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A solution to the degree-*d* twisted rabbit problem

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Abstract. We solve generalizations of Hubbard's twisted rabbit problem for analogs of the rabbit polynomial of degree $d \geq 2$. The twisted rabbit problem asks: when a certain quadratic polynomial, called the Douady rabbit polynomial, is twisted by a cyclic subgroup of a mapping class group, to which polynomial is the resulting map equivalent (as a function of the power of the generator)? The solution to the original quadratic twisted rabbit problem, given by Bartholdi and Nekrashevych, depended on the 4-adic expansion of the power of the mapping class by which we twist. In this paper, we provide a solution to a degree-*d* generalization that depends on the d^2-adic expansion of the power of the mapping class element by which we twist.

Key words: holomorphic dynamics, topological polynomials, branched covers, mapping class groups
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1. *Introduction*

Let $f : \mathbb{C} \to \mathbb{C}$ be an orientation-preserving branched cover. Let $C_f \subset \mathbb{C}$ be the set of points for which f is locally non-injective (the set of critical points). The post-critical set $P_f = \{f^n(c) \mid c \in C_f, n \geq 1\}$ is the forward orbit of C_f. If P_f is finite, f is said to be *post-critically finite*. Let $g : \mathbb{C} \to \mathbb{C}$ be another post-critically finite branched cover with post-critical sets P_g. We say that f, g are *equivalent* or *combinatorially equivalent* (or *Thurston equivalent*) if there exist orientation-preserving homeomorphisms $h_0, h_1 : (\mathbb{C}, P_f) \to (\mathbb{C}, P_g)$ such that $h_0 f = g h_1$, and h_0 and h_1 are homotopic relative to P_f. Thurston proved that a post-critically finite branched cover $\mathbb{C} \to \mathbb{C}$ is either equivalent

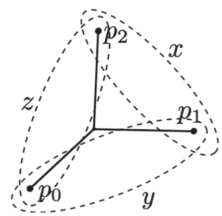

FIGURE 1. The Hubbard tree of the rabbit polynomial along with the simple closed curves x, y, and z.

to a polynomial or has a certain kind of topological obstruction [6]. Over the past decades, much work has been directed toward determining a holomorphic map to which an unobstructed branched cover $\mathbb{C} \to \mathbb{C}$ (or $S^2 \to S^2$) is equivalent [1–5, 11, 12, 14, 15, 17–19].

In the 1980s, Hubbard posed the *twisted rabbit problem*, which presented the challenge of classifying certain branched covers by the polynomial to which they were equivalent. The rabbit polynomial is the quadratic polynomial $R(z) = z^2 + c$, where $c \approx -0.122561 + 0.744862i$ for which the critical point 0 is 3-periodic. The post-critical set consists of three points: $\{0, R(0), R^2(0)\}$. Let x be a curve surrounding $R(0)$ and $R^2(0)$, and let D_x be the Dehn twist about x (see Figure 1). The composition $D_x^m R$ is a branched cover $\mathbb{C} \to \mathbb{C}$, and by the Bernstein–Levy theorem [13] (also [10, Ch. 10]), it is equivalent to a polynomial. Hubbard's twisted rabbit problem is as follows: for $m \in \mathbb{Z}$, find a function in terms of m that determines to which polynomial $D_x^m R$ is equivalent. After remaining open for nearly 25 years, Bartholdi and Nekrachevych solved the twisted rabbit problem in 2006 [2].

Belk, Lanier, Margalit, and the second author solved a generalization of the twisted rabbit problem in which they compose quadratic polynomials where the critical point is n-periodic with powers of an analogous Dehn twist for $n \geq 3$ [3]. Recently, Lanier and the second author extended their work to unicritical cubic polynomials (with any number of post-critical points).

1.1. *The degree-d twisted rabbit problem.* In this paper, we generalize Hubbard's twisted rabbit problem to higher degree analogs of the rabbit polynomial. That is: for each $d \geq 2$, there is a unicritical polynomial $R_d(z) = z^d + c_d$ that naturally generalizes the rabbit polynomial. The critical point 0 is 3-periodic, as in the quadratic case. Let x_d be the curve that is homotopic to the boundary of a neighborhood of the straight line segment between $R_d(0)$ and $R_d^2(0)$. In Theorem 1.1, we describe a function that determines to which polynomial $D_{x_d}^m R_d$ is equivalent in terms of m.

1.2. *The polynomials.* Before we state the solution to the degree-d twisted rabbit problem, we explain the notation we use for the polynomials that appear. For each degree d, there exist $d+1$ equivalence classes of polynomials that have a critical point that is 3-periodic. One of these is R_d, the generalization of the rabbit polynomial (the degree-d rabbit); see Figure 2(a) for the Julia set when $d = 5$. The complex conjugate of the rabbit polynomial is called the degree-d corabbit polynomial $\overline{R}_d$, see Figure 2(b) for its Julia

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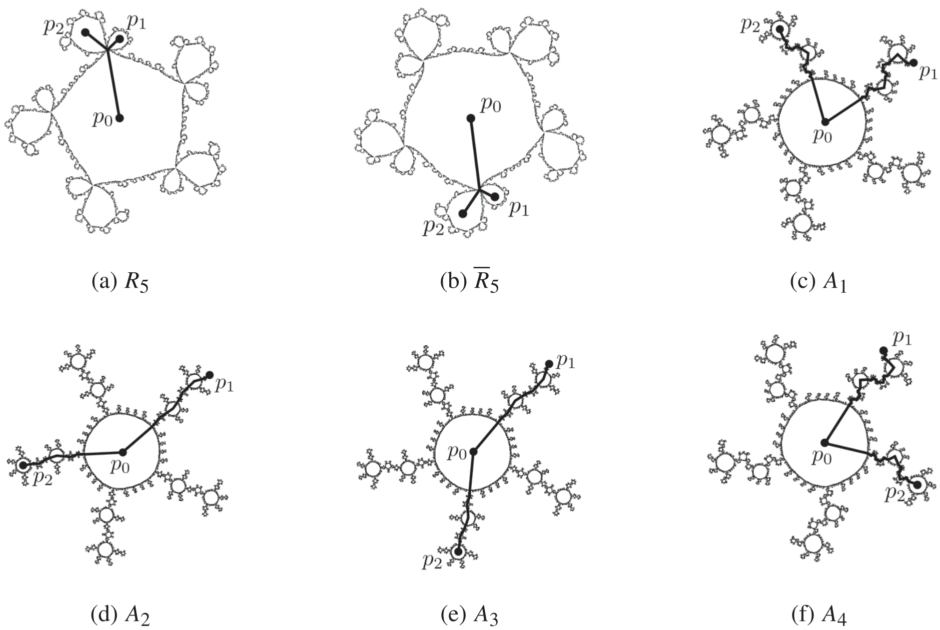

FIGURE 2. The Julia sets and Hubbard trees for the unicritical polynomials of degree 5 with 3-periodic critical point.

set when $d = 5$. The remaining $d - 1$ polynomials are all generalizations of the airplane polynomial. These remaining $d - 1$ polynomials have a Hubbard tree with two edges that meet at the critical point and can be indexed by the angle made by these two edges at the critical point with a fixed orientation. For $1 \leq i \leq d - 1$, we denote by $A_{d,i}$ the degree-d generalization of the airplane for which this angle is $2\pi i/d$; see Figures 2(c)–2(f) for the Julia sets of the four degree-d airplanes when $d = 5$.

The solution to the original (quadratic) twisted rabbit problem depends on the 4-adic expansion of the power by which we twist. Similarly, the solution to the degree-d twisted rabbit problem will depend on the d^2-adic expansion of the power by which we twist.

Any integer m has a d^2-adic expansion of the form $m_s m_{s-1} m_{s-2} \cdots m_1$ if $m \geq 0$, or $\overline{d^2 - 1} m_s m_{s-1} m_{s-2} \cdots m_1$ if $m < 0$, with $m_i \in \{0, 1, \ldots, d^2 - 1\}$ for all $i \in \{1, 2, \ldots, s\}$. Let $\sigma_{d^2}(m)$ be the least value of s such that all digits of the d^2-adic expansion of m to the left of m_s are repeating (that is, for all $t > s$, $m_t = 0$ if $m \geq 0$ or $m_t = d^2 - 1$ if $m < 0$).

THEOREM 1.1. *Let R_d be the degree-d rabbit polynomial and D_x the Dehn twist about the curve $x = x_d$.*

If $(d + 1) | m_i$ for all $i \in \{1, 2, \ldots, \sigma_{d^2}(m)\}$, then

$$D_x^m R_d \simeq \begin{cases} R_d & \text{if } m \geq 0, \\ \overline{R}_d & \text{if } m < 0. \end{cases}$$

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TABLE 1. Base cases for $d = 5$.

m	$D_x^m R_5$	m	$D_x^m R_5$	m	$D_x^m R_5$	m	$D_x^m R_5$	m	$D_x^m R_5$
0	R_5	5	$A_{5,4}$	10	$A_{5,3}$	15	$A_{5,2}$	20	$A_{5,1}$
1	$A_{5,1}$	6	R_5	11	$A_{5,4}$	16	$A_{5,3}$	21	$A_{5,2}$
2	$A_{5,2}$	7	$A_{5,1}$	12	R_5	17	$A_{5,4}$	22	$A_{5,3}$
3	$A_{5,3}$	8	$A_{5,2}$	13	$A_{5,1}$	18	R_5	23	$A_{5,4}$
4	$A_{5,4}$	9	$A_{5,3}$	14	$A_{5,2}$	19	$A_{5,1}$	24	R_5

Otherwise, let i be the least index such that m_i is not divisible by $d + 1$. We may write m_i uniquely as $d\ell + n$, where $\ell, n \in \{0, 1, \ldots, d - 1\}$. Since $(d + 1) \nmid m_i$, we have that $\ell \neq n$. Then,

$$D_x^m R_d \simeq \begin{cases} A_{d,n-\ell} & \text{if } n > \ell, \\ A_{d,d-(\ell-n)} & \text{if } n < \ell. \end{cases}$$

In particular, in the set $\{D_x^m R_d \mid |m| \leq N\}$, the airplane polynomials $A_{d,1}, \ldots, A_{d,d-1}$ occur with equal frequency. Moreover, as N tends to infinity, the probability that $D_x^m R_d$ (with $|m| \leq N$) is equivalent to the rabbit or corabbit polynomial approaches zero.

COROLLARY 1.2. *Fix a degree d greater than 1. For $S \geq 1$, let*

$$\Sigma_S = \{m \in \mathbb{Z} \mid \sigma_{d^2}(m) \leq S\}.$$

With the uniform distribution on Σ_S, the probability that for $m \in \Sigma_S$, $D_x^m R_d$ is equivalent to a map in the collection $\{A_{d,i} \mid 1 \leq i \leq d - 1\}$ is given by $1 - 1/d^S$. In particular, for $m \in \Sigma_S$, we have the following.

- *For any $i \in \{1, 2, \ldots, d - 1\}$, the probability that $D_x^m R_d \simeq A_{d,i}$ is $1/(d - 1) - 1/(d - 1)d^S$.*
- *The probability that $D_x^m R_d \simeq R_d$ is $1/2d^S$.*
- *The probability that $D_x^m R_d \simeq \overline{R}_d$ is $1/2d^S$.*

Example 1.3. To make Theorem 1.1 concrete, we give the polynomials to which $D_x^m R_5$ is equivalent for $0 \leq m \leq 24$ in Table 1. We observe that $A_{5,1}, \ldots, A_{5,4}$ each appear four times and R_d appears five times. The next m for which $D_x^m R_5$ is equivalent to the rabbit is $m = 150$, which has 25-adic expansion $m_2 m_1$, with $m_2 = m_1 = 6$. We also note that $\overline{R}_5$ does not appear in the table because it only occurs when $m < 0$. For $-25 \leq m \leq 0$, the polynomial $\overline{R}_d$ occurs when $m = -1, -7, -13, -19, -25$.

1.3. *Methods.* We follow the strategy of Bartholdi and Nekrashevych [2]: in §3, we find formulas that reduce $D_x^m R_d$ to R_d post-composed with one of a finite set of maps. Then, in §4, we determine a polynomial equivalent to each of these 'base cases' using the lifting algorithm of Belk, Lanier, Margalit, and the second author in [3].

1.4. *Other twisted rabbit problems.* As mentioned above, the (quadradic) twisted rabbit problem remained open for over two decades. When Bartholdi and Nekrashevych solved the problem, it shifted the techniques used to study holomorphic dynamics. However, they

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gave more than just a solution to the problem that Hubbard originally posed: for instance, they gave an algorithm to determine the polynomial to which gR_2 was equivalent for any pure mapping class g. Their work opened up a world of possible generalizations: in this paper, we follow their lead by increasing the *degree* of the polynomial by which we twist. Belk, Lanier, Margalit, and the second author generalize the (quadratic) twisted rabbit problem by increasing the size of the post-critical set. In concurrent work of Lanier and the second author, they solve several of these problems for the cubic rabbit polynomial. In particular, they give a closed-form solution to $D_x^m R_3$ that agrees with our solution in that case. They also generalize this case to give a closed-form solution to determine the equivalence class of $D_x^m R_{3,n}$, where $R_{3,n}$ is a specific cubic polynomial with n post-critical points that is a natural generalization of R_3. By pairing our paper with theirs, we have a 2-parameter family of generalizations of the twisted rabbit problem: by varying both the degree d and the number of post-critical points. In this paper, we prove that we can obtain any equivalence class of a degree d-unicritical polynomial with a 3-periodic critical point by twisting the degree d rabbit polynomial by a power of D_x. Lanier and the second author show that the same is not true when the critical point has period greater than 3 (at least in the cubic case). Further investigation into this 2-parameter family of twisted rabbit problems may reveal deeper structure or patterns within the set of unicritical polynomials with periodic critical point.

2. *Hubbard trees and degree-d polynomials*

Throughout this paper, we rely heavily on the established theory of Hubbard trees [7, 8]. We will use a modification of Poirier's conditions for Hubbard trees to allow us to define (topological) Hubbard trees for unobstructed branched covers $\mathbb{C} \to \mathbb{C}$ [6, 16].

2.1. *Hubbard trees.*

Let $f : \mathbb{C} \to \mathbb{C}$ be a post-critically finite branched cover with post-critical set P_f. For our purposes, a tree T is a finite graph with no cycles, embedded in $\mathbb{C}$ such that:

(1) P_f is contained in the vertex set of T and

(2) all leaves of T are in P_f.

The preimage $f^{-1}(T)$ does not, in general, satisfy the definition of a tree given above. The *lift* of a tree T, denoted $\widetilde{T}$, is the hull of $f^{-1}(T)$ relative to P_f. That is, any edge of $f^{-1}(T)$ that is not contained in a path between a pair of points in P_f is contracted to a point. Thus, $\widetilde{T}$ is a tree. We say two trees T and T' are *isomorphic* if they are homeomorphic as topological spaces.

Let f be a polynomial of degree $d \geq 2$. Let P_f be the post-critical set for f. The *Hubbard tree* H_f for f is a subset of $\mathbb{C}$ comprising regulated arcs of the filled Julia set of f that connect pairs of points of P_f (see [16] or [10]). An important feature of the Hubbard tree of f is that it is invariant under f, that is, $f(H_f) \subseteq H_f$. Likewise, it is also true that $f^{-1}(H_f) \subseteq H_f$ and that H_f is isotopic to $\widetilde{H}_f$.

Let g be a branched cover $\mathbb{C} \to \mathbb{C}$ that is equivalent to f. Then, we can define the (topological) Hubbard tree for g, denoted H_g, as the pullback of H_f under the equivalence between f and g. In this case, the abstract trees for H_f and H_g are isomorphic.

Angle assignments. The combinatorial structure of a Hubbard tree is not sufficient to distinguish the combinatorial equivalence class of a branched cover $\mathbb{C} \to \mathbb{C}$. That is, there are inequivalent polynomials that have isomorphic Hubbard trees. There is additional information that one can provide to distinguish post-critically finite branched covers. Following Poirier (see [16]), we use an invariant angle assignment, which we define as follows. Because we think of a (Hubbard) tree as a subset of $\mathbb{C}$, the edges that meet at any vertex have an associated cyclic order. An *angle assignment for a vertex v* is a set of angle measures between each pair of (not necessarily distinct) edges that have endpoint v and are adjacent in the cyclic order around v such that the measures of the angles sum to 2π. An *angle assignment for a tree T* is the union of angle assignments at each of the vertices of T.

The tree lift $\widetilde{T}$ inherits an angle assignment from T, as follows. Let $\angle$ be an angle of T adjacent to the vertex $v \in T$ with measure $|\angle|$. Let $\widetilde{v} \in f^{-1}(v)$, and let $\nu(\widetilde{v})$ denote the local degree of $\widetilde{v}$ under f. There exists a preimage of the angle $\angle$ at $\widetilde{v}$, denoted $\widetilde{\angle}$. Assign the measure of $|\angle|/\nu(\widetilde{v})$ to the angle $\widetilde{\angle}$. By applying this process to each angle in T, we define an angle assignment on $f^{-1}(T)$. To define an angle assignment on $\widetilde{T}$, consider any edge e in $f^{-1}(T)$ that is contracted to a point of $\widetilde{T}$. The edge e is the side of two angles $\angle_1$ and $\angle_2$. When e is contracted to a point of $\widetilde{T}$, the two angles $\angle_1$ and $\angle_2$ are replaced with a new angle $\angle'$. Assign the measure of $\angle'$ to be $|\angle_1| + |\angle_2|$. We say an angle assignment on T is *invariant* if:

(1) T is isotopic to $\widetilde{T}$ and
(2) each angle of T has the same measure as the corresponding angle (under the isotopy) of $\widetilde{T}$.

Given a branched cover $f : \mathbb{C} \to \mathbb{C}$, it follows from the work of Douady and Hubbard [7, 8] or Poirier [16] that a Hubbard tree H_f, an invariant angle assignment on H_f, and the restriction of f to $\widetilde{H}_f$ suffice to determine the equivalence class of f.

2.2. *The degree-d polynomials with a 3-periodic critical cycle.* In this section, we describe the Hubbard trees for all unicritical polynomials of degree d with a 3-periodic critical point. We first count the equivalence classes of such polynomials.

Counting parameter rays. Every unicritical polynomial of degree d is affine conjugate to $z^d + c$ for some $c \in \mathbb{C}$. There exist exactly $d^2 - 1$ polynomials in this family that have a critical cycle of period 3. Indeed, this may be seen by counting angles of parameter rays in the Mandelbrot set. The parameter rays that land on a hyperbolic component of the Mandelbrot set that contains a polynomial with a critical cycle of period dividing 3 exactly comprise the set of parameter rays with angles in $\{i/d^3 - 1 \mid i \in \{0, 1, 2, \ldots, d^3 - 2\}\}$. Among such rays, the rays at the angles $0, 1/(d-1), 2/(d-1), \ldots, (d-2)/(d-1)$ land on the unique hyperbolic component of period 1. Of the remaining $d^3 - d$ rays, groups of d rays each land on the same hyperbolic component. Thus, there are $(d^3 - d)/d = d^2 - 1$ hyperbolic components that contain unicritical polynomials with critical cycle of period 3. However, the affine conjugacy class (and therefore the equivalence class) of each polynomial $z^d + c$ consists of the $d - 1$ polynomials $z^d + \omega^j c$, where $\omega = e^{2\pi i/(d-1)}$ and $j \in \{1, 2, \ldots, d-1\}$. Thus, there are exactly $d + 1$ equivalence classes of unicritical polynomials that have a critical cycle of exact period 3.

Alternatively, we observe that every polynomial of the form $z^d + c$ with $c \neq 0$ is conjugate to the polynomial $p_\lambda(z) = \lambda(1 + z/d)^d$, where $\lambda = dc^{d-1}$. In the polynomial family $\{p_\lambda\}$, the polynomials p_{λ_1} and p_{λ_2} are conjugate if and only if $\lambda_1 = \lambda_2$. In this family, there are exactly $d + 1$ distinct solutions to the equation $p_\lambda^{\circ 3}(0) = 0$.

2.2.1. *Hubbard trees of polynomials with three critical points.* Let f be a unicritical polynomial of degree d with a 3-periodic critical point. Let p_0 be the critical point of f, let $p_1 = f(p_0)$ be the critical value, and let $p_2 = f^2(p_0)$.

The $d + 1$ equivalence classes of degree d polynomials that have a critical point of period 3 can be distinguished by their Hubbard trees and an invariant angle assignment. There are only two combinatorial structures for (Hubbard) trees with three post-critical points:

(1) a tripod, that is, a graph where $\{p_0, p_1, p_2\}$ are all leaves and there is an unmarked trivalent vertex or
(2) a path of length 2 (two of the post-critical points are leaves).

2.2.2. *The rabbit R_d and corabbit $\overline{R}_d$.* The rabbit polynomial R_d and corabbit polynomial $\overline{R}_d$ are the two polynomials (up to equivalence) that have a tripod as their Hubbard tree. Both polynomials cyclically permute edges. Therefore, the only possible invariant angle assignment is $2\pi/3$ at all angles, in both cases. They can be distinguished, however, because R_d rotates the edges of its Hubbard tree counterclockwise and $\overline{R}_d$ rotates the edges of its Hubbard tree clockwise. Because a polynomial is determined by its Hubbard tree, the invariant angle assignment, and the action of the map on the edges of its Hubbard tree, we may take this to be the definition of R_d and $\overline{R}_d$. Where $d = 5$, the Hubbard tree for R_5 is shown as a subset of the filled Julia set in Figure 2(a) and the Hubbard tree for $\overline{R}_5$ is a subset of the filled Julia set in Figure 2(b).

2.2.3. *The airplanes $A_{d,i}$.* The remaining $d - 1$ equivalence classes of polynomials have Hubbard trees that are paths of length 2. Moreover, the vertex of valence 2 must be p_0 because the critical point is the only point for which f maps a neighborhood d-to-1 to its image. The polynomials are distinguished by the invariant angle assignment on their Hubbard tree. For any $i \in \{1, 2, \ldots, d - 1\}$, there is a polynomial such that $\angle p_1 p_0 p_2$ has an invariant angle assignment of $2\pi i/d$. Define $A_{d,i}$ to be the polynomial where $\angle p_1 p_0 p_2$ is $2\pi i/d$. Figures 2(c)–2(f) show the filled Julia sets and Hubbard trees (with angles) for $A_{5,1}$ through $A_{5,4}$, respectively.

3. *Reduction formulas*

We prove Theorem 1.1 in two steps, following the original proof of Bartholdi and Nekraschevych. In Lemma 3.2, we give an algorithm that determines a map to which $D_{x_d}^m R_d$ is equivalent and that either belongs to a finite set (of base cases) or is of the form $D_{x_d}^k R_d$, where $k \leq m$. In the next section, we find a Hubbard tree for each of the base cases.

3.1. *Lifting.* Let P_d be the post-critical set of R_d. A homeomorphism $h : (\mathbb{C}, P_d) \to (\mathbb{C}, P_d)$ is said to be liftable under R_d if there exists a homeomorphism $\widetilde{h} : (\mathbb{C}, P_d) \to (\mathbb{C}, P_d)$ such that $R_d\widetilde{h} = hR_d$. In this case, hR_d is equivalent to $\widetilde{h}R_d$. We use the notation

$$h \rightsquigarrow \widetilde{h}$$

to describe the (directional) equivalence between hR_d and $\widetilde{h}R_d$.

The isotopy classes of homeomorphisms that lift under R_d form a finite index subgroup of $\mathrm{PMod}(\mathbb{C}, P_d)$ called the *liftable mapping class group* $\mathrm{LMod}(\mathbb{C}, P_d)$. As described above, there is a homomorphism $\psi : \mathrm{LMod}(\mathbb{C}, P_d) \to \mathrm{PMod}(\mathbb{C}, P_d)$ defined for $h \in \mathrm{LMod}(\mathbb{C}, P_d)$ as $\psi(h) = \widetilde{h}$.

Following Bartholdi and Nekrashevych, we use ψ to define a version of lifting for any $h \in \mathrm{PMod}(\mathbb{C}, P_d)$. For any $h \in \mathrm{PMod}(\mathbb{C}, P_d)$, there exists $g \in \mathrm{PMod}(\mathbb{C}, P_d)$ such that $g^{-1}h \in \mathrm{LMod}(\mathbb{C}, P_d)$. Then, $g^{-1}hR_d$ is equivalent to $\psi(g^{-1}h)R_d$. Moreover, hR_d is equivalent to $\psi(g^{-1}h)gR_d$ [3, Lemma 5.1]. We use the notation

$$h \overset{g}{\rightsquigarrow} \psi(g^{-1}h)g$$

to denote the equivalence between hR_d and $\psi(g^{-1}h)gR_d$ obtained by lifting $g^{-1}h$ under R_d. In particular, the superscript g indicates which coset representative of $h\,\mathrm{LMod}(\mathbb{C}, P_d)$ we choose. If $h \in \mathrm{LMod}(\mathbb{C}, P_d)$, the notation $h \rightsquigarrow \widetilde{h}$ is a special case of the notation

$$h \overset{g}{\rightsquigarrow} \psi(g^{-1}h)g,$$

where g is the identity, which we suppress.

3.2. *Branch cuts.* Let $f : \mathbb{C} \to \mathbb{C}$ be a post-critically finite branched cover of degree $d \geq 2$. A *branch cut* B is a union of arcs such that:

- each endpoint of each arc in B is a critical value of f (possibly infinity);
- each critical value of f is an endpoint of an arc of B; and
- the complement of $f^{-1}(B)$ in $\mathbb{C}$ consists of d components.

If f is unicritical, B can be chosen to be a single arc b joining the unique critical value to infinity. A *special branch cut* for f is a branch cut such that all points in the post-critical set P_f for f are contained in the closure of a single component of $\mathbb{C} \setminus f^{-1}(b)$.

3.3. *Intersections with a branch cut.* Let γ be an arc in $(\mathbb{C}, P_f)$ with endpoints in P_f and let b be a branch cut for f. The preimage $f^{-1}(\gamma)$ intersects the preimage $f^{-1}(b)$ at $d|\gamma \cap b|$ points. Moreover, if we assign an orientation to b and γ, the points of intersection $\gamma \cap b$ will inherit an orientation. The orientation of each point of $\gamma \cap b$ will lift to an orientation of the corresponding d preimages of $f^{-1}(\gamma)$. The *geometric intersection* of γ and b is the minimum of $|\gamma \cap b|$ over all arcs homotopic to γ. The *algebraic intersection* of γ and b is the sum of the signed (from the orientation) intersections of the homotopy class representative of γ that realizes the algebraic intersection.

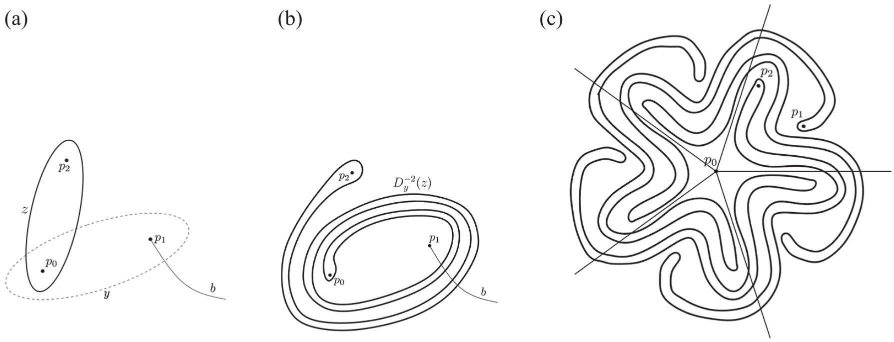

FIGURE 3. (a) The simple closed curves y and z. (b) The simple closed curve $D_y^{-2}(z)$. (c) The preimage of $D_y^{-2}(z)$ under R_d when $d = 5$. All components of the preimage are peripheral.

3.4. *Defining arcs.* Let c be a simple closed curve that bounds a disk that contains two points p, q in P_f. Then, c is homotopic to the boundary of a neighborhood of a simple arc γ_c with endpoints at p and q. We call γ_c the *defining arc* of c.

A simple closed curve in $\mathbb{C} \setminus P_f$ is called *trivial* if it is homotopic to a point relative to P_f or bounds a single point of P_f. The next lemma gives a condition for when the preimage of a (non-trivial) curve is trivial.

LEMMA 3.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a post-critically finite branched cover of degree d and let b be a special branch cut for f. The complement of $f^{-1}(b)$ in $\mathbb{C}$ consists of d (open) components. Label these components counterclockwise as $\Gamma_0, \ldots, \Gamma_{d-1}$, where such Γ_0 is the component such that its closure contains the -critical set.*

Let c be a curve in $(\mathbb{C}, P_f)$ that bounds a disk containing two points in P_f such that neither point is the critical value. Let γ_c be the defining arc of c. Suppose the algebraic intersection of γ_c and b is i. Then, for each $0 \leq j \leq d-1$, there is a path lift of γ_c such that the endpoints of the path lift are in Γ_j and $\Gamma_{(j-|i|) \bmod d}$. In particular, each component of $f^{-1}(c)$ is trivial if and only if $d \nmid i$.

Proof. For the first statement, induct on i.

The defining arc of each connected component of $f^{-1}(c)$ is a path lift of γ_c. We note that a lift of γ_c is the defining arc of a non-trivial curve if and only if both endpoints are in $P_f \subset \Gamma_0$. Moreover, the endpoints of a lift of γ_c are in the same component Γ_j if and only if $d \mid i$. □

Lemma 3.2 is the key step that allows us to write $D_{x_d}^m R_d$ as a map with a lower power than m. Let $x = x_d$, $y = y_d$, and $z = z_d$ be the curves in Figure 1.

Let c be a simple closed curve in $(\mathbb{C}, R_d)$. Recall that the Dehn twist about c is trivial if and only if c is trivial. For example, in Figure 3, we illustrate the curve $D_y^{-2}(z)$ and its lift under R_5. By Lemma 3.1, all components of $R_5^{-1}(D_y^{-2}(z))$ are trivial since $5 \nmid -2$; therefore, any lift of the $D_{D_y^{-2}(z)}$ under R_5 is trivial.

LEMMA 3.2. *Let $m \in \mathbb{Z}$, $m = d^2k + d\ell + n$ with $k \in \mathbb{Z}$, and $\ell, n \in \{0, 1, \ldots, d-1\}$. Note that k, ℓ, and n are unique. We have*

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$$D_x^m R_d \simeq \begin{cases} D_x^k R_d, & \ell = n, \\ D_y^{\ell-n} R_d, & \ell \neq n. \end{cases}$$

Proof. We note that for any r, the defining arcs of the curves $D_x^r(z)$ and $D_y^r(z)$ have algebraic intersection $\pm r$ with the branch cut. Then, by Lemma 3.1, for all r such that $d \nmid r$, the curves $D_x^r(z)$ and $D_y^r(z)$ lift to trivial curves. Thus, $D_{D_x^r(z)}$ and $D_{D_y^r(z)}$ are trivial for all r such that $d \nmid r$.

We consider three cases.

(1) If $n = \ell = 0$, then we have

$$D_x^m = D_x^{d^2k} \rightsquigarrow D_y^{dk} \rightsquigarrow D_z^k \rightsquigarrow D_x^k.$$

(2) If $n = 0$, $\ell \neq 0$, then we have

$$D_x^m = D_x^{d^2k+d\ell} \rightsquigarrow D_y^{dk+\ell} \overset{D_y^\ell}{\rightsquigarrow} D_z^k D_y^\ell \overset{D_y^\ell}{\rightsquigarrow} \psi(D^k_{D_y^{-\ell}(z)}) D_y^\ell = D_y^\ell.$$

The last equality holds by Lemma 3.1 since $D_y^{-\ell}(z)$ has algebraic intersection ℓ with the branch cut, and $d \nmid \ell$.

(3) If $n \neq 0$, then we have

$$\begin{aligned} D_x^{d^2k+d\ell+n} &\overset{D_x^n}{\rightsquigarrow} D_y^{dk+\ell} D_x^n = D_y^{dk+\ell-1} D_x^{-1} D_z^{-1} D_x^n \\ &\overset{D_y^{dk+\ell-1} D_x^{n-1}}{\rightsquigarrow} \psi(D^{-1}_{D_x^{-n}(z)}) D_y^{dk+\ell-1} D_x^{n-1} \\ &= D_y^{dk+\ell-1} D_x^{n-1}. \end{aligned}$$

The first equality follows from the lantern relation $D_x D_y D_z = \mathrm{id}$. The last line holds because $1 \leq n \leq d-1$, so $d \nmid n$ and we may apply Lemma 3.1.

The calculation above shows that $D_y^{dk+\ell} D_x^n R_d$ is equivalent to $D_y^{dk+\ell-1} D_x^{n-1} R_d$. Performing this step n times, we have that $D_y^{dk+\ell} D_x^n R_d$ is equivalent to

$$D_y^{dk+\ell-n} D_x^{n-n} R_d = D_y^{dk+\ell-n} R_d.$$

Finally, we have that

$$\begin{aligned} D_x^{d^2k+d\ell+n} R_d &\simeq D_y^{dk+\ell-n} R_d \\ &\simeq \begin{cases} D_x^k R_d, & \ell = n, \\ D_y^{\ell-n} R_d, & \ell \neq n. \end{cases} \end{aligned}$$

The last equivalence follows by cases (1) and (2). □

Example 3.3. We illustrate the reduction formulas where $d = 5$ and $m = 23425$:

$$\begin{aligned} 23425 &= 5^2(937) + 5(0) + 0 \\ 937 &= 5^2(37) + 5(2) + 2 \\ 37 &= 5^2(1) + 5(2) + 2 \\ 1 &= 5^2(0) + 5(0) + 1. \end{aligned}$$

Therefore, $D_x^m R_d \simeq D_x^{937} R_d \simeq D_x^{37} R_d \simeq D_x R_d \simeq D_y^{-1} R_d$.

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3.5. *Reduction to base cases.* More generally, for $m = d^2k + d\ell + n$ with $\ell = n$, Lemma 3.2 returns a branched cover $D_x^k R_d$ with $k \leq m$. We can then repeatedly apply Lemma 3.2 until one of three 'base cases' occurs (we describe the process precisely in the proof of Theorem 1.1):

(1) we obtain a map $D_y^{\ell - n} R_d$ for $0 \leq |\ell - n| \leq d - 1$ to which $D_x^m R_d$ is equivalent;
(2) $m \geq 0$ and $D_x^m R_d$ reduces to $D_x^0 R_d$; or
(3) $m < 0$ and $D_x^m R_d$ reduces to $D_x^{-1} R_d$.

It therefore suffices to compute $D_x^{-1} R_d$, and $D_y^i R_d$ for $1 \leq i \leq d - 1$ and $-(d-1) \leq i \leq -1$.

4. *Invariant trees*

In this section, we will find the Hubbard trees for $D_y^i R_d$ for $1 \leq i \leq d - 1$ and $-(d-1) \leq i \leq -1$, and for $D_x^{-1} R_d$, which determines the polynomials in the base cases above.

For each base case, we find an invariant tree using the tree lifting algorithm of Belk, Lanier, Margalit, and the second author [3]. However, we will not show this process. We need only verify that each tree that we find is invariant under lifting and has an invariant angle assignment under the corresponding map, thus satisfying the conditions of Poirier. This proves that the invariant tree we found is indeed a Hubbard tree for the corresponding map.

PROPOSITION 4.1. *Let R_d be the degree-d rabbit polynomial. Then,*

$$D_y^i R_d \simeq \begin{cases} A_{d,d-i}, & 1 \leq i \leq d-1, \\ A_{d,-i}, & -(d-1) \leq i \leq -1. \end{cases}$$

The Hubbard tree H_- for $D_y^i R_d$ for $-(d-1) \leq i \leq -1$ is the tree in Figure 4. The invariant angle between e_1 and e_2 (measured counterclockwise) has measure $-2\pi i/d = 2\pi |i|/d$. The Hubbard tree H_+ for $D_y^i R_d$ for $1 \leq i \leq d-1$ is the tree in Figure 5. The invariant angle between e_1' and e_2 (measured counterclockwise) has measure $2\pi - 2\pi i/d$. Figure 4 demonstrates that H_- is invariant under $R_5^{-1} D_y$, and Figure 5 demonstrates that H_+ is invariant under $R_5^{-1} D_y^{-1}$; the proof below explains that the figures generalize for all $d \geq 2$, and for $-(d-1) \leq i \leq -1$ and $1 \leq i \leq d-1$, respectively.

Proof. Let b be a special branch cut for R_d (for instance, the arc b in Figure 4). By the definition of special branch cut, the preimage $R_d^{-1}(b)$ consists of d arcs from p_0 to ∞ such that the complement of $R_d^{-1}(b)$ in $\mathbb{C}$ contains d components and one of them contains both points p_1 and p_2. Label these complementary components by $\Gamma_0, \ldots, \Gamma_{d-1}$ counterclockwise where Γ_0 is the component that contains p_1 and p_2. The set $R_d^{-1}(p_0)$ contains d points; name them $\widetilde{p}_0, \ldots, \widetilde{p}_{d-1}$ such that $\widetilde{p}_j \in \Gamma_j$.

We will show that H_- is the Hubbard tree for $D_y^i R_d$ when $-(d-1) \leq i \leq 1$ and H_+ is the Hubbard tree for $D_y^i R_d$ when $1 \leq i \leq d-1$ using the same basic strategy. First, we take the lift of the tree $D_y^{-i}(H_\pm)$ under R_d by computing the path lifts of the edges of $H_\pm$ (the edges of H_- are e_1 and e_2 in H_-, the edges of H_+ are e_1' and e_2) and determining

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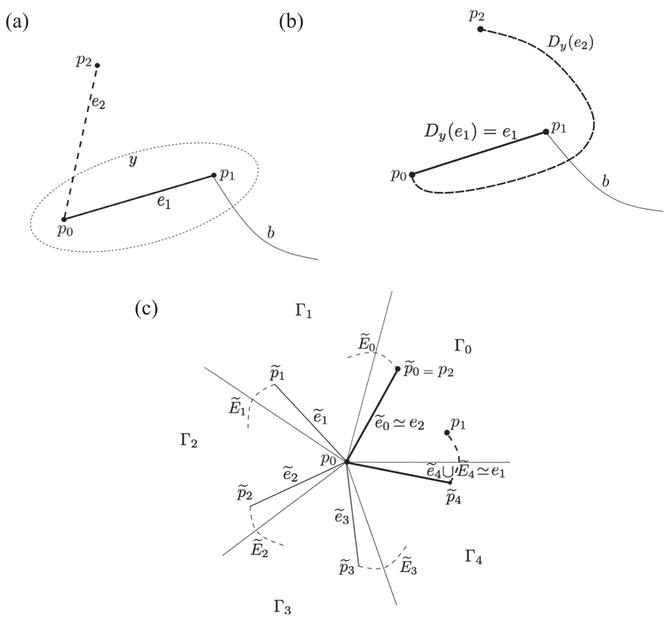

FIGURE 4. (a) The Hubbard tree H_- for $D_y^i R_d$ with $-(d-1) \le i \le -1$. (b) A tree that is homotopic to $D_y(H_-)$. (c) A tree that is homotopic to the preimage of $D_y(H_-)$ under R_5.

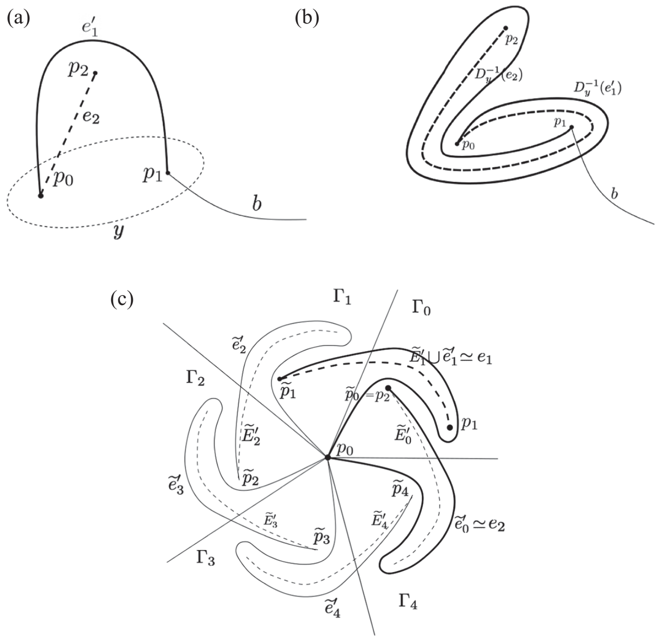

FIGURE 5. (a) The Hubbard tree H_+ for $D_y^i R_d$ with $1 \le i \le d-1$. (b) A tree that is homotopic to $D_y^{-1}(H_+)$. (c) A tree that is homotopic to the preimage of $D_y^{-1}(H_+)$ under R_5.

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which are in the hull of p_0, p_1, and p_2. We then verify the desired angle assignment is invariant under lifting.

We first treat the case where $-(d-1) \leq i \leq -1$. Note that i is negative so D_y^{-i} is a positive (left-handed) twist. Observe that e_1 is invariant under D_y^{-i} for all i. The edge $D_y^{-i}(e_2)$ intersects the branch cut b with both algebraic and geometric intersection i (with appropriately chosen orientations of b and e_2). That is, all intersections of $D_y^{-i}(e_2)$ and b have the same orientation (this an arc version of [9, Proposition 3.2]).

For each $0 \leq j \leq d-1$, there is a path lift of $D_y^{-i}(e_1) = e_1$ based at $\widetilde{p}_j$, call it $\widetilde{e}_j$. Because the interior of e_1 is disjoint from the branch cut b, each $\widetilde{e}_j$ is a straight line segment from $\widetilde{p}_j$ to $p_0 = R_d^{-1}(p_1)$ that is contained in the closure of Γ_j, as in Lemma 3.1. In particular, $\widetilde{e}_0$ is contained in the closure of Γ_0 and has endpoints p_0 and p_2. Since $\widetilde{e}_0$ is homotopic to a straight line segment relative to $\{p_0, p_1, p_2\}$, it is homotopic to e_2.

The arc $D_y^{-i}(e_2)$ twists counterclockwise $|i|$ times around e_1 (when oriented from p_0 to p_2). For each $0 \leq j \leq d-1$, there is a path lift of $D_y^{-i}(e_2)$ based at $\widetilde{p}_j$ with opposite endpoint in $R_d^{-1}(p_2)$; call this $\widetilde{E}_j$. Each $\widetilde{E}_j$ comprises a distinct component of $R_d^{-1}(D_y^{-i}(e_2))$. Moreover, each $\widetilde{E}_j$ intersects $R_d^{-1}(b)$ at $|i|$ points, rotating counterclockwise from $\widetilde{p}_j$ to a point in $R^{-1}(p_2)$. We may then apply Lemma 3.1 to determine that the opposite endpoint of $\widetilde{E}_j$ is in $\Gamma_{(j+|i|) \bmod d}$.

The preimage $R_d^{-1}(D_y^{-i}(H_-))$ comprises the union of $\{\widetilde{e}_j \mid 0 \leq j \leq d-1\}$ and $\{\widetilde{E}_j \mid 0 \leq j \leq d-1\}$. For each j, the union of $\widetilde{e}_j$ and $\widetilde{E}_j$ is a path between an element of $R_d^{-1}(p_2)$ and p_0 (via $\widetilde{p}_j$). The only such path that is in the hull of $\{p_0, p_1, p_2\}$ in $R_d^{-1}(D_y^{-i}(H_-))$ is $\widetilde{e}_{d+i} \cup \widetilde{E}_{d+i}$, which contains p_1, $\widetilde{p}_{d+i}$, and p_0. This edge is homotopic to e_1. The other edge in the hull of $\{p_0, p_1, p_2\}$ in $R_d^{-1}(D_y^{-i}(H_-))$ is $\widetilde{e}_0$, which has endpoints p_2 and p_0, and is homotopic to e_2. Thus, the tree H_- is invariant under lifting.

To see that the angle between e_1 and e_2 (measured counterclockwise) of $-2\pi i/d$ is invariant under $D_y^i R_d$ for $-(d-1) \leq i \leq -1$, we track the preimage of all angles under the lifting process. In particular, we observe that the preimage of the angle between e_1 and e_2 is an angle at an unmarked vertex (of valence 2) in $R_d^{-1}(D_y^{-i}(H_-))$, which is therefore irrelevant. However, because $R_d^{-1}(p_1) = p_0$, the preimage of the angle of 2π at p_1 (measured from e_1 to itself) consists of d angles between the d path lifts of e_1, each of measure $2\pi/d$. The path lifts of e_1 in the hull of $R_d^{-1}(D_y^{-i}(H_-))$ relative to $\{0, p_1, p_2\}$ are $\widetilde{e}_0$ and $\widetilde{e}_{d+i}$. There are $|i-1|$ path lifts of e_1 under R_d between $\widetilde{e}_{d+i}$ and $\widetilde{e}_0$ in the cyclic counterclockwise ordering of vertices adjacent to p_0. Therefore, the angle between $\widetilde{e}_0$ and $\widetilde{e}_{d+i}$ is $2\pi|i|/d$, when measured counterclockwise. Since i is negative, the counterclockwise angle between the edges of the lift homotopic to e_1 and e_2 is $-2\pi i/d$, and we have shown that $D_y^i R_d$ is equivalent to $A_{d,-i}$ for $-(d-1) \leq i \leq -1$.

Now we show that the tree H_+ in Figure 5 is invariant under $D_y^i R_d$ for $1 \leq i \leq d-1$. Both edges e_1' and e_2 (in Figure 5) meet at the critical point p_0. Orient both edges away from p_0. The arcs $D_y^{-i}(e_1')$ and $D_y^{-i}(e_2)$ each intersect b in i points and all intersections have the same orientation (i.e., both arcs are directed clockwise at the points of intersection with b). For each $0 \leq j \leq d-1$, there is a path lift of $D_y^{-i}(e_1)$ under R_d based at $\widetilde{p}_j$ and a path lift of $D_y^{-i}(e_2)$ based at $\widetilde{p}_j$; call these $\widetilde{e}_j'$ and $\widetilde{E}_j'$, respectively. Each $\widetilde{e}_j'$ and $\widetilde{E}_j'$ intersects $R_d^{-1}(b)$ at i points in a clockwise direction until it reaches its other endpoint.

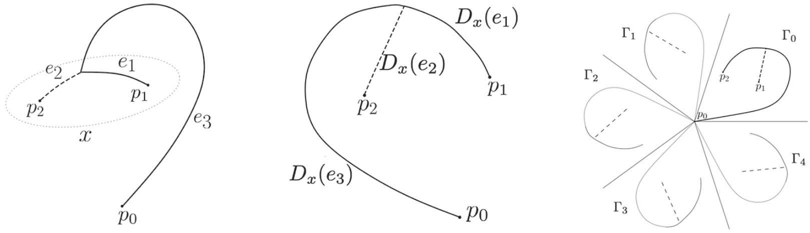

FIGURE 6. (a) The Hubbard tree H for $D_x^{-1}R_d$. (b) $D_x(H)$. (c) $R_5^{-1}(D_x(H))$.

The other endpoint of $\widetilde{E}'_j$ is an element of $R_d^{-1}(p_2)$ and by Lemma 3.1, this endpoint is in $\Gamma_{(j-i) \bmod d}$. The other endpoint of $\widetilde{e}'_j$ is p_0 for all j. Then, $\widetilde{e}'_0$, which has endpoints p_2 and p_0, is in the hull of $\{p_0, p_1, p_2\}$ in $R_d^{-1}(D_y^{-i}(H_+))$. In fact, as long as $1 \leq i \leq d-1$, $\widetilde{e}'_0$ is homotopic to a straight line segment, that is, e_2. Moreover, the union of $\widetilde{E}'_i$ and $\widetilde{e}'_i$ forms an edge from p_1 to p_0 (via $\widetilde{p}_i$). As long as $1 \leq i \leq d-1$, this edge is homotopic to e'_1.

As in the case where $-(d-1) \leq i \leq -1$, to verify that the angle $2\pi - 2\pi i/d$ between e'_1 and e_2 (measured counterclockwise) is invariant under $R_d^{-1}D_y^{-i}$, we need only consider the angle between the path lifts of e'_1 that are in the hull of $\{p_0, p_1, p_2\}$. Indeed, we saw above that $\widetilde{e}'_0$ is always homotopic to e_2 and $\widetilde{e}'_i$ is part of the path in $R_d^{-1}(D_y^{-i}(H_+))$ that is homotopic to e'_1. The angle between $\widetilde{e}'_i$ and $\widetilde{e}'_0$ (measured counterclockwise) then has measure $2\pi - 2\pi i/d$, as desired. Therefore, H_+ with angle measure $2\pi - 2\pi i/d$ between e'_1 and e_2 is invariant under $R_d^{-1}D_y^{-i}$ when $1 \leq i \leq d-1$. Thus, $D_y^i R_d$ is equivalent to $A_{d,d-i}$ for $1 \leq i \leq d-1$. □

LEMMA 4.2. *For all $d \geq 2$, the branched cover $D_x^{-1}R_d$ is equivalent to $\overline{R}_d$.*

Proof. Figure 6 shows the Hubbard tree H for $D_x^{-1}R_d$ for all $d \geq 2$. Figure 6 also shows that the lift of $D_x(H)$ under R_5 is homotopic to H; a similar calculation verifies that the lift of $D_x(H)$ under R_d is homotopic to H for all $d \geq 2$. The angle assignment of $2\pi/3$ at each angle of the trivalent vertex is invariant under lifting. There are no Julia edges, so Poirier's conditions verify that H is indeed the Hubbard tree of $D_x^{-1}R_d$. Moreover, $D_x^{-1}R_d$ rotates the edges clockwise relative to the trivalent vertex (one edge and its lift is dashed to assist tracking the rotation). Therefore, $D_x^{-1}R_d$ is equivalent to $\overline{R}_d$ for $d \geq 2$. □

5. *Proof of main theorem*

We now combine the reduction formulas from §3 and the base cases from §4 to prove Theorem 1.1.

Proof of Theorem 1.1. As in the theorem, we consider the map $D_x^m R_d$. Consider the d^2-adic expansion of m, which is $m_s m_{s-1} \cdots m_1$ if $m \geq 0$ and $\overline{d^2-1}\,m_s m_{s-1} \cdots m_1$ if $m < 0$.

We may write m uniquely as

$$m = d^2 k_1 + d\ell_1 + n_1,$$

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where $k_1 \in \mathbb{Z}, \ell_1, n_1 \in \{0, 1, \ldots, d-1\}$. We note that

$$k_1 = \frac{m - m_1}{d^2} \quad \text{and} \quad m_1 = d\ell_1 + n_1.$$

For $2 \leq i \leq s$, there exist integers $k_i \in \mathbb{Z}, 0 \leq \ell_i, n_i \leq d-1$ such that

$$\begin{aligned} k_i &= \frac{k_{i-1} - m_i}{d^2}, \\ m_i &= d\ell_i + n_i, \quad \text{and} \\ k_{i-1} &= d^2 k_i + d\ell_i + n_i. \end{aligned}$$

That is, the d^2-adic expansion of k_i is obtained from m by dropping the right-most i digits. In particular, if $m \geq 0$, then $k_s = 0$. If $m < 0$, then k_s has d^2-adic expansion $\overline{d^2 - 1}$ and therefore $k_s = -1$. Furthermore, we note that $\ell_i = n_i$ if and only if $(d+1)|m_i$.

By applying Lemma 3.2 to $D_x^m R_d$, we have

$$D_x^m R_d \simeq \begin{cases} D_x^{k_1} R_d & \text{if } (d+1)|m_1, \\ D_y^{\ell_1 - n_1} R_d & \text{otherwise.} \end{cases}$$

Thus, if $(d+1) \nmid m_1$, we may apply Proposition 4.1, to obtain

$$D_x^m R_d \simeq D_y^{\ell_1 - n_1} R_d = \begin{cases} A_{d, n_1 - \ell_1} & \text{if } n_1 > \ell_1, \\ A_{d, d-(\ell_1 - n_1)} & \text{if } n_1 < \ell_1. \end{cases}$$

So we can deduce the equivalence class of $D_x^m R_d$ directly if $\ell_1 \neq n_1$. Otherwise, we right-shift the d^2-adic expansion of m and consider $D_x^{k_1} R_d$ instead. We repeat this process for $k_1, \ldots, k_s$. Then, one of the following will occur.

- If $(d+1)|m_i$ for all i, then $\ell_i = n_i$ for all i. In particular, $\ell_s = n_s$ and we have

  $$D_x^m R_d \simeq D_x^{k_1} R_d \simeq D_x^{k_2} R_d \simeq \cdots \simeq D_x^{k_{s-1}} R_d \simeq D_x^{k_s} R_d.$$

  If $m \geq 0$, then $k_s = 0$, so $D_x^m R_d \simeq D_x^{k_s} R_d \simeq R_d$.If $m < 0$, then $k_s = \overline{d^2 - 1} = -1$ and $D_x^m R_d \simeq D_x^{k_s} R_d \simeq D_x^{-1} R_d$. By Lemma 4.2, $D_x^{-1} R_d$ is equivalent to $\overline{R}_d$.
- If there exists i such that $d+1$ does not divide m_i, choose the minimal such i and write $m_i = d\ell_i + n_i$, with $\ell_i, n_i \in \{0, 1, \ldots, d-1\}$. Then, $\ell_i \neq n_i$. Thus, by Proposition 4.1, we have

  $$D_x^m R_d \simeq D_y^{\ell_i - n_i} R_d \simeq \begin{cases} A_{d, n_i - \ell_i} & \text{if } n_i > \ell_i, \\ A_{d, d-(\ell_i - n_i)} & \text{if } n_i < \ell_i. \end{cases}$$

This completes the proof of the theorem. □

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REFERENCES

[1] L. Bartholdi and D. Dudko. Algorithmic aspects of branched coverings. *Ann. Fac. Sci. Toulouse Math. (6)* **26**(5) (2017), 1219–1296.

[2] L. Bartholdi and V. Nekrashevych. Thurston equivalence of topological polynomials. *Acta Math.* **197**(1) (2006), 1–51.

[3] J. Belk, J. Lanier, D. Margalit and R. Winarski. Recognizing topological polynomials by lifting trees. *Duke Math. J.* **171** (2022), 3401–3480.

[4] B. Bielefeld, Y. Fisher and J. Hubbard. The classification of critically preperiodic polynomials as dynamical systems. *J. Amer. Math. Soc.* **5**(4) (1992), 721–762.

[5] S. Bonnot, M. Braverman and M. Yampolsky. Thurston equivalence to a rational map is decidable. *Mosc. Math. J.* **12**(4) (2012), 747–763, 884.

[6] A. Douady and J. Hubbard. A proof of Thurston's topological characterization of rational functions. *Acta Math.* **171**(2) (1993), 263–297.

[7] A. Douady and J. H. Hubbard. *Étude Dynamique des Polynômes Complexes. Partie I (Publications Mathématiques d'Orsay [Mathematical Publications of Orsay], 84)*. Université de Paris-Sud, Département de Mathématiques, Orsay, 1984.

[8] A. Douady and J. H. Hubbard. *Étude Dynamique des Polynômes Complexes. Partie II (Publications Mathématiques d'Orsay [Mathematical Publications of Orsay], 85)*. Université de Paris-Sud, Département de Mathématiques, Orsay, 1985; with the collaboration of P. Lavaurs, Tan Lei and P. Sentenac.

[9] B. Farb and D. Margalit. *A Primer on Mapping Class Groups*. Princeton University Press, Princeton, NJ, 2011.

[10] J. H. Hubbard. *Teichmüller Theory and Applications to Geometry, Topology, and Dynamics. Volume 2: Surface Homeomorphisms and Rational Functions*. Matrix Editions, Ithaca, NY, 2016.

[11] J. H. Hubbard and D. Schleicher. The spider algorithm. *Complex Dynamical Systems (Cincinnati, OH, 1994) (Proceedings of Symposia in Applied Mathematics, 49)*. Ed. R. Devaney. American Mathematical Society, Providence, RI, 1994, pp. 155–180.

[12] G. Kelsey and R. Lodge. Quadratic Thurston maps with few postcritical points. *Geom. Dedicata* **201** (2019), 33–55.

[13] S. V. F. Levy. Critically finite rational maps (Thurston). *PhD Thesis*, Princeton University, ProQuest LLC, Ann Arbor, MI, 1985.

[14] V. Nekrashevych. Combinatorics of polynomial iterations. *Complex Dynamics*. A. K. Peters, Wellesley, MA, 2009, pp. 169–214.

[15] V. Nekrashevych. Combinatorial models of expanding dynamical systems. *Ergod. Th. & Dynam. Sys.* **34**(3) (2014), 938–985.

[16] A. Poirier. Hubbard trees. *Fund. Math.* **208**(3) (2010), 193–248.

[17] K. Rafi, N. Selinger and M. Yampolsky. Centralizers in mapping class groups and decidability of Thurston equivalence. *Arnold Math. J.* **6**(2) (2020), 271–290.

[18] A. Shepelevtseva and V. Timorin. Invariant spanning trees for quadratic rational maps. *Arnold Math. J.* **5**(4) (2019), 435–481.

[19] D. P. Thurston. A positive characterization of rational maps. *Ann. of Math. (2)* **192**(1) (2020), 1–46.