GEOMETRY OF A SIMPLEX INSCRIBED IN A NORMAL RATIONAL CURVE

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In 1959, Professor N. A. Court [2] generated synthetically a twisted cubic C circumscribing a tetrahedron T as the poles for T of the planes of a coaxal family whose axis is called the Lemoine axis of C for T. Here is an analytic attempt to relate a normal rational curve r^n of order n, whose natural home is an n-space $[n]$, with its Lemoine $[n-2]$ L such that the first polars of points in L for a simplex S inscribed to r^n pass through r^n anf the last polars of points on r^n for S pass through L. Incidently we come across a pair of mutually inscribed or Moebius simplexes but as a privilege of odd spaces only. In contrast, what happens in even spaces also presents a case, not less interesting, as considered here.

1. Polarity for a simplex

(a) If P be a point $(p_0, p_1, \cdots, p_n)$ referred to a simplex $S = A_0 A_1 \cdots A_n$, the first polar of P for S is the primal $(P) \equiv \sum (p_i/x_i) = 0$ of order n, and the last or n^{th} polar is the prime $p \equiv \sum (x_i/p_i) = 0$ $(i = 0, 1, \cdots, n)$ as a well known fact. Thus: If the polar prime $q \equiv \sum (x_i/q_i) = 0$ of a point $Q(q_i)$ for S pass through P; i.e., $(p_i/q_i) = 0$, (P) passes through Q. Or, *(P) is the locus of the poles for S of the primes through P.*

(b) Let the secant through P to an edge $A_i A_j$ of S and its opposite $[n-2]$ a^{ij} meet the edge in a point P_{ij}, and Q_{ij} be the point on this edge as the harmonic conjugate of P_{ij} w.r.t. the pair of the vertices A_i, A_j. That is, $H(A_i A_j, P_{ij} Q_{ij})$ or $(A_i P_{ij} A_j Q_{ij}) = -1$. The $\binom{n+1}{2}$ points Q_{ij} then all lie in the polar prime p of P for S [4; 7–11]. Conversely, if a prime p cuts $A_i A_j$ in Q_{ij} and P_{ij} be such that $H(A_i A_j, P_{ij} Q_{ij})$, the $\binom{n+1}{2}$ primes $a^{ij} p_{ij}$ concur at the pole P of p for S.

Hence, if p pass through A_i, Q_{ij} and therefore P_{ij} both coincide at A_i which then becomes the pole of p for S. Or, *the pole of a prime through a vertex of S for S lies at this vertex.*

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2. Normal rational curve

(a) The normal rational curve (n.r.c.) r^n is generated by the corresponding primes of n related pencils whose n vertices $[n-2]$'s form its chordals [14]. As the prime p in 1(b) varies in a pencil cutting the n edges A_iA_j of the simplex S through its vertex A_i in the n points Q_{ij}, the n corresponding primes $a^{ij}P_{ij}$ of the n pencils with vertices as the $[n-2]$'s a^{ij} of the prime a^i of S opposite A_i generate r^n as the locus of the poles of primes p of the given pencil for S. From the symmetry of the result follows the following:

THEOREM 1. *The locus of the poles of the primes of pencil for a simplex S in $[n]$ is an n.r.c. r^n through its vertices.*

(b) Conversely we may have the following:

THEOREM 2. *The polar primes of the points of an n.r.c. r^n circumscribing a simplex S for S form a coaxal family.*

PROOF 1. Following Court [2], we can prove synthetically the proposition by induction. For it is true in plane ($n = 2$) and solid ($n = 3$).

PROOF 2. Let r^n be represented parametrically by the $n+1$ coordinates $x_i = 1/(k-u_i)$ of a point P on r^n, k being the parameter [14; p. 220]. The polar prime p of P for S by 1(a) is

$$\text{(i)} \qquad \sum (k-u_i)x_i = 0, \quad \text{or} \quad k\sum x_i - \sum u_i x_i = 0.$$

This equation shows that p passes through the $[n-2]$ L common to the 2 primes: $\sum x_i = 0$, $\sum u_i x_i = 0$, thus proving the proposition.

REMARK 1. Theorem 1 could be proved by taking the vertex $[n-2]$ of the pencil as L above and deduce the parametric equations $x_i=1/(k-u_i)$ of the r^n.

DEFINITION. L is said to be the *Lemoine* $[n-2]$ of r^n for the simplex S.

THEOREM 3. *Any $n+3$ general points in $[n]$ determine an n.r.c. r^n in $\binom{n+3}{2}$ ways by choosing any $n+1$ of them to form a simplex inscribed to it thus giving us $\binom{n+3}{2}$ Lemoine $[n-2]$'s, one for each simplex.*

PROOF. Theorem 2 tells us that an r^n is determined by $n+3$ points, $n+1$ forming a simplex S and the other two points being the poles for S of a couple of primes through the Lemoine $[n-2]$ of r^n for S.

3. Polar and Cevian quadrics

The polar quadric of a point P on an r^n circumscribing a simplex S with coordinates $x_i = 1/(k-u_i)$ for S is

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(ii) $$\sum (k-u_i)(k-u_j)x_i x_j = 0$$

or

$$k^2 \sum x_i x_j - k \sum (u_i+u_j)x_i x_j + \sum u_i u_j x_i x_j = 0,$$

showing that it belongs to a *special net* [5] determined by the 3 quadrics:

$$\sum x_i x_j = 0, \quad \sum (u_i+u_j)x_i x_j = 0, \quad \sum u_i u_j x_i x_j = 0.$$

The cevian quadric [10] of P for S touching the edges of S at the feet thereat of its bicevians through P is

$$\sum (k-u_i)^2 x_i^2 - 2 \sum (k-u_i)(k-u_j)x_i x_j = 0,$$

or,

(iii) $$4 \sum (k-u_i)(k-u_j)x_i x_j - (\sum \overline{k-u_i} x_i)^2 = 0$$

showing that it too belongs to a *special net*, and has ring contact with the corresponding quadric of the net (ii) along the polar prime p (i) of P for S. Thus we have

THEOREM 4. *The polar as well as cevian quadrics of the points of an n.r.c. r^n circumscribing a simplex S for S belong respectively to two special nets such that the pair of quadrics corresponding to a point P on r^n have ring contact along the polar prime p of P for S.*

4. Lemoine axes

THEOREM 5. *The Lemoine $[q-2]$'s of the n.r. curves in the $[q]$'s of a simplex S in $[n]$, which are projections therein of an n.r.c. r^n circumscribing S from the opposite $[n-q-1]$'s, all lie in the Lemoine $[n-2]$ L of r^n. In particular, the Lemoine axes of the cubic projections of r^n in the solids of S from the opposite $[n-4]$'s and the Lemoine points of the conic projections of r^n in the planes of S from the opposite $[n-3]$'s lie in L.*

PROOF. The polar prime p of a point P for simplex S in $[n]$ passes through the polar $[q-1]$ p_q of the projection P_q of P in a $[q]$ of S from its opposite $[n-q-1]$ for its q-simplex in this $[q]$. If p varies in a pencil through an $[n-2]$ L, p_q too varies in a pencil through the $[q-2]$ L_q which is a section of L by the $[q]$. Thus P_q traces an n.r.c. r^q, as a projection of r^n traced by P from the chordal $[n-q-1]$, having Lemoine $[q-2]$ as L_q.

Conversely we have

THEOREM 6. *If the Lemoine $[q-2]$'s of certain n.r.c.s. in the $[q]$'s of a simplex S in $[n]$ all lie in an $[n-2]$ L, every such r_q is then the projection of an r^n circumscribing S from its $[n-q-1]$ opposite its $[q]$ of the r^q.*

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5. First polars

THEOREM 7. *The $n-1$ first polars for a simplex S in $[n]$ of any $n-1$ independent points determining an $[n-2]$ L determine or have an n.r.c. r^n common such that the first polar of any point of L for S passes through r^n.*

PROOF. The first polar of a point for a simplex in $[n]$ is a primal of order n and dimension $n-1$, and contains the $\binom{n+1}{2}$ $[n-2]$'s of S once, the $\binom{n+1}{3}$ $[n-3]$'s twice, $\cdots$, the $\binom{n+1}{r}$ $[n-r]$'s $(r-1)$-times, $\cdots$ and $\binom{n+1}{2}$ edges of S $(n-1)$-times. Thus the intersection of the first polars of 2 points for S is of dimension $n-2$ but order $n^2-\binom{n+1}{2}=\binom{n}{2}$, that of 3 independent points is of dimension $n-3$ but order $n\binom{n}{2}-2\binom{n+1}{3}=\binom{n}{3}, \cdots$, that of r independent points is of dimension $n-r$ but order $n\binom{n}{r-1}-(r-1)\binom{n+1}{r}=\binom{n}{r}, \cdots$ and that of $n-1$ independent points is of dimension 1 but order $\binom{n}{n-1}=n$.

THEOREM 8. *L of the preceding theorem is the Lemoine $[n-2]$ of the r^n for the simplex S.*

PROOF. Let us take L to be the $[n-2]$ given by the pair of linear equations: $\sum x_i=0$, $\sum u_i x_i=0$, and P be a point $(p_0, p_1, \cdots, p_n)$ in L such that $\sum p_i=0=\sum u_i p_i$. Now the first polar of P is $(P)\equiv\sum(p_i/x_i)=0$ which obviously passes through the r^n given by the coordinates $x_i=1/(k-u_i)$ of any point on it because of the two conditions satisfied by P. Hence, by the definition of the Lemoine $[n-2]$ of an r^n, follows the theorem.

6. Tangents

THEOREM 9. *The meets of the primes a^i of a simplex S in $[n]$ with the tangents, at its opposite vertices A_i, of an n.r.c. r^n circumscribing S are the poles of the $[n-2]$ projections therein, of the Lemoine $[n-2]$ L of r^n for S from A_i, for the respective $(n-1)$-simplexes of S.*

PROOF. The equations of the tangent line of an n.r.c. r^n at any point with coordinates $x_i=(k-u_i)^{-1}$ on it are given by

$$\text{(iv)} \qquad 0=\begin{pmatrix} x_0 & \cdots & x_i & \cdots & x_n \\ (k-u_0)^{-1} & \cdots & (k-u_i)^{-1} & \cdots & (k-u_n)^{-1} \\ (k-u_0)^{-2} & \cdots & (k-u_i)^{-2} & \cdots & (k-u_n)^{-2} \end{pmatrix}_2$$

following the notations of Professor T. G. Room [14]. To find the tangents at the vertices of the simplex S of reference, we may write (iv) as

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$$\text{(v)} \qquad 0 = \begin{pmatrix} x_0(k-u_0)^2 & \cdots & x_i(k-u_i)^2 & \cdots & x_n(k-u_n)^2 \\ (k-u_0) & \cdots & (k-u_i) & \cdots & (k-u_n) \\ 1 & \cdots & 1 & \cdots & 1 \end{pmatrix}_2$$

and put $k = u_i$ in (v) to find one at the vertex A_i of S. Thus the tangent of r^n at A_i is given by the equations

$$x_0(u_i-u_0) = \cdots = x_{i-1}(u_i-u_{i-1}) = x_{i+1}(u_i-u_{i+1}) = \cdots = x_n(u_i-u_n)$$

meeting the opposite prime $x_i = 0$ of S in the point A'_i whose n coordinates other than x_i are then $x_j = (u_i-u_j)^{-1}$.

The equation of the $[n-2]$ projection in the prime $x_i = 0$ of S, of the Lemoine $[n-2]$ of the r^n for S from the opposite vertex A_i is found to be $\sum_{j\neq i} (u_i-u_j)x_j = 0$ showing it to be the last polar (1a) of A'_i for the $(n-1)$-simplex of S in the prime under consideration.

REMARK 2. r^n being the locus (Theorem 1) of the poles, for S, of the primes through L, A_i being the pole of the prime LA_i for S (1b) and the tangent of r^n at A_i being the limit of the chords of r^n through A_i, the Theorem 9 follows immediately from the definition of the pole and polar for a simplex (2; 4; 7—11).

THEOREM 10. *The n tangents of the n r^{n-1} projections of an n.r.c. r^n circumscribing a simplex S in $[n]$, in its n primes through a vertex A_i of S from the opposite vertices, at their common point A_i meet its n opposite $[n-2]$'s in the n points A'_{ij} which form a Cevian $(n-1)$-simplex of the $(n-1)$-simplex of S opposite A_i for the meet A'_i of its prime a^i with the tangent of r^n at A_i* [10].

PROOF. The tangent of the n.r.c. r^{n-1} projection of r^n, in the prime $x_j = 0$ of S from the opposite vertex A_j, at the vertex A_i meets the opposite $[n-2]$ a^{ij} (1b) in the point A'_{ij} whose coordinates referred to S are $x_i = 0 = x_j$, $x_k = 1/(u_i-u_k)$ for all values of k other than i, j (7a). Thus A_j, A'_i, A'_{ij} ($\neq A'_{ji}$) are collinear.

REMARK 3. In view of Remark 2, Theorem 10 can also be deduced from the definition of the pole and polar for a simplex [2].

7. Even spaces

If we put down the $n+1$ coordinates (6a) of the meet A'_i of a prime a^i of the simplex S of reference with the tangent of an n.r.c. r^n circumscribing S at its opposite vertex A_i as the ith row of a matrix M ($i = 0, \cdots, n$), we find M to be skew symmetric such that its determinant $|M| = 0$, thus showing that the $n+1$ points A'_i are co-primal if n is even. Hence follows the following:

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THEOREM 11. *The $2m+1$ meets of the $2m+1$ primes of a simplex S in $[2m]$ with the tangents of an n.r.c. r^{2m} circumscribing S at its opposite vertices all lie in a prime which coincides with the Lemoine axis of a triangle for a conic circumscribing it when $m = 1$* [11].

8. Odd spaces

THEOREM 12. *The $2m$ meets of the $2m$ primes of a simplex S in $[2m-1]$ with the tangents of an n.r.c. r^{2m-1} circumscribing S at its opposite vertices form another simplex S' Moebius or mutually inscribed with S* [1–3; 6; 12].

PROOF. The first minor of a skew symmetric matrix obtained by crossing its i^{th} row and i^{th} column is also skew symmetric. Hence if we substitute the $n+1$ coordinates $x_i = 1$, $x_j = 0$ (for all $j \neq i$) of a vertex A_i of a simplex S in the ith row of the matrix M of the preceding section, we find $|M| = 0$ thus showing that A_i lies in the prime determined by the n points A'_j if n is odd.

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