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From Steklov to Neumann and Beyond, via Robin: The Szegő Way

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Abstract. The second eigenvalue of the Robin Laplacian is shown to be maximal for the disk among simply-connected planar domains of fixed area when the Robin parameter is scaled by perimeter in the form $\alpha/L(\Omega)$, and α lies between -2π and 2π. Corollaries include Szegő's sharp upper bound on the second eigenvalue of the Neumann Laplacian under area normalization, and Weinstock's inequality for the first nonzero Steklov eigenvalue for simply-connected domains of given perimeter.

The first Robin eigenvalue is maximal, under the same conditions, for the degenerate rectangle. When area normalization on the domain is changed to conformal mapping normalization and the Robin parameter is positive, the maximiser of the first eigenvalue changes back to the disk.

1 Introduction

The eigenvalue problem for the Robin Laplacian on a domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary is

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \alpha u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where α is a real parameter and ν is the outward unit normal. The corresponding eigenvalues, denoted $\lambda_k(\Omega;\alpha)$ for $k = 1, 2, \dots$, are increasing and continuous as functions of the Robin parameter α, and for each fixed α satisfy

$$\lambda_1(\Omega;\alpha) < \lambda_2(\Omega;\alpha) \le \lambda_3(\Omega;\alpha) \le \cdots \longrightarrow \infty.$$

Isoperimetric eigenvalue inequalities in the literature typically assume an area normalization of the domain; see, for instance, [2,9,14] and the survey [4], which includes many related results on Robin eigenvalues.

While this area normalization is natural for Dirichlet and Neumann problems, it provides only part of the story for Robin, because the rescaling relation $t^2\lambda(t\Omega;\alpha/t) = \lambda(\Omega;\alpha)$ shows that the area-normalized product $|\Omega|\lambda(\Omega;\alpha)$ is not scale-invariant. This observation has prompted us to look for natural, scale-invariant isoperimetric inequalities for eigenvalues of problem (1.1). We claim the most natural formulation

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for planar domains is to keep the domain normalized by area while considering the Robin parameter scaled by the perimeter of the domain. The eigenvalues under consideration thus become

$$\lambda_1\Big(\Omega; \frac{\alpha}{L(\Omega)}\Big) < \lambda_2\Big(\Omega; \frac{\alpha}{L(\Omega)}\Big) \le \lambda_3\Big(\Omega; \frac{\alpha}{L(\Omega)}\Big) \le \cdots \longrightarrow \infty,$$

where $L(\Omega)$ denotes the length of the boundary $\partial\Omega$. Under this new scaling, the behavior of eigenvalues changes dramatically with regard to the existence and characterization of extremal domains. One consequence is that area-normalized eigenvalues can now remain bounded from both above and below; we prove in Theorem A that for each real α the scaled and normalized first eigenvalue is maximal for the degenerate rectangle, and for each positive α the eigenvalue is bounded below (since it is positive). In Theorem E we show that if one normalizes not the area of the domain but rather its conformal mapping radius, while maintaining the perimeter scaling, then the disk is promoted to maximise the first eigenvalue.

The above result for the first eigenvalue hints at a possible prolongation of the Szegő–Weinberger upper bound [30, 31] for the second eigenvalue from $\alpha = 0$ to $\alpha \neq 0$. The first Neumann eigenvalue is zero for all domains and so has no preferred extremal domain. The second Neumann eigenvalue is maximal for the disk by the Szegő–Weinberger result, and one hopes for this to extend to the Robin eigenvalues, at least when $|\alpha|$ is small. Indeed, for $\alpha \in [-2\pi, 2\pi]$ we show in Theorem B that the second eigenvalue is maximal for the disk among simply-connected planar domains when the Robin parameter is scaled by perimeter and the domain is normalized by area.

Hence, we unify two results: Weinstock's upper bound on the first nonzero Steklov eigenvalue for domains with given perimeter and Szegő's upper bound on the first nonzero Neumann eigenvalue for domains with given area. We also provide an estimate on the value of $\alpha > 0$ after which the disk can no longer remain the maximal domain.

When $\alpha < 0$, maximality of the disk for $\lambda_2(\Omega; \alpha/L(\Omega))$ implies maximality of the disk for the unscaled eigenvalue $\lambda_2(\Omega; \alpha)$, as we will show in Corollary C. The point is that the unscaled eigenvalue under area normalization is equivalent to $\lambda_2(\Omega; \alpha/|\Omega|^{1/2})$, where the Robin parameter is scaled not by perimeter but by the square root of area, and then the (geometric) isoperimetric inequality can be applied. This corollary recovers a planar case of our earlier result that the ball maximizes the second Robin eigenvalue among domains of fixed volume [14]. Thus, under some circumstances, length scaling of the Robin parameter yields a stronger result than for the unscaled problem.

In this paper we concentrate on the 2-dimensional problem, but our proposed scaling and normalization extend naturally to the general dimension n by considering quantities of the form,

$$|\Omega|^{2/n}\lambda\Big(\Omega; \frac{|\Omega|^{1-2/n}}{|\partial\Omega|}\alpha\Big).$$

The upper bound on the first eigenvalue in Theorem A extends to higher dimensions in this manner, with an analogous proof. For maximising the second eigenvalue, we raise a higher dimensional conjecture for convex domains in section 3, where some other open problems are discussed too.

2 Notation and Main Results

We consider the quantity

$$\lambda_k\big(\Omega;\alpha/L(\Omega)\big)\,A(\Omega), \qquad k=1,2,\dots,$$

in which each eigenvalue is multiplied by the area $A(\Omega)$, and the Robin parameter is scaled by the perimeter $L(\Omega)$. This quantity is scale invariant; *i.e.*, its value does not change when Ω is scaled by a positive constant factor t, thanks to the rescaling relation $t^2\lambda(t\Omega;\alpha/t)=\lambda(\Omega;\alpha)$. In terms of Rayleigh quotients, the one associated with $\lambda_k(\Omega;\alpha)$ in equation (1.1) is

$$Q[u]=\frac{\int_\Omega|\nabla u|^2\,dx+\alpha\int_{\partial\Omega}u^2\,ds}{\int_\Omega u^2\,dx},$$

where $u\in H^1(\Omega)$. After multiplying by area and replacing α with $\alpha/L(\Omega)$, the Rayleigh quotient takes an appealing "mean value" form

$$\overline{Q}[u]=\frac{A(\Omega)\,⨍_\Omega|\nabla u|^2\,dx+\alpha\,⨍_{\partial\Omega}u^2\,dS}{⨍_\Omega u^2\,dx},$$

where we observe that each of the three terms is scale invariant by itself.

The distinction between the *normalizing factor* that multiplies the eigenvalue and the *scale factor* that divides the Robin parameter is central to this paper. These two distinct factors lie behind the unification (in Corollary D) of Weinstock's bound on the first Steklov eigenvalue for given perimeter and Szegő's bound on the first (nontrivial) Neumann eigenvalue for given area.

The First Eigenvalue

Under normalization by A and scaling by L, the first eigenvalue is bounded from above on general domains for all α, being maximal in the limiting case of a degenerate rectangle. This upper bound is elementary, yet suggestive of the different type of results we should expect now that the Robin parameter is appropriately scaled. The theorem also has the virtue of holding for all domains and for both positive and negative values of the parameter α.

Theorem A (Sharp upper bound on λ_1 for all α) *Fix $\alpha\neq 0$. If Ω is a bounded, Lipschitz planar domain, then*

$$\lambda_1\big(\Omega;\alpha/L(\Omega)\big)\,A(\Omega)<\alpha$$

with equality holding in the limit for rectangular domains that degenerate to a line segment.

In the omitted case of vanishing α, equality holds for all domains, since $\lambda_1(\Omega;0)=0$.

Although sharp among all domains, the theorem is not sharp for a fixed domain Ω in the limit as α approaches $\pm\infty$, in the sense that the first Robin eigenvalue for any given domain approaches a finite number (the Dirichlet eigenvalue) as $\alpha \to +\infty$, and approaches $-\infty$ quadratically rather than linearly as $\alpha \to -\infty$, by the asymptotic formula of Lacey *et al.* [17, Theorem 4.14].

Theorem A in the nonstrict, unscaled form $\lambda_1(\Omega;\alpha) \le \alpha L(\Omega)/A(\Omega)$ was noted by several authors previously. The novelty here consists rather of the scaling form and the asymptotic sharpness of the strict inequality.

As mentioned in the introduction, in n dimensions, Theorem A can be generalized in a straightforward fashion to apply to $\lambda_1(\Omega;\alpha V^{1-2/n}/S)V^{2/n}$ where V is volume and S is surface area.

The Second Eigenvalue

A *Jordan domain* is a simply-connected, bounded planar domain Ω whose boundary is a Jordan curve. A *Jordan–Lipschitz domain* is a Jordan domain with Lipschitz boundary.

Theorem B (perimeter scaling $\Rightarrow \lambda_2$ maximal for disk) *Fix $\alpha \in [-2\pi, 2\pi]$. If Ω is a Jordan–Lipschitz domain then the scale invariant quantity*

$$\lambda_2\big(\Omega;\alpha/L(\Omega)\big)A(\Omega)$$

is maximal for the disk. Equivalently,

$$\lambda_2\big(\Omega;\alpha/L(\Omega)\big) \le \lambda_2\big(D;\alpha/L(D)\big)$$

where D is a disk with the same area as Ω. Equality holds if and only if Ω is a disk.

The endpoint value $\alpha = -2\pi$ is special, because it is where $\lambda_2(D;\alpha/L(D)) = 0$; indeed, by Proposition 4, the disk D of radius R and perimeter $L(D) = 2\pi R$ has repeated second eigenvalue $\lambda_2(D;-1/R) = \lambda_3(D;-1/R) = 0$. The corresponding eigenfunctions are $u = x_1$ and $u = x_2$.

The interval of α-values on which Theorem B holds could perhaps be expanded. The theorem must fail as $\alpha \to \infty$, though, due to Dirichlet eigenvalues being arbitrarily large on long thin domains. In fact, such domains show that the theorem definitely fails for $\alpha \ge 32.7$, as explained at the end of section 7. Notice here the reason we can state the interval in terms of absolute constants $\pm 2\pi$ and state the counterexample with absolute constant 32.7 is because α was divided by $L(\Omega)$. Otherwise the perimeter would need to be included in all the relevant intervals and constants.

The Lipschitz assumption on the boundary in Theorem B could be weakened somewhat, since it is used only to guarantee compactness of the imbedding $H^1 \hookrightarrow L^2$ and existence of the trace operator on the boundary, and to ensure the chord–arc condition in Case (ii) of section 7.

A corollary with fixed negative α (not scaled by perimeter) follows easily from the theorem with the help of the isoperimetric inequality. Let $R(\Omega) = \sqrt{A(\Omega)/\pi}$ be the radius of the disk D having the same area as Ω.

Corollary C (no scaling ⇒ λ_2 maximal for disk) *If Ω is a Jordan–Lipschitz domain and $\alpha \in [-1/R(\Omega), 0]$, then $\lambda_2(\Omega;\alpha) \le \lambda_2(D;\alpha)$, with equality if and only if Ω is a disk.*

This corollary is a special case of our earlier result [14, Theorem A] for arbitrary domains in all dimensions, which was proved using a Weinberger-type method. Thus, for the second Robin eigenvalue on simply-connected planar domains, the Szegő method gives a definitely stronger inequality (Theorem B) than the Weinberger method (Corollary C). On the other hand, the Weinberger method offers additional flexibility, which we exploited in [14, Theorem A] to prove the result of Corollary C for a larger range of α-values, all the way down to $-3/2R(\Omega)$. Further, Weinberger's method works regardless of connectivity, whereas Theorem B fails for certain doubly connected domains (annuli), as explained below.

We shall now relate our results to the Neumann and Steklov spectra. To this end write $0 = \mu_0 < \mu_1 \le \mu_2 \le \cdots$ for the spectrum of the Neumann Laplacian, and $0 = \sigma_0 < \sigma_1 \le \sigma_2 \le \cdots$ for the Steklov spectrum (corresponding to harmonic functions with $\partial u/\partial \nu = \sigma u$ on the boundary). For an introduction to Steklov spectral geometry, we highly recommend Girouard and Polterovich's survey paper [16].

The next result unifies Weinstock's upper bound on σ_1 under perimeter normalization with Szegő's upper bound on μ_1 under area normalization. Until now these results have been regarded as different due to their different normalizing factors, although the proofs are clearly closely related [15]. By inspecting the horizontal and vertical intercepts of $\alpha \mapsto \lambda_2(\Omega;\alpha/L)A$, we discover that the Steklov and Neumann inequalities are in fact two facets of one underlying result, Theorem B.

Corollary D (Weinstock [32], Szegő [30]) *For Ω a Jordan–Lipschitz domain, the scale invariant quantities*

$$\sigma_1(\Omega)L(\Omega) \quad \textit{and} \quad \mu_1(\Omega)A(\Omega)$$

are maximal for the disk, and only for the disk.

The Weinstock inequality on $\sigma_1(\Omega)L(\Omega)$ fails for certain annuli [17, Example 5.14]. Hence, the above corollary and Theorem B both fail for general domains that are not simply connected. On the other hand, by weakening the normalization to area and considering $\sigma_1(\Omega)\sqrt{A(\Omega)}$, Brock did obtain a result valid for all domains, and which extends to all dimensions [3]. The Szegő inequality on $\mu_1(\Omega)A(\Omega)$ likewise holds for all domains and extends to all dimensions, as was shown by Weinberger [31]. These Brock and Weinberger inequalities are unified by our recent work on the Robin spectrum under volume normalization with no scaling of the Robin parameter [14, Corollary B].

Other Normalizations

If instead of normalizing the Robin eigenvalue with area we normalize with the square of the conformal mapping radius, then for positive α a geometrically sharp result can be obtained for the first eigenvalue. The Robin parameter continues to be scaled by perimeter in what follows.

Theorem E (conformal radius normalization $\Rightarrow \lambda_1$ maximal for the disk) *Suppose $F : \mathbb{D} \to \Omega$ is a conformal map of the unit disk onto a Jordan–Lipschitz domain Ω. If $\alpha > 0$, then the scale invariant quantity*

$$\lambda_1\big(\Omega; \alpha/L(\Omega)\big)\,|F'(0)|^2$$

is maximal if and only if F is linear and Ω is a disk.

By letting $\alpha \to \infty$, one recovers the result of Pólya and Szegő [28, §5.8] that the first Dirichlet eigenvalue normalized by conformal mapping radius, $\lambda_1^{\mathrm{Dir}}|F'(0)|^2$, is maximal for the disk.

3 Open Problems and Conjectures

A stronger result than Corollary D is known to hold, namely, that the normalized harmonic means

$$\frac{L}{(\sigma_1^{-1} + \sigma_2^{-1})/2} \qquad \text{and} \qquad \frac{A}{(\mu_1^{-1} + \mu_2^{-1})/2}$$

of the first two Steklov and Neumann eigenvalues are maximal for the disk, among simply-connected domains; see [31, p. 634]. A natural question is whether Theorem B can be strengthened in a similar way to handle the harmonic mean of the Robin eigenvalues λ_2 and λ_3.

Another open problem is to generalize Theorem B to higher dimensions, where convexity might provide a reasonable substitute for simply connectedness. Given a domain Ω in higher dimensions, write V for its volume and S for its surface area. Let $\mathbb{B}$ be the unit ball.

Conjecture 1 (perimeter-volume scaling $\Rightarrow \lambda_2$ maximal for ball) *The ball maximises the scale invariant quantity*

$$\lambda_2(\Omega; \alpha V^{1-2/n}/S)\, V^{2/n}$$

among all convex bounded domains in $\mathbb{R}^n$, when $\alpha \in [-S(\mathbb{B})/V(\mathbb{B})^{1-2/n}, 0]$.

Consequently, $\lambda_2(\Omega;\alpha) \le \lambda_2(B;\alpha)$ for all $\alpha \in [-1/R, 0]$, where $B = B(R)$ is a ball having the same volume as Ω.

Taking $n = 2$ reduces the conjecture back to $\lambda_2(\Omega; \alpha/L)A$, as in Theorem B.

Maximality of the ball among convex domains for the normalized Steklov eigenvalue $\sigma_1 S/V^{1-2/n}$ would follow from Conjecture 1, by arguing as in the plane for Corollary D. In fact, this maximality of the Steklov eigenvalue at the ball, among convex domains, has been proved directly by Bucur *et al.* [5], and one would like to extend their method to the Robin eigenvalue in order to prove Conjecture 1.

Does Theorem E also hold for the second Robin eigenvalue? It does in the limit $\alpha \to \infty$, because Ashbaugh and Benguria [1, §4] proved for the second Dirichlet eigenvalue that $\lambda_2^{\mathrm{Dir}}|F'(0)|^2$ is maximal for the disk. Curiously, this result was not proved by employing conformal mapping to create trial functions for the second eigenvalue.

Instead, they combined their sharp PPW inequality on the ratio of the first two eigenvalues with Pólya and Szegő's bound on the first eigenvalue, using the decomposition

$$\lambda_2^{\mathrm{Dir}}|F'(0)|^2 = \frac{\lambda_2^{\mathrm{Dir}}}{\lambda_1^{\mathrm{Dir}}}\big(\lambda_1^{\mathrm{Dir}}|F'(0)|^2\big),$$

where each factor on the right side is maximal for the disk. In view of this Dirichlet result, it seems natural to conjecture that the second Robin eigenvalue is maximised at the disk.

Conjecture 2 (conformal radius normalization $\Rightarrow \lambda_2$ maximal for the disk) *Suppose $F : \mathbb{D} \to \Omega$ is a conformal map of the unit disk onto a Jordan–Lipschitz domain Ω. If $\alpha > 0$, then the scale invariant quantity*

$$\lambda_2\big(\Omega; \alpha/L(\Omega)\big)|F'(0)|^2$$

is maximal when F is linear and Ω is a disk.

We already discussed the limit $\alpha \to \infty$. At the other extreme, when $\alpha = 0$ the conjecture says $\mu_1|F'(0)|^2$ is maximal when F is linear and Ω is the disk, where μ_1 is the first positive eigenvalue of the Neumann Laplacian. This claim is certainly true, as it follows from Szegő's theorem [30] maximising $\mu_1 A$ for the disk, noting that the ratio $|F'(0)|^2/A = |F'(0)|^2/\int_{\mathbb{D}}|F'(z)|^2\,|dz|^2$ is maximal when F' is constant, that is, when F is linear and Ω is a disk.

Theorem E could perhaps be generalized to cone metrics on the disk and other geometric situations considered in the Dirichlet case by Laugesen and Morpurgo [23].

Eigenvalue Sums

The methods of this paper do not seem to extend to eigenvalue sums of the form $\lambda_1 + \cdots + \lambda_m$, because composition with a conformal map does not preserve L^2-orthogonality of trial functions, while pre-composition with a Möbius transformation of the disk can help only to the extent of a few degrees of freedom.

Composition with a linear transformation, on the other hand, does preserve L^2-orthogonality. That observation has generated a number of sharp upper bounds on sums of Robin and magnetic Robin eigenvalues for domains that are linear images of rotationally symmetric domains, in work by Laugesen *et al.* [22, Theorem 3.2] and Laugesen and Siudeja [24, Theorem 3.3], [25, Theorem 3], with generalizations to starlike domains as well [26, Theorem 3.5]. The Robin parameter in these results is scaled by various geometric factors of the domain such as its moment of inertia [24, Lemma 5.3], and thus the scaling is more complicated than the perimeter factor used in this paper.

The methods of this paper also do not appear to extend to reciprocal sums of the form $1/\lambda_1 + \cdots + 1/\lambda_m$ or to spectral zeta functions, because the numerator $\int_\Omega |\nabla u|^2\,dx + \alpha\int_{\partial\Omega} u^2\,ds$ of the Robin Rayleigh quotient is not conformally invariant.

Lower Bounds—Literature and Discussion

To complete the context for this paper's upper bounds on eigenvalues, we mention the Faber–Krahn type lower bound on the first eigenvalue, $\lambda_1(\Omega;\alpha)A(\Omega) \geq \lambda_1(D;\alpha) A(D)$, proved for $\alpha > 0$ by Bossel [2] and extended to the n-dimensional case by Daners [9]. An alternative approach via the calculus of variations was found more recently by Bucur and Giacomini [7, 8], with a quantitative version by Bucur *et al.* [6]. Among the family of rectangles of given area, the square is the minimizer [12, Theorem 4.1], as shown by Keady and Wiwatanapataphee [19] by using appealing convexity arguments (see also [12, Theorem 4.1]). Many more results for rectangles, and conjectures for general domains, are presented by Laugesen [21].

For the reverse inequality when $\alpha < 0$, which is known as the Bareket conjecture, a great deal is now known for domains near the disk by Ferone *et al.* [11], and for general domains when $|\alpha|$ is small by Freitas and Krejčiřík [13], while annular counterexamples have been discovered for large $|\alpha|$. References and a fuller discussion are provided in our earlier paper [14, §1].

For a lower bound on the second eigenvalue, Kennedy [20] observed that Krahn's two-disk argument for the Dirichlet Laplacian carries across to the Robin case as a corollary of Bossel's inequality for the first eigenvalue. For more on spectral shape optimization, we recommend the survey volume edited by Henrot [17].

4 Proof of Theorem A

Substituting the constant trial function $u(x) \equiv 1$ into the Rayleigh quotient gives the upper bound

$$\lambda_1(\Omega;\alpha/L)A \leq \frac{0 + (\alpha/L)\int_{\partial\Omega} 1^2\,ds}{\int_\Omega 1^2\,dx} A = \alpha.$$

We show that this inequality must be strict. If equality held, then the constant trial function u would be a first eigenfunction, and so $\lambda_1(\Omega;\alpha/L)u = -\Delta u = 0$, which means $\lambda_1(\Omega;\alpha/L) = 0$. From equality holding we would deduce $\alpha = 0$, contradicting a hypothesis in the theorem. Hence, equality cannot hold and the inequality is strict.

To show equality is attained asymptotically for rectangles degenerating to a line segment, consider the family of rectangles Ω_t having side lengths t and $1/t$, area $A(t) = 1$, and perimeter $L(t) = 2(t + t^{-1})$, where $t \geq 1$. By separation of variables and using a known lower bound on the first eigenvalue of an interval [12, Appendix A.1], one gets for fixed $\alpha > 0$ that

$$\lambda_1\big(\Omega_t;\alpha/L(t)\big)A(t) \geq \alpha - O_\alpha(t^{-2}) \qquad \text{as } t \to \infty.$$

Hence,

$$\lambda_1\big(\Omega_t;\alpha/L(t)\big)A(t) \to \alpha \qquad \text{as } t \to \infty, \tag{4.1}$$

and so equality is attained asymptotically in the theorem.

The argument is similar when $\alpha < 0$, by using hyperbolic trigonometric instead of trigonometric functions for the separated eigenfunctions. ∎

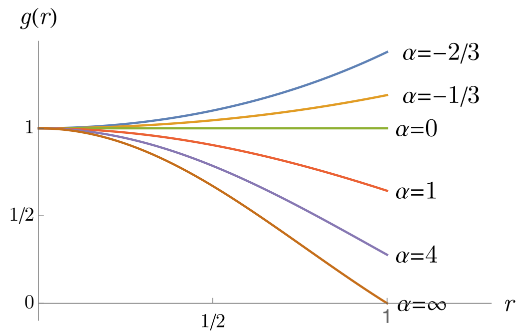

Figure 1: Plot of the first Robin eigenfunction $g(r)$ of the unit disk, for various values of α, normalized with $g(0) = 1$. When $\alpha = 0$, one sees $g(r)$ is the constant Neumann eigenfunction with eigenvalue 0, and when $\alpha = \infty$ it is the Dirichlet eigenfunction $J_0(j_{0,1}r)$ with eigenvalue $j_{0,1}^2$. Between these extremes, $g(r) = J_0(\sqrt{\lambda_1}\, r)$ where $\lambda_1 = \lambda_1(\mathbb{D};\alpha) > 0$ is the eigenvalue.

5 The Robin Spectrum on the Disk

The proof of Theorem B will require some properties of the Robin eigenvalues and eigenfunctions on the unit disk $\mathbb{D}$. Separating variables in the Robin eigenvalue problem (1.1) with $u(r,\theta) = g(r)T(\theta)$ implies that the angular part satisfies $T''(\theta) + \kappa^2 T(\theta) = 0$, where $\kappa \geq 0$ is an integer. When $\kappa = 0$ (giving a constant function T) the eigenfunctions on the disk are purely radial. For positive values of κ the angular function $T(\theta)$ equals $\cos\kappa\theta$ or $\sin\kappa\theta$, and the eigenvalues have multiplicity 2.

The radial part g satisfies the Bessel-type equation

$$g''(r) + \frac{1}{r}g'(r) + \Big(\lambda - \frac{\kappa^2}{r^2}\Big)g(r) = 0$$

due to the eigenfunction equation $-\Delta u = \lambda u$, while the boundary condition

$$\frac{\partial u}{\partial \nu} + \alpha u = 0$$

at $r = 1$ implies $g'(1) + \alpha g(1) = 0$. The key facts about the first and second eigenvalues and eigenfunctions are summarized in the next propositions and in Figure 1 and Figure 2, which are taken from [14, Section 5], where the ball was handled in all dimensions. The spectral curves for the disk are illustrated in [14, Figure 3].

For simplicity, since the domain is fixed in this section, we do not rescale α by the perimeter 2π of the disk. Thus, the range $\alpha \in [-2\pi, 2\pi]$ in Theorem B corresponds here to $\alpha \in [-1,1]$.

Proposition 3 (First Robin eigenfunction of the disk) *The first eigenvalue of* $\mathbb{D}$ *is simple, and changes sign at* $\alpha = 0$ *according to*

$$\lambda_1(\mathbb{D};\alpha)\begin{cases} < 0 \text{ when } \alpha < 0,\\ = 0 \text{ when } \alpha = 0,\\ > 0 \text{ when } \alpha > 0.\end{cases}$$

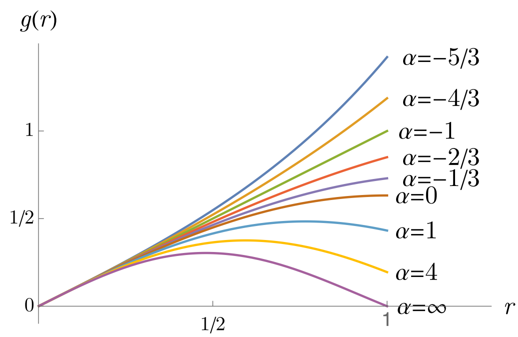

Figure 2: Plot of the radial part $g(r)$ of the second Robin eigenfunction of the unit disk, for various values of α, normalized with $g'(0) = 1$. When $\alpha = -1$, it is the straight line $g(r) = r$ and $\lambda_2(\mathbb{D}; -1) = 0$. When $\alpha > -1$, one has $g(r) = (\text{const.})J_1(\sqrt{\lambda_2}\, r)$ where $\lambda_2 = \lambda_2(\mathbb{D}; \alpha) > 0$ is the eigenvalue. The eigenfunctions are $g(r)\cos\theta$ and $g(r)\sin\theta$.

The first eigenfunction is radial ($\kappa = 0$), with $g(0) > 0$ and $g'(0) = 0$. If $\alpha < 0$, then $g'(r) > 0$; if $\alpha = 0$, then $g'(r) = 0$; and if $\alpha > 0$, then $g'(r) < 0$, when $r \in (0,1)$.

Proposition 4 (Second Robin eigenfunctions of the disk) *The eigenfunctions for the double eigenvalue $\lambda_2(\mathbb{D}; \alpha) = \lambda_3(\mathbb{D}; \alpha)$ have angular dependence ($\kappa = 1$), meaning they take the form*

$$g(r)\cos\theta \qquad \text{and} \qquad g(r)\sin\theta.$$

The radial part has $g(0) = 0, g'(0) > 0, g(r) > 0$ for $r \in (0,1)$, and $g(1) > 0$. When $\alpha \le 0$, one finds $g(r)$ is strictly increasing, with $g'(r) > 0$. When $\alpha > 0$, the derivative g' is positive on some interval $(0, r_\alpha)$ and negative on $(r_\alpha, 1)$, for some number $r_\alpha \in (0,1)$.

The eigenvalue changes sign at $\alpha = -1$, with

$$\lambda_2(\mathbb{D}; \alpha) = \lambda_3(\mathbb{D}; \alpha) \begin{cases} < 0 \text{ when } \alpha < -1, \\ = 0 \text{ when } \alpha = -1, \\ > 0 \text{ when } \alpha > -1. \end{cases}$$

A couple of the assertions in Proposition 4 when $\alpha > 0$ are not included in [14, Section 5], and so we justify them here. The radial part of the second Robin eigenfunction is $g(r) = (\text{const.})J_1(\sqrt{\lambda_2(\mathbb{D}; \alpha)}\, r)$. As α increases from 0 to ∞, the eigenvalue increases from the second Neumann eigenvalue to the second Dirichlet eigenvalue of the unit disk, and so $j'_{1,1} < \sqrt{\lambda_2(\mathbb{D}; \alpha)} < j_{1,1}$. (Numerically, $j'_{1,1} \simeq 1.84$ and $j_{1,1} \sim 3.83$.) The Bessel function J_1 vanishes at 0 and at $j_{1,1}$, and has positive derivative on $(0, j'_{1,1})$ and negative derivative on $(j'_{1,1}, j_{1,1})$. Hence, $g(1) > 0$, and g' is positive on the interval $(0, r_\alpha)$ and negative on $(r_\alpha, 1)$, where the number $r_\alpha = j'_{1,1}/\sqrt{\lambda_2(\mathbb{D}; \alpha)}$ lies between 0 and 1.

6 Center of Mass Argument

In this section, Ω is a simply-connected planar domain, and $g(r)$ is a continuous function for $0 \le r \le 1$ with $0 = g(0) < g(1)$. Define continuous functions

$$u_2 = g(r)\cos\theta \quad \text{and} \quad u_3 = g(r)\sin\theta,$$

on the unit disk $\mathbb{D}$. The following center of mass result will be used in proving Theorem B.

Lemma 5 (Center of mass) *If v_1 is an integrable real-valued function on Ω with $\int_\Omega v_1\,dx > 0$, then a conformal map f to $\mathbb{D} \to \Omega$ can be chosen such that the functions $v_2 = u_2 \circ f^{-1}$ and $v_3 = u_3 \circ f^{-1}$ are orthogonal to v_1:*

$$\int_\Omega v_2 v_1\,dx = 0 \qquad \text{and} \qquad \int_\Omega v_3 v_1\,dx = 0.$$

Szegő [30, Section 2.5] treated the case $v_1 \equiv 1$ by an approximate identity argument and elementary index theory. Hersch [18] reformulated the argument more geometrically, avoiding Szegő's use of approximate identities. For the sake of completeness, we include a version of Hersch's proof below.

Proof of Lemma 5 Fix a conformal map $F : \mathbb{D} \to \Omega$, and let $H(z) = g(r)e^{i\theta}$ where $z = re^{i\theta} \in \overline{\mathbb{D}}$. Note that H is continuous on the closed disk, including at the origin, since $g(0) = 0$. Define a complex-valued function (vector field) on the disk by

$$V(\zeta) = \int_\Omega H\big(M_\zeta\big(F^{-1}(x)\big)\big)\,v_1(x)\,dx, \qquad \zeta \in \mathbb{D},$$

where

$$M_\zeta(z) = \frac{z+\zeta}{1+z\overline{\zeta}}, \qquad z \in \mathbb{D},$$

is a Möbius map of the unit disk $\mathbb{D}$ to itself.

Notice that $M_\zeta(z)$ remains continuous as a function of $(\zeta, z) \in \overline{\mathbb{D}} \times \mathbb{D}$ (where now we allow $|\zeta| = 1$), taking values in $\overline{\mathbb{D}}$. Thus, the vector field $V(\zeta)$ is well defined for $\zeta \in \overline{\mathbb{D}}$, and is continuous at each point by a simple application of dominated convergence, using continuity and boundedness of H. The boundary behavior is easily determined: when $\zeta = e^{i\phi}$, one has $M_\zeta(z) = e^{i\phi}$ for all $z \in \mathbb{D}$, and so

$$V(e^{i\phi}) = g(1)e^{i\phi}\int_\Omega v_1\,dx, \qquad \phi \in [0, 2\pi].$$

Thus, the continuous vector field V points radially outward on the unit circle, because $g(1)\int_\Omega v_1\,dx > 0$ by construction.

Index theory, or the Brouwer fixed point theorem, implies that V vanishes somewhere in the interior of the disk. That is, $V(\zeta) = 0$ for some $\zeta \in \mathbb{D}$, which means $H \circ f^{-1}$ is orthogonal to v_1, where $f = F \circ M_\zeta^{-1}$. Because $H = u_2 + iu_3$ by definition, we conclude that $u_2 \circ f^{-1}$ and $u_3 \circ f^{-1}$ are orthogonal to v_1. ∎

7 Proof of Theorem B

After rescaling, we can suppose Ω has area π, so that D is the unit disk $\mathbb{D}$. Our goal is to show

$$\lambda_2\big(\Omega; \alpha/L(\Omega)\big) \le \lambda_2\big(\mathbb{D}; \alpha/2\pi\big), \qquad \alpha \in [-2\pi, 2\pi].$$

Note on the right side that $\lambda_2\big(\mathbb{D}; \alpha/2\pi\big) \ge 0$ by Proposition 4, since $\alpha/2\pi \ge -1$.

Let u_2 and u_3 be the second Robin eigenfunctions of the unit disk with Robin parameter $\alpha/2\pi$, which from Proposition 4 have the form

$$u_2 = g(r)\cos\theta \quad \text{and} \quad u_3 = g(r)\sin\theta,$$

where g is smooth with $0 = g(0) < g(1)$. Take a conformal map f from $\mathbb{D}$ onto Ω, and define

$$v_2 = u_2 \circ f^{-1} \qquad \text{and} \qquad v_3 = u_3 \circ f^{-1}.$$

These functions belong to $H^1(\Omega)$, because they are bounded and smooth with

$$\int_\Omega |\nabla v_k|^2\,dx = \int_{\mathbb{D}} |\nabla u_k|^2\,dx < \infty, \qquad k = 2, 3, \tag{7.1}$$

by conformal invariance of the Dirichlet integral. Note v_2 and v_3 extend continuously to $\partial\Omega$, since f^{-1} extends continuously (using that $\partial\Omega$ is a Jordan curve).

The conformal map can be chosen by Lemma 5 to ensure the orthogonality relations

$$\int_\Omega v_2 v_1\,dx = 0 \qquad \text{and} \qquad \int_\Omega v_3 v_1\,dx = 0,$$

where v_1 is the first Robin eigenfunction on Ω for Robin parameter $\alpha/L(\Omega)$; note here that v_1 does not change sign, and so we can assume its integral is positive. Thus v_2 and v_3 are valid trial functions for $\lambda_2(\Omega;\alpha/L(\Omega))$. Taking v_2 as a trial function in the Rayleigh principle for the second eigenvalue shows that

$$\lambda_2\big(\Omega;\alpha/L(\Omega)\big)\int_\Omega v_2^2\,dx \le \int_\Omega |\nabla v_2|^2\,dx + \frac{\alpha}{L(\Omega)}\int_{\partial\Omega} v_2^2\,ds.$$

This formula pulls back under the conformal map to

$$\lambda_2\big(\Omega;\alpha/L(\Omega)\big)\int_{\mathbb{D}} u_2^2|f'|^2\,dx \le \int_{\mathbb{D}} |\nabla u_2|^2\,dx + \frac{\alpha}{L(\Omega)}\int_{\partial\Omega} v_2^2\,ds,$$

due to the conformal invariance in (7.1). Substituting the definition $u_2 = g(r)\cos\theta$ gives

$$\lambda_2\big(\Omega;\alpha/L(\Omega)\big)\int_{\mathbb{D}} g(r)^2(\cos\theta)^2|f'|^2\,dx \le \int_{\mathbb{D}} \big(g'(r)^2\cos^2\theta + r^{-2}g(r)^2\sin^2\theta\big)\,dx + \frac{\alpha}{L(\Omega)}\int_{\partial\Omega} v_2^2\,ds.$$

An analogous formula holds for u_3, with the roles of cos and sin interchanged. Adding that formula to the preceding one and using that $\cos^2\theta + \sin^2\theta = 1$ and hence $v_2^2 + v_3^2 = g(1)^2$ on $\partial\Omega$, we deduce

$$\lambda_2\big(\Omega;\alpha/L(\Omega)\big)\int_{\mathbb{D}} g(r)^2|f'|^2\,dx \le \int_{\mathbb{D}} \big(g'(r)^2 + r^{-2}g(r)^2\big)\,dx + \alpha g(1)^2. \tag{7.2}$$

Equality holds if Ω is the unit disk and f is the identity map, since u_2 and u_3 are the second eigenfunctions of the disk, which means

$$\lambda_2(\mathbb{D};\alpha/2\pi)\int_{\mathbb{D}} g(r)^2\,dx = \int_{\mathbb{D}} \big(g'(r)^2 + r^{-2}g(r)^2\big)\,dx + \alpha g(1)^2. \tag{7.3}$$

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Combining (7.2) and (7.3) gives that

$$\lambda_2\big(\Omega;\alpha/L(\Omega)\big)\int_{\mathbb{D}} g(r)^2|f'|^2\,dx \le \lambda_2(\mathbb{D};\alpha/2\pi)\int_{\mathbb{D}} g(r)^2\,dx. \tag{7.4}$$

To make further progress, we will compare the integrals of g^2 on the left and right.

Suppose Ω is not a disk. We will show when $\alpha \le 2\pi$ that

$$\int_{\mathbb{D}} g(r)^2\,dx < \int_{\mathbb{D}} g(r)^2|f'|^2\,dx. \tag{7.5}$$

Szegő proved inequality (7.5) under the assumption that g is increasing, which in our Robin situation holds when $\alpha \le 0$. We will extend his method to handle $\alpha \le 2\pi$. To start with,

$$\begin{aligned}\int_{\mathbb{D}} g(r)^2|f'|^2\,dx - \int_{\mathbb{D}} g(r)^2\,dx &= -\int_0^1 g(r)^2\frac{d}{dr}\Big(\pi r^2 - \int_{\mathbb{D}(r)}|f'|^2\,dx\Big)\,dr\\ &= \int_0^1 2g(r)g'(r)\Big(\pi r^2 - \int_{\mathbb{D}(r)}|f'|^2\,dx\Big)\,dr\end{aligned}$$

by integration by parts, noting that the boundary terms vanish, because

$$\pi = A(\Omega) = \int_{\mathbb{D}}|f'|^2\,dx. \tag{7.6}$$

Hence,

$$\int_{\mathbb{D}} g(r)^2|f'|^2\,dx - \int_{\mathbb{D}} g(r)^2\,dx = \int_0^1 2g(r)g'(r)\pi r^2\big(1-M(r)\big)\,dr, \tag{7.7}$$

where

$$M(r) = \frac{1}{\pi r^2}\int_{\mathbb{D}(r)}|f'|^2\,dx$$

is the mean value function. The mean value is increasing due to subharmonicity of $|f'|^2$. More directly, one can write $f(z) = \sum_{n=0}^\infty a_n z^n$ as a power series and substitute into $M(r)$ to obtain

$$M(r) = \frac{1}{\pi r^2}\int_{\mathbb{D}(r)}|f'(\rho e^{i\theta})|^2\,\rho\,d\rho d\theta = \sum_{n=1}^\infty n|a_n|^2 r^{2(n-1)}, \tag{7.8}$$

which plainly increases as a function of r. Further, since Ω is not a disk we have $f(z) \not\equiv a_0 + a_1 z$, and so $a_n \ne 0$ for some $n \ge 2$, which implies by (7.8) that $M(r)$ is strictly increasing as a function of r.

The area normalization (7.6) gives $M(1) = 1$, and so $M(r) < 1$ for $r \in (0,1)$.

If $\alpha \le 0$, then g and g' are both positive on $(0,1)$ by Proposition 4, and so inequality (7.5) follows from (7.7).

Next assume $0 < \alpha \le 2\pi$. Define

$$G(r) = \int_0^r 2g(\rho)g'(\rho)\pi\rho^2\,d\rho, \qquad r \in [0,1].$$

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Formula (7.7) becomes

$$\int_{\mathbb{D}} g(r)^2 |f'|^2 \, dx - \int_{\mathbb{D}} g(r)^2 \, dx = \int_0^1 G'(r)\big(1 - M(r)\big) \, dr$$
$$= \int_0^1 G(r) M'(r) \, dr$$

after integrating by parts, since $G(0) = 0$ and $M(1) = 1$. We want this last integral to be positive, so that (7.5) holds. Because $M'(r) > 0$, it suffices to show that $G(r) > 0$ for $r \in (0,1)$. Recall from Proposition 4 that when $\alpha > 0$, the function g is positive on $(0,1)$, while g' is positive on some interval $(0, r_\alpha)$ and negative on $(r_\alpha, 1)$. Thus, G' is positive on $(0, r_\alpha)$ and negative on $(r_\alpha, 1)$, and so to show $G(r)$ is positive for $r \in (0,1)$, we need only show that $G(1) \geq 0$. The next lemma shows that indeed $G(1) \geq 0$, and so inequality (7.5) is proved when $0 < \alpha \leq 2\pi$.

Lemma 6 *If $0 < \alpha \leq 2\pi$ and g is the radial part of the eigenfunction for $\lambda_2(\mathbb{D}; \alpha/2\pi)$, then*

$$\int_0^1 2g(r)g'(r)\pi r^2 \, dr \geq 0.$$

Proof By section 5, one has $g(r) = J_1(\sqrt{\lambda_2} r)$, where $\lambda_2 = \lambda_2(\mathbb{D}; \alpha/2\pi) > 0$. Applying this formula for g and making a change of variable, we are reduced to showing that

$$\int_0^{\sqrt{\lambda_2}} 2J_1(r)J_1'(r) \, r^2 \, dr \geq 0.$$

The antiderivative for the left side is $J_0(r)J_2(r)r^2$, as one can check using standard Bessel formulas [27, Eq. (10.6.1) and (10.6.2)]. Hence, the inequality to be proved is

$$J_0(\sqrt{\lambda_2})J_2(\sqrt{\lambda_2})\lambda_2 \geq 0.$$

Note that the second Robin eigenvalue λ_2 of the disk is less than the second Dirichlet eigenvalue $j_{1,1}^2$ of the disk, which in turn is less than $j_{2,1}^2$. Therefore, $J_2(\sqrt{\lambda_2}) > 0$, and so the last displayed inequality holds if and only if $J_0(\sqrt{\lambda_2}) \geq 0$. Thus, we want to show $\sqrt{\lambda_2} \leq j_{0,1}$, or $\lambda_2(\mathbb{D}; \alpha/2\pi) \leq j_{0,1}^2$.

The Robin eigenvalue increases with α, and since $\alpha \leq 2\pi$ by hypothesis, it suffices to take $\alpha = 2\pi$ and show $\lambda_2(\mathbb{D}; 1) \leq j_{0,1}^2$. For this, observe that $u = J_1(j_{0,1} r)\cos\theta$ is a nonradial eigenfunction of the Laplacian on the unit disk with eigenvalue $j_{0,1}^2$. We confirm that u satisfies the Robin boundary condition with $\alpha = 1$, namely, $\partial u/\partial \nu + u = 0$ at $r = 1$, by computing

$$j_{0,1}J_1'(j_{0,1}) + J_1(j_{0,1}) = -j_{0,1}J_0''(j_{0,1}) - J_0'(j_{0,1}) = 0,$$

where we used the relation $J_1 = -J_0'$ and the Bessel equation $r^2 J_0''(r) + rJ_0'(r) + r^2 J_0(r) = 0$. Since u is nonradial it is not the first eigenfunction, and so $\lambda_k(\mathbb{D}; 1) = j_{0,1}^2$ for some $k \geq 2$, which implies $\lambda_2(\mathbb{D}; 1) \leq j_{0,1}^2$, as needed. ∎

To complete the proof of Theorem B, we divide into two cases.

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Case (i): $-2\pi < \alpha \leq 2\pi$. The positivity of $\lambda_2(\mathbb{D};\alpha/2\pi)$ when $\alpha > -2\pi$ enables us to use inequality (7.5) on the right side of (7.4), concluding that $\lambda_2\big(\Omega;\alpha/L(\Omega)\big) < \lambda_2\big(\mathbb{D};\alpha/2\pi\big)$ when Ω is not a disk. That proves the theorem.

Case (ii): $\alpha = -2\pi$. In this case $\lambda_2\big(\mathbb{D};\alpha/2\pi\big) = \lambda_2(\mathbb{D};-1) = 0$ and so (7.4) gives that $\lambda_2\big(\Omega;-2\pi/L(\Omega)\big) \leq 0 = \lambda_2(\mathbb{D};-1)$, which is the inequality in the theorem. Suppose now that equality holds, so that

$$\lambda_2\big(\Omega;-2\pi/L(\Omega)\big) = 0.$$

The remaining task is to show Ω is a disk.

This part of the argument follows Weinstock's equality case [32, §3]. He assumed the boundary of Ω to be analytic, whereas we assume only Lipschitz smoothness. We invoke subtle results from complex analysis to ensure that the harmonic function $\log|f'|$ equals the Poisson integral of its boundary values. The need for such care in the Lipschitz case may not have been recognized in earlier treatments [15, Theorem 1.3].

Suppose equality holds above, meaning $\lambda_2(\mathbb{D};\alpha/2\pi) = 0$. Then $\alpha = -2\pi$ and the eigenfunctions u_2 and u_3 for the disk are the coordinate functions x_1 and x_2, by the case "$\alpha = -1$" in Proposition 4, with $g(r) = r$.

Equality holds in (7.2), because both sides of the inequality equal 0. By the Rayleigh principle, the trial functions v_2 and v_3 used to derive (7.2) must therefore be eigenfunctions on Ω with eigenvalue 0. Thus, v_2 and v_3 satisfy the (weak form of) the Robin boundary condition, which we proceed to investigate.

Since $\partial\Omega$ is a rectifiable Jordan curve, the derivative f' of the conformal map belongs to the analytic Hardy space and the boundary values $f'(e^{i\theta})$ provide the Jacobian factor for arclength (see [10, Theorem 3.12] and remarks following it). That is, $ds = |f'(e^{i\theta})|\,d\theta$ where ds denotes the arclength element on $\partial\Omega$. We will show that $|f'(e^{i\theta})|$ is constant a.e.

The weak formulation of the eigenfunction equation for v_2 on Ω, with $\alpha = -2\pi$ and eigenvalue 0 as above, says

$$\int_\Omega \nabla v_2 \cdot \nabla\psi\,dx - \frac{2\pi}{L(\Omega)}\int_{\partial\Omega} v_2\psi\,ds = 0, \qquad \psi \in H^1(\Omega).$$

Pulling back to $\mathbb{D}$, we deduce by conformal invariance that

$$\int_{\mathbb{D}} \nabla u_2 \cdot \nabla\phi\,dx = \frac{2\pi}{L(\Omega)}\int_{\partial\mathbb{D}} u_2\phi|f'|\,d\theta, \quad \phi \in C^\infty(\overline{\mathbb{D}}),$$

where we note that $\psi = \phi\circ f^{-1}$ belongs to $H^1(\Omega)$. Recall $u_2 = x_1 = r\cos\theta$. By applying Green's theorem on the left side of the last equation, we find

$$\int_0^{2\pi}(\cos\theta)\phi(e^{i\theta})\,d\theta = \frac{2\pi}{L(\Omega)}\int_0^{2\pi}(\cos\theta)\phi(e^{i\theta})|f'(e^{i\theta})|\,d\theta, \quad \phi \in C^\infty(\overline{\mathbb{D}}).$$

Since ϕ is arbitrary, it follows that

$$\cos\theta = \frac{2\pi}{L(\Omega)}(\cos\theta)|f'(e^{i\theta})|$$

for almost every θ, which means $|f'(e^{i\theta})| = L(\Omega)/2\pi$ a.e. Thus, $|f'|$ is constant a.e. on the unit circle.

We will show $|f'|$ is constant on the unit disk. We start by proving the Jordan curve $J = \partial\Omega$ has the chord–arc property, meaning

$$\text{length}\big(J(x,y)\big) \le C|x-y|, \qquad x,y \in J,$$

for some constant C, where $J(x,y)$ is the shorter arc of J between x and y. Suppose the chord-arc property fails. By considering $C = 1,2,3,\dots$, one constructs sequences $x_n, y_n \in J$ such that

$$\text{length}\big(J(x_n,y_n)\big) > n|x_n - y_n|. \tag{7.9}$$

Notice $|x_n - y_n| \to 0$, since the length of $J(x_n,y_n)$ is bounded by the length of J, which is finite. Further, by compactness we can assume the sequences x_n and y_n converge to some point $x \in J$. The domain Ω has Lipschitz boundary by hypothesis, and so the curve J can be represented near x as the graph of a Lipschitz function. That is, after suitably rotating the coordinate system, there is a disk B centered at x and a Lipschitz function $b : \mathbb{R} \to \mathbb{R}$ such that $B \cap J = B \cap \{(t, b(t)) : t \in \mathbb{R}\}$. Let β be the Lipschitz constant. For all n large enough that the points x_n and y_n lie in the disk B, choose s_n and t_n such that $x_n = (s_n, b(s_n))$ and $y_n = (t_n, b(t_n))$. Then

$$\text{length}\big(J(x_n,y_n)\big) \le \sqrt{1+\beta^2}\,|s_n - t_n| \le \sqrt{1+\beta^2}\,|x_n - y_n|,$$

which contradicts (7.9) as $n \to \infty$. Therefore, J must satisfy the chord–arc property.

The chord–arc property of $\partial\Omega$ implies that Ω is Ahlfors-regular [29, Proposition 7.7], and hence the conformal map f satisfies the Smirnov condition [29, Proposition 7.5 and Theorem 7.6], which says that on $\mathbb{D}$ the harmonic function $\log|f'|$ equals the Poisson integral of its boundary values. Its boundary values are constant a.e., by our work above, and so $\log|f'|$ is constant on $\mathbb{D}$. Thus, $|f'|$ is constant, and so f' is constant, which means f is linear and Ω is a disk, as we wanted to show. ∎

Next we justify the claim made earlier in the paper that Theorem B fails when $\alpha > 32.7$. Specifically, we show that

$$\lambda_2\big(\mathbb{D}; \alpha/L(\mathbb{D})\big)A(\mathbb{D}) < \alpha \qquad \text{when } \alpha > 32.7, \tag{7.10}$$

so that by (4.1) the disk $\mathbb{D}$ gives a smaller value than a long thin rectangle Ω_t, for large t, and hence the disk is not the maximizer.

To prove (7.10), recall from section 5 (see Figure 2) that the second eigenfunction of the disk with positive Robin parameter $\alpha/L(\mathbb{D}) = \alpha/2\pi$ has radial part $g(r) = J_1(\sqrt{\lambda_2}\,r)$ where $\sqrt{\lambda_2} \in (j'_{1,1}, j_{1,1})$ is chosen to satisfy the Robin boundary condition $g'(1) + (\alpha/2\pi)g(1) = 0$. That condition can be rearranged to say

$$\alpha = -2\pi\frac{\sqrt{\lambda_2}J_1'(\sqrt{\lambda_2})}{J_1(\sqrt{\lambda_2})}. \tag{7.11}$$

Since λ_2 is a strictly increasing function of α, we can invert and regard α as a function of λ_2 (see [14, Section 5] with $n = 2$ and $\kappa = 1$). By the last formula, the condition $\alpha > \lambda_2\pi$ in (7.10) is equivalent to

$$-2\frac{J_1'(x)}{xJ_1(x)} > 1,$$

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where $x = \sqrt{\lambda_2} \in (j'_{1,1}, j_{1,1})$. Solving numerically, the inequality holds for $3.2261 \le x < j_{1,1}$ where we rounded the root up to 3.2261. Substituting this root into (7.11) and again rounding up, we obtain a range $32.7 \le \alpha < \infty$ on which (7.10) holds.

8 Proof of Corollary C

We can assume Ω has area π, after rescaling, and so the task is to show $\lambda_2(\Omega;\alpha) \le \lambda_2(\mathbb{D};\alpha)$ when $\alpha \in [-1,0]$.

The isoperimetric inequality $L(\Omega) \ge 2\pi$ implies $\alpha \le 2\pi\alpha/L(\Omega)$ when $\alpha \le 0$, and so

$$\lambda_2(\Omega;\alpha) \le \lambda_2\big(\Omega; 2\pi\alpha/L(\Omega)\big),$$

because the Robin eigenvalues are increasing functions of α. The assumption $\alpha \in [-1,0]$ ensures $2\pi\alpha \in [-2\pi, 0]$, and so Theorem B can be applied with α replaced by $2\pi\alpha$, giving

$$\lambda_2\big(\Omega; 2\pi\alpha/L(\Omega)\big) \le \lambda_2\big(\mathbb{D}; 2\pi\alpha/L(\mathbb{D})\big) = \lambda_2(\mathbb{D};\alpha).$$

Combining the last two inequalities proves the corollary.

If equality holds, then Ω must be a disk, by the equality statement in Theorem B.

9 Proof of Corollary D

That $\mu_1(\Omega)A(\Omega)$ is maximal for the disk, under area normalization, is the case $\alpha = 0$ of Theorem B.

Weinstock's result, saying the disk maximises the first nontrivial Steklov eigenvalue under perimeter normalization, requires a little more explanation. The Steklov spectrum of the Laplacian is denoted $0 = \sigma_0 < \sigma_1 \le \sigma_2 \le \cdots$, where the eigenvalue problem is

$$\begin{aligned} \Delta u &= 0 \quad &&\text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= \sigma u \quad &&\text{on } \partial\Omega. \end{aligned}$$

Thus, σ belongs to the Steklov spectrum exactly when 0 belongs to the Robin spectrum with $\alpha = -\sigma$.

After rescaling Ω we can suppose it has area π. The task is to prove $\sigma_1(\Omega)L(\Omega) \le 2\pi$, since $\sigma_1(\mathbb{D}) = 1$. Choosing $\alpha = -2\pi$ in Theorem B yields

$$\lambda_2\big(\Omega; -2\pi/L(\Omega)\big) \le \lambda_2\big(\mathbb{D}; -2\pi/L(\mathbb{D})\big) = \lambda_2(\mathbb{D}; -1) = 0.$$

Also, $\lambda_2(\Omega;0) = \mu_1(\Omega) > 0$. Since the Robin eigenvalues vary continuously with α, a value $\widetilde{\alpha} \in [-2\pi, 0)$ must exist for which $\lambda_2\big(\Omega; \widetilde{\alpha}/L(\Omega)\big) = 0$. Choose $\widetilde{\alpha}$ to be the greatest such number, so that $\lambda_2\big(\Omega; \alpha/L(\Omega)\big) > 0$ for all $\alpha > \widetilde{\alpha}$. Then $-\widetilde{\alpha}/L(\Omega)$ belongs to the Steklov spectrum of Ω, and is in fact the smallest positive Steklov eigenvalue, $\sigma_1(\Omega)$. Hence, $\sigma_1(\Omega)L(\Omega) = -\widetilde{\alpha} \le 2\pi$, as we needed to show.

If equality holds then $\widetilde{\alpha} = -2\pi$, and so the equality statement in Theorem B (with $\alpha = -2\pi$) implies that Ω is a disk.

10 Proof of Theorem E

Fix $\alpha > 0$. The Robin eigenfunction on the disk corresponding to $\lambda_1(\mathbb{D}; \alpha/2\pi)$ has radial form $u_1 = g(r)$, by Proposition 3. Adapting Pólya and Szegő's method [28], we define $v_1 = u_1 \circ F^{-1}$ on Ω. This function is smooth and bounded, and belongs to $H^1(\Omega)$ by conformal invariance of the Dirichlet integral. Employing v_1 as a trial function in the Rayleigh principle for the first eigenvalue yields

$$\lambda_1\big(\Omega; \alpha/L(\Omega)\big) \int_\Omega v_1^2 \, dx \le \int_\Omega |\nabla v_1|^2 \, dx + \frac{\alpha}{L(\Omega)} \int_{\partial\Omega} v_1^2 \, ds,$$

which pulls back under the conformal map F to

$$\lambda_1\big(\Omega; \alpha/L(\Omega)\big) \int_{\mathbb{D}} u_1^2 |F'|^2 \, dx \le \int_{\mathbb{D}} |\nabla u_1|^2 \, dx + \frac{\alpha}{L(\Omega)} \int_{\partial\Omega} v_1^2 \, ds,$$

by conformal invariance of the Dirichlet integral.

Substituting $u_1 = g(r)$, which in particular gives $v_1 = g(1)$ on $\partial\Omega$, we obtain

$$\lambda_1\big(\Omega; \alpha/L(\Omega)\big) \int_{\mathbb{D}} g(r)^2 |F'|^2 \, dx \le \int_{\mathbb{D}} g'(r)^2 \, dx + \alpha g(1)^2. \tag{10.1}$$

For the left side of the inequality, note that

$$|F'(0)|^2 \int_{\mathbb{D}} g(r)^2 \, r \, dr d\theta \le \int_{\mathbb{D}} g(r)^2 |F'(re^{i\theta})|^2 \, r \, dr d\theta, \tag{10.2}$$

because

$$|F'(0)|^2 = \Big| \frac{1}{2\pi} \int_0^{2\pi} F'(re^{i\theta}) \, d\theta \Big|^2 \le \frac{1}{2\pi} \int_0^{2\pi} |F'(re^{i\theta})|^2 \, d\theta. \tag{10.3}$$

Multiply inequality (10.2) by $\lambda_1(\Omega; \alpha/L(\Omega))$, which is positive, since $\alpha > 0$, and then substitute into (10.1), getting

$$\lambda_1\big(\Omega; \alpha/L(\Omega)\big) \, |F'(0)|^2 \le \frac{\int_{\mathbb{D}} g'(r)^2 \, dx + (\alpha/2\pi) \int_{\partial\mathbb{D}} g(1)^2 \, ds}{\int_{\mathbb{D}} g(r)^2 \, dx} = \lambda_1(\mathbb{D}; \alpha/2\pi),$$

as we wanted to prove.

If equality holds in the theorem, then equality must hold in (10.2), and hence also in (10.3) for $r \in (0,1)$. By substituting the power series for F into (10.3) and setting the two sides equal, we deduce that F' is constant, and hence F is linear.

Note added in proof. The third Robin eigenvalue of a simply connected planar domain is maximal for the disjoint union of two equal disks (which are approached in a suitable limiting sense), under the same scaling and normalization as imposed in this paper, when $\alpha \in [-4\pi, 0]$. See A. Girouard and R. S. Laugesen, *Robin spectrum: two disks maximize the third eigenvalue*, preprint, https://arxiv.org/abs/1907.13173.

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