted shift A be unicellular is that Λ be infinite.*

PROOF. By lemma 2.1 this condition is necessary. To prove sufficiency choose an integer r in Λ, $r > 0$, and choose g in $\mathscr{B}$ satisfying $k(g) = r-1$. We know that $\mathscr{C}(g) = \text{span}\,\{A^n g\}_{n=0}^{\infty}$. Now span $\{A^n g\}_{n=1}^{\infty} = \text{span}\,\{A^n(Ag)\}_{n=0}^{\infty}$, and $k(Ag) = r$. So by the hypothesis span $\{A^n g\}_{n=1}^{\infty} = M_r$. Now span $\{g, M_r\} = M_{r-1}$, so it soon follows that $\mathscr{C}(g) = M_{r-1}$, and this is true for any g satisfying $k(g) = r-1$. We see that r in Λ implies $r-1$ is also in Λ. Since Λ is infinite it follows that Λ contains each non-negative integer. So lemma 2.1 applies and shows that A is unicellular.

REMARK. To prove unicellularity of Donoghue's operator, i.e. the weighted shift D with weights $\alpha_n = 2^{-n}$, it is sufficient to prove that $\mathscr{C}(f) = M_0$ whenever $k(f) = 0$. This is because $D^n U^k f = 2^{-nk} U^k D^n f$ where U is the weighted shift whose weights are all equal to 1, n and k are any non-negative integers and f is any vector in $\mathscr{B}$. This implies

$$\text{span}\,\{D^n U^k f\}_{n=0}^{\infty} = U^k\,\text{span}\,\{D^n f\}_{n=0}^{\infty}, \tag{2.3}$$

for any non-negative integer k and for any f in $\mathscr{B}$. Since (2.3) is not always true when D is replaced by any other weighted shift, such a simplification is not possible in the general case.

If one of the weights, α_n say, is zero then e_n is an eigenvector of A. So a unicellular weighted shift has no zero weight. Henceforth we assume that each weight α_n of the weighted shift A is non-zero. For each n let $z_n = A^n e_0$ and let $S_n = \|z_n\|$. We have:

$$S_0 = 1 \text{ and } S_n = |\alpha_0 \alpha_1 \cdots \alpha_{n-1}|, \qquad n > 0. \tag{2.4}$$

Also $z_n = S_n \lambda_n e_n$, where each λ_n has modulus 1, and in terms of the z_n A is defined by:

$$A z_n = z_{n+1}, \qquad n \geqq 0. \tag{2.5}$$

We now assume that $\mathscr{B}$ is one of the separable spaces C^0 or l^p. For each non-zero f in $\mathscr{B}$, we can write $f = \sum_{t=k(f)}^{\infty} f_t z_t$, and this series converges in norm to f. We identify each non-zero f in $\mathscr{B}$ with a certain linear transformation $f(A)$, defined in the following way:

(i) $\operatorname{dom}[f(A)] = \{x : x \in \mathscr{B}, \sum_{t=k(f)}^{\infty} f_t A^{t-k(f)} P_{k(f)} x$ converges in $\mathscr{B}\}$;

(ii) for x in $\operatorname{dom}[f(A)]$, $f(A)x$ is the above sum.

The transformation $f(A)$ is not necessarily bounded, but $\operatorname{dom}[f(A)]$ is a dense subset of $\mathscr{B}$, because $f(A)z_r = 0$ for any r less than $k(f)$, and for any non-negative n,

$$\begin{aligned} f(A)z_{n+k(f)} &= \sum_{t=k(f)}^{\infty} f_t A^{t-k(f)} z_{n+k(f)} \\ &= \sum_{t=k(f)}^{\infty} f_t z_{t+n} \qquad (2.6) \\ &= A^n f. \end{aligned}$$

Let $\Gamma = \{r : k(f) = r$ implies $f(A)$ maps M_r continuously into a dense subset of itself$\}$. We shall prove the following

LEMMA 2.7. *If Γ is infinite then A is unicellular.*

PROOF. Choose any r in Γ and any f in $\mathscr{B}$ for which $k(f) = r$. Then $f(A)M_r$ is dense in M_r. Now

$$f(A)M_r \subseteq \operatorname{span}\{f(A)z_{n+k(f)}\}_{n=0}^{\infty} = \operatorname{span}\{A^n f\}_{n=0}^{\infty} = \mathscr{C}(f)$$

by equation (2.6). Therefore $\overline{f(a)M_r} = M_r \subseteq \mathscr{C}(f)$. The reverse inclusion is obvious, so $M_r = \mathscr{C}(f)$. It follows that Γ is contained in Λ, so Λ is also infinite. Therefore by lemma 2.2, A is unicellular.

We now consider certain uniformly closed algebras of operators acting on $\mathscr{B}$ which are related to A.

For each k we have $P_k A P_k = A P_k$, and more generally, for any polynomial p with zero constant term:

$$p(AP_k) = p(A)P_k. \qquad (2.8)$$

For each k let $\mathscr{A}_k$ be the uniform closure of the set of all polynomials in AP_k whose constant terms are zero. Then each $\mathscr{A}_k$ is a commutative Banach algebra of operators mapping $\mathscr{B}$ into M_k. It follows from (2.8) that if Q is in M_0, then QP_k is in M_k for each k. If A is quasinilpotent, then so is each AP_k and each $\mathscr{A}_k$ is a radical algebra. For the theory of the radical in a commutative Banach algebra, see for example [5], § 4.

Let $\Omega = \{r : k(f) = r+1$ implies $f(A)AP_r$ is in $\mathscr{A}_r\}$. We shall prove the following

LEMMA 2.9. *If Ω is infinite then A is unicellular.*

PROOF. Choose any r in Ω and any g in $\mathscr{B}$ for which $k(g) = r$. Then $g = g_r z_r + f$ where $k(f) = r+1$, and

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$$g(A) = (g_r I + \sum_{t=r+1}^{\infty} g_t A^{t-r}) P_r$$
$$= g_r P_r + (\sum_{t=r+1}^{\infty} g_t A^{t-(r+1)}) A P_r.$$

Since

$$A P_r = P_{r+1} A P_r,$$

(2.10)
$$g(A) = g_r P_r + (\sum_{t=r+1}^{\infty} g_t A^{t-(r+1)}) P_{r+1} A P_r$$
$$= g_r P_r + f(A) A P_r.$$

By the hypothesis $f(A)AP_r$ is in $\mathscr{A}_r$, so it is bounded and quasinilpotent. So by equation (2.10) the restriction of $g(A)$ to M_r is the sum of a non-zero scalar and a quasinilpotent operator. Thus $g(A)$ is bounded and $g(A)M_r = M_r$. So r is in Γ, and hence Γ contains Ω. Therefore Γ is infinite and by lemma 2.7 A is unicellular.

3. The main theorem

We now impose restrictions on the weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ of the weighted shift A so that the conditions of lemma 2.9 are fulfilled. From (2.4) we see that a restriction on the sequence $\{|\alpha_n|\}_{n=0}^{\infty}$ is also a restriction on the sequence $\{S_n\}_{n=0}^{\infty}$. Since $|\alpha_n| = S_{n+1}/S_n$ the converse is also true.

If $\nu(A)$ denotes the spectral radius of the weighted shift A, then on each of spaces C^0 and l^p

(3.1)
$$\nu(A) = \lim_{n\to\infty} (\sup_r |\alpha_r \alpha_{r+1} \cdots {}_{r+n-1}|^{1/n}).$$

In terms of $\{S_n\}_{n=0}^{\infty}$ this is:

(3.2)
$$\nu(A) = \lim_{n\to\infty} (\sup_r (S_{n+r}/S_r)^{1/n}).$$

Let $\mathscr{B}^*$ denote the Banach dual of $\mathscr{B}$. The dual of each of the sequence spaces C^0 and l^p, $1 \leqq p \leqq \infty$, is a sequence space; to wit $(C^0)^* = l^1$ and $(l^p)^* = l^q$ where $q = p(p-1)^{-1}$ if $p \neq 1$ and $q = \infty$ if $p = 1$. The following theorem is given in terms of the dual space $\mathscr{B}^*$.

THEOREM 3.3. *Suppose that $\mathscr{B}$ is any of the sequence spaces C_0 or l^p, and that A is a weighted shift with bounded non-zero weights acting on $\mathscr{B}$. Then if*

(i) *A is quasinilpotent and*

(ii) *for infinitely many positive integers r, there are positive integers $t(r)$ such that the sequence*

$$\{\|A^n P_{t(r)}\|/S_{n+r}\}_{n=0}^{\infty}$$

is in $\mathscr{B}^*$, then A is unicellular.

PROOF. First note that

$$\|A^n P_{t(r)}\| = \sup_{t \geqq t(r)} |\alpha_t \alpha_{t+1} \cdots \alpha_{t+n-1}| = \sup_{t \geqq t(r)} S_{t+n}/S_t.$$

We shall show that each of the integers, whose existence is guaranteed by (ii), is in Ω. Thus Ω is infinite, and by lemma 2.9 A is unicellular.

Choose such an integer r, and choose f in $\mathscr{B}$ for which $k(f) = r+1$. Write $f = \sum_{t=r+1}^{\infty} f_t z_t$. For each positive integer N let

$$X_N = \Big(\sum_{t=r+1}^{r+N} f_t A^{t-r}\Big) P_r.$$

Clearly each X_N, being a polynomial in AP_r by (2.8), is in $\mathscr{A}_r$, and we will show that $\{X_N\}_{N=1}$ is a Cauchy sequence in $\mathscr{A}_r$. Choose positive integers N and N' so that N is greater than N' and choose x in M_r. Write $x = \sum_{n=r}^{\infty} x_n z_n$. Then $(X_N - X_{N'})x = \sum_{n=r}^{\infty} x_n (X_N - X_{N'}) z_n$. If $\mathscr{B} = l^p$, $1 < p < \infty$ we obtain

$$\|(X_N - X_{N'})x\|_p \leqq \sum_{n=r} |x_n| \, \|(X_N - X_{N'})z_n\|_p$$
$$\leqq \Big(\sum_{n=r}^{\infty} |x_n S_n|^p\Big)^{1/p} \Big(\sum_{n=r}^{\infty} [\|(X_N - X_{N'})z_n\|_p / S_n]^q\Big)^{1/q}$$

where $1/p + 1/q = 1$. So we have:

$$\|X_N - X_{N'}\|_p \leqq \Big(\sum_{n=r} [\|(X_N - X_{N'})z_n\|_p / S_n]^q\Big)^{1/q}. \tag{3.4}$$

If $\mathscr{B} = l^1$ the corresponding equation is:

$$\|X_N - X_{N'}\|_1 \leqq \sup_{n \geqq r} \|(X_N - X_{N'})z_n\|_1 / S_n, \tag{3.5}$$

and if $\mathscr{B} = C^0$ we have:

$$\|X_N - X_{N'}\|_\infty \leqq \sum_{n=r}^{\infty} \|(X_N - X_{N'})z_n\|_\infty / S_n. \tag{3.6}$$

We next obtain estimates for the norm of $(X_N - X_{N'})z_n$ in the various sequence spaces. For n not less than r,

$$(X_N = X_{N'})z_n = \sum_{t=N'+1}^{N} f_{r+t} A^t z_n$$
$$= \sum_{t=N'+1}^{N} f_{r+t} z_{n+t}.$$

We now assume that N' is greater than $t(r)$. Since $S_{n+t} = (S_{n+t}/S_{r+t}) S_{r+t}$, it follows that

$$S_{n+t} \leqq \|A^{n-r} P_{t(r)}\| S_{r+t} \tag{3.7}$$

for each t greater than N'.

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For p satisfying $1 < p < \infty$, and for $N > N' > t(r)$,

$$\|(X_N - X_{N'})z_n\|_p = (\sum_{t=N'+1}^{N} |f_{r+t} S_{n+t}|^p)^{1/p} \leqq \|A^{n-r} P_{t(r)}\| (\sum_{t=N'+1}^{N} |f_{r+t} S_{r+t}|^p)^{1/p}.$$

So (3.4) becomes:

$$\|X_N - X_{N'}\|_p \leqq (\sum_{n=r}^{\infty} [\|A^{n-r} P_{t(r)}\|/S_n]^q)^{1/q} (\sum_{t=N'+1}^{N} f_{r+t} S_{r+t}|^p)^{1/p}. \tag{3.8}$$

If $p = 1$, and if $N > N' > t(r)$,

$$\|(X_N - X_{N'})z_n\|_1 \leqq \|A^{n-r} P_{t(r)}\| \sum_{t=N'+1}^{N} |f_{r+t} S_{r+t}|.$$

So (3.5) becomes

$$\|X_N - X_{N'}\|_1 \leqq \sup_{n \geqq r} \|A^{n-r} P_{t(r)}\|/S_n \cdot \sum_{t=N'+1}^{N} |f_{r+t} S_{r+t}|. \tag{3.9}$$

By (3.7) again, for $N > N' > t(r)$, we have

$$\|(X_N - X_{N'})z_n\|_\infty \leqq \|A^{n-r} P_{t(r)}\| \sup_{t > N'} |f_{r+t} S_{r+t}|.$$

So (3.6) becomes

$$\|X_N - X_{N'}\|_\infty \leqq \sum_{n=r}^{\infty} \|A^{n-r} P_{t(r)}\|/S_n \cdot \sup_{t > N'} |f_{r+t} S_{r+t}|. \tag{3.10}$$

The second hypothesis of the theorem implies that the first factor on the right in each of (3.8), (3.9), and (3.10) is finite. Because f is in the space l^p, $1 < p < \infty$, l^1 and C^0 respectively, in each case the second term approaches zero as N' becomes large. So it follows that if the underlying space $\mathscr{B}$ is any of the spaces C^0 or l^p, $1 \leqq p < \infty$, then the corresponding sequence $\{X_N\}_{N=1}^{\infty}$ is Cauchy in $\mathscr{A}_r$. Since $\mathscr{A}_r$ is complete, it contains the limit $(\sum_{t=r+1}^{\infty} f_t A^{t-r}) P_r$. This limit is $f(A)AP_r$, so by earlier remarks the theorem is proved.

NOTE. Nikolskii [8, Theorem 1] states that the condition $\liminf_n S_n^{1/n} = 0$ is necessary for unicellularity of a weighted shift acting on any of the spaces C^0 and l^p, $1 \leqq p < \infty$. (Halmos [6, page 240] had proved this when the underlying space is the Hilbert space l^2). We shall show that this condition, with condition (ii) of theorem 3.3, implies that A is quasinilpotent. So condition (i) of the theorem could be replaced by the necessary condition: $\liminf_n S_n^{1/n} = 0$. To prove this suppose that A is not quasinilpotent; then neither is AP_k for any k. So for any k there is some positive δ and for some positive integer N such that $\|A^n P_k\|^{1/n} > \delta$ for each $n > N$. Since $\liminf_n S_n^{1/n} = 0$, for any r $S_{n+r}^{1/n} < \delta/2$ for infinitely many n. So $\|A^n P_k\| > 2^n S_{n+r}$ for infinitely many n. Thus $\{\|A^n P_k\|/S_{n+r}\}_{n=0}^{\infty}$ is not bounded for any r and any k so condition (ii) of the theorem is not fulfilled.

It is not known if every unicellular weighted shift is necessarily quasinilpotent.

EXAMPLE 3.11. Consider the weighted shift A with weights α_n given by:

$$\alpha_n = \begin{cases} 1 & \text{if } n = 2^p, \quad p = 1, 2, 3, \cdots \\ 2^{-(2n-1)} & \text{if } n = 2^p+1, \ p = 1, 2, 3, \cdots \\ 2^{-n} & \text{otherwise.} \end{cases}$$

We suppose that A acts on any of the spaces C^0 and l^p, $1 \leqq p < \infty$.

There is no positive integer r such that all the arithmetic subsequences with constant difference r are monotonic, so Nikolskii's condition does not apply. However we have

$$S_n = \begin{cases} 2^{-(n-2)(n-1)/2} & \text{if } n = 2^p+1, \ p = 1, 2, 3, \cdots \\ 2^{-n(n-1)/2} & \text{otherwise.} \end{cases}$$

So for each n $S_n \geqq 2^{-n(n-1)/2}$, and for positive n and k, $\|A^n P_k\| \leqq 2^{-(n-1)(2k+n-2)}$ so $\|A^n\|^{1/n} \leqq 2^{-(n-1)(n-2)/2n} \to 0$ as $n \to \infty$. Thus A is quasinilpotent.

Also for any positive r, let $t(r) = r+2$. Then

$$\begin{aligned} \|A^n P_{t(r)}\|/S_{n+r} &\leqq 2^{\{(n+r)(n+r-1)-(n-1)(n+2r+2)\}/2} \\ &\leqq 2^{-n+(r^2+r+2)/2}. \end{aligned}$$

So $\{\|A^n P_{t(r)}\|/S_{n+r}\}_{n=0}^{\infty}$ is bounded and in each l^p, $1 \leqq p < \infty$. The theorem shows that on each of the spaces C^0 and l^p, $1 \leqq p < \infty$, the weighted shift is unicellular.

It can be shown that theorem 3.3 includes Nikolskii's sufficient condition. Example 3.11 shows that they are not equivalent.

4. Acknowledgement

The author wishes to express his appreciation to Professor J. B. Miller for introducing him to these interesting questions of unicellularity of operators. The author also gratefully acknowledges the support of a Commonwealth Post Graduate Award.

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