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REGULAR WALLMAN COMPACTIFICATIONS OF RIM-COMPACT SPACES

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Abstract

A compact Hausdorff space is regular Wallman if it possesses a separating ring of regular closed sets, an s-ring. It was proved by P. C. Baayen and J. van Mill [*General Topology and Appl.* **9** (1978), 125–129] that if a locally compact Hausdorff space possesses an s-ring, then every Hausdorff compactification with zero-dimensional remainder is regular Wallman.

In this paper the reasoning leading to this result is modified to work in a more general setting. Let αX be a Hausdorff compactification of a space X, and let $\mathcal{C}_\alpha$ be the family of those closed sets in αX whose boundaries are contained in X. A main result is the following: If $\mathcal{C}_\gamma \cap X$ contains an s-ring for some Hausdorff compactification γX, then every larger Hausdorff compactification αX for which $\mathcal{C}_\alpha \cap (\alpha X - X)$ is a base for the closed sets on $\alpha X - X$, is regular Wallman. Various consequences concerning compactifications of a class of rim-compact spaces (called totally rim-compact spaces) are discussed.

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0. Introduction

A compact Hausdorff space is called *regular Wallman* if it possesses a separating ring of regular closed sets, an *s-ring*. Such a space is a Wallman compactification of every dense subspace. (See [11].) In [1] P. C. Baayen and J. van Mill discuss conditions for a Hausdorff compactification of a completely regular space to be regular Wallman. A main result is the following: If a *locally compact* space possesses an s-ring, then every Hausdorff compactification with zero-dimensional remainder is regular Wallman.

In this paper we show how the reasoning leading to this result may be modified to work in a more general setting. A central role is played by the family $\mathcal{C}_\alpha$ of those closed subsets of the compactification αX whose boundaries are contained in X. Among our results is the following: If $\mathcal{C}_\gamma \cap X$ contains an s-ring for *some* Hausdorff compactification γX, then every *larger* Hausdorff compactification αX for which $\mathcal{C}_\alpha \cap (\alpha X - X)$ is a base for the closed sets of $\alpha X - X$, is regular Wallman. Applications of this result are discussed. In particular the result for locally compact spaces quoted above follows as a rather direct consequence.

1. Basic concepts

In the following, X shall always denote a completely regular Hausdorff space. By a compactification we shall always mean a Hausdorff compactification.

Let αX be an arbitrary compactification of X. Let $\mathcal{O}_\alpha$ and $\mathcal{C}_\alpha$ denote the families of open, respectively closed, subsets of αX with boundaries contained in X. We note that a set A in $\mathcal{C}_\alpha$ is a neighbourhood in αX of every point in $A \cap \rho_\alpha X$, where $\rho_\alpha X$ denotes the remainder $\alpha X - X$. We shall say that X is *rim-included* in αX if $\mathcal{O}_\alpha \cap X$ is a base for the open sets in X. (When $X \subset Y$ and $\mathcal{F}$ is a family of subsets of Y, we write $\mathcal{F} \cap X$ for $\{F \cap X \colon F \in \mathcal{F}\}$.) We shall say that $\rho_\alpha X$ is *rim-excluded* in αX if $\mathcal{O}_\alpha \cap \rho_\alpha X$ is a base for the open sets in $\rho_\alpha X$. Clearly if $\rho_\alpha X$ is rim-excluded in αX, then $\rho_\alpha X$ is zero-dimensional (in the sense of small inductive dimension).

We shall write int, cl, ∂ (respectively int_α, cl_α, ∂_α) for interior, closure and boundary in X (respectively in αX).

LEMMA 1. *If $\rho_\alpha X$ is rim-excluded in αX, then $\mathcal{O}_\alpha$ contains a neighbourhood base in αX for every point in $\rho_\alpha X$.*

PROOF. Let $x_0 \in \rho_\alpha X$, $x_0 \in A$, A open in αX, and let W_1 be an open set in αX such that $\alpha X - A \subset W_1$, and $x_0 \notin \mathrm{cl}_\alpha W_1$. By assumption there is a set O in $\mathcal{O}_\alpha$ such that $x_0 \in O$, $\mathrm{cl}_\alpha W_1 \cap \rho_\alpha X \subset (\alpha X - O) \cap \rho_\alpha X = (\alpha X - \mathrm{cl}_\alpha O) \cap \rho_\alpha X$. Clearly the set $W_2 = \alpha X - \mathrm{cl}_\alpha O$ belongs to $\mathcal{O}_\alpha$. Write $U = W_1 \cup W_2$. As $\rho_\alpha X \cap \mathrm{cl}_\alpha W_2 \cap O = \varnothing$ by definition, and $\rho_\alpha X \cap \mathrm{cl}_\alpha W_1 \cap O \subset \rho_\alpha X \cap W_2 \cap O$, we conclude that $\mathrm{cl}_\alpha U \cap \rho_\alpha X \cap O = (\mathrm{cl}_\alpha W_1 \cup \mathrm{cl}_\alpha W_2) \cap \rho_\alpha X \cap O = \varnothing$. Clearly $\partial_\alpha U \cap \rho_\alpha X \cap (\alpha X - O) = \varnothing$, and consequently $\partial_\alpha U \subset X$. It follows that the set $V = \alpha X - \mathrm{cl}_\alpha U$ belongs to $\mathcal{O}_\alpha$. Now $x_0 \in V \subset \alpha X - W_1 \subset A$, from which we conclude that $\mathcal{O}_\alpha$ contains a neighbourhood base for x_0. This completes the proof.

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LEMMA 2. *If X is rim-included in αX, then $\mathcal{O}_\alpha$ contains a neighbourhood base in αX for every point in X.*

Proof. Let $x_0 \in X$, $x_0 \in A$, A open in αX, and let B be a closed set in αX such that $x_0 \notin B$, $\alpha X - A \subset B$, $\text{cl}_\alpha \text{ int}_\alpha B = B$. Let $y \in B$, and let V be an arbitrary neighbourhood of y. Then $V \cap \text{int}_\alpha B \neq \varnothing$, hence $(B \cap X) \cap V \neq \varnothing$, from which follows that $y \in \text{cl}_\alpha(B \cap X)$. Thus $B = \text{cl}_\alpha(B \cap X)$.

The set $U = (\alpha X - B) \cap X$ is a neighbourhood of x_0 in X, and $U \subset A$. By assumption there is a W in $\mathcal{O}_\alpha$ such that $x_0 \in W$, $W \cap X \subset U$. Clearly $W \cap (B \cap X) = \varnothing$, hence also $W \cap \text{cl}_\alpha(B \cap X) = \varnothing$. Thus $W \subset \alpha X - \text{cl}_\alpha(B \cap X) = \alpha X - B \subset A$, which means that $\mathcal{O}_\alpha$ contains a neighbourhood base for x_0. This completes the proof.

We recall that $\rho_\alpha X$ is said to be *zero-dimensionally embedded* in αX if $\mathcal{O}_\alpha$ (resp. $\mathcal{C}_\alpha$) is a base for the open (resp. closed) sets in αX. A countable remainder is always zero-dimensionally embedded (see [10, p. 273]).

COROLLARY 1. *$\rho_\alpha X$ is zero-dimensionally embeded in αX if and only if X is rim-included in αX and $\rho_\alpha X$ is rim-excluded in αX.*

PROOF. Follows immediately from Lemma 1 and Lemma 2.

REMARK 1. A locally compact space X is rim-included in every compactification αX. The remainder $\rho_\alpha X$ (of a locally compact space X) is rim-excluded in αX if and only if it is zero-dimensional (in which case it is also zero-dimensionally embedded).

We recall that a space is called *rim-compact* if the family of open (resp. closed) subsets with compact boundaries form a base for the open (resp. closed) sets on X. We formulate as a proposition the following fact.

PROPOSITION 1. *The space X is rim-compact if and only if it is rim-included in some compactification.*

PROOF. The sufficiency follows immediately from Lemma 2, since boundaries in αX are compact. The necessity follows from the well-known result that a rim-compact space X has at least one compactification in which the remainder is zero-dimensionally embedded, namely the Freudenthal compactification ϕX (see, for example, [4, p. 189], [7, p. 223]).

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We shall briefly illustrate by some examples the concepts we have been discussing so far.

First consider the situation where X is rim-included and $\rho_\alpha X$ is rim-excluded in αX. Examples of this situation are of course exactly those where the remainder is zero-dimensionally embedded (Corollary 1), among those the locally compact spaces with zero-dimensional remainder. In the following example X is not locally compact (though of course rim-compact).

Example 1. Let S denote the closed unit square $[0, 1] \times [0, 1]$ with standard metric topology, let S^0 denote the open unit square $(0, 1) \times (0, 1)$, and let T denote the subspace of S consisting of all points both coordinates of which are rational. Define $X = S^0 - T$, and let Y be the one-point compactification of S^0. Then Y is a compactification αX of X, and $\rho_\alpha X$ is easily seen to be zero-dimensionally embedded in αX. (Actually αX is the Freudenthal compactification of X, see [6, p. 658].)

Also consider the space $X = S - T$ with $\alpha X = S$. Also in this situation $\rho_\alpha X$ is zero-dimensionally embedded in αX.

Next consider the situation where X is rim-included while $\rho_\alpha X$ is not rim-excluded in αX. Examples of this kind have to be found among rim-compact spaces with compactifications where the remainder is not zero-dimensionally embedded (Corollary 1 and Proposition 1). In particular all locally compact spaces with compactifications where the remainder is not zero-dimensional, give examples. In the following example, X is not locally compact.

Example 2. Let S, S^0, T and X be as in Example 1. The space S is a compactification αX of X. Clearly X is rim-included in αX, while $\rho_\alpha X$ contains $S - S^0$ and is thus not zero-dimensional.

Next consider the situation where $\rho_\alpha X$ is rim-excluded while X is not rim-included in αX. Examples of this kind have to be found among spaces with compactifications where the remainder is zero-dimensional but not zero-dimensionally embedded (Corollary 1). Such examples are not so easily constructed as the foregoing. We shall only indicate where to find relevant material.

Example 3. In [8] Yu. M. Smirnov described a space X where (in our terminology) $\rho_\beta X$ is rim-excluded but not zero-dimensionally embedded in βX (the Stone-Čech-compactification). Details of a similar construction is given in [5, p. 118].

We note that X is not rim-compact. This is a consequence of the fact that for a rim-compact space, the maximal compactification with zero-dimensional remainder (which in this example evidently is βX) has zero-dimensionally embedded remainder (see, for example, [6, p 658]).

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Finally consider the situation where X is not rim-included and $\rho_\alpha X$ is not rim-excluded in αX. Examples have to be found among spaces which are not locally compact, with compactifications where the remainder is not zero-dimensionally embedded. In the following example the space is rim-compact.

EXAMPLE 4. Let X be the spaces $S^0 \cap T$, where S^0 and T are as in Example 1, and let αX be the compactification S. Here X is obviously not rim-included in αX. Furthermore $\rho_\alpha X$ contains the connected set $S - S^0$, and is therefore not rim-excluded in αX.

We note that the space X is zero-dimensional, and thus in particular rim-compact.

2. Preliminary results

For further reference we discuss some simple relationships between sets in X and in αX.

LEMMA 4. *Let* $C \in \mathcal{C}_\alpha$. *Then* $\mathrm{cl}_\alpha(C \cap X) = C$.

PROOF. Obviously $\mathrm{cl}_\alpha(C \cap X) \subset C$. Let $x_0 \in C \cap \rho_\alpha X$, and let V be an arbitrary neighbourhood of x_0 in αX. Since $C \in \mathcal{C}_\alpha$, C is a neighbourhood of x_0 in αX. Then $V \cap C \neq \varnothing$, hence $V \cap C \cap X \neq \varnothing$. This means that $x_0 \in \mathrm{cl}_\alpha(C \cap X)$, and consequently $C \cap \rho_\alpha X \subset \mathrm{cl}_\alpha(C \cap X)$. It follows that $C = (C \cap X) \cup (C \cap \rho_\alpha X) \subset \mathrm{cl}_\alpha(C \cap X)$, which completes the proof.

LEMMA 5. *Let B be a closed set in X, and let* $C \in \mathcal{C}_\alpha$. *Then* $\mathrm{cl}_\alpha(B \cap C) = (\mathrm{cl}_\alpha B) \cap C$.

PROOF. Clearly $\mathrm{cl}_\alpha(B \cap C) \subset (\mathrm{cl}_\alpha B) \cap C$. Let $x_0 \in (\mathrm{cl}_\alpha B) \cap C \cap \rho_\alpha X$. Since C is a neighbourhood of x_0 and $x_0 \in \mathrm{cl}_\alpha B$, we have $V \cap (B \cap C) = (V \cap C) \cap B \neq \varnothing$ for every neighbourhood V of x_0. Thus $(\mathrm{cl}_\alpha B) \cap C \cap \rho_\alpha X \subset \mathrm{cl}_\alpha(B \cap C)$, and consequently

$$\begin{aligned}(\mathrm{cl}_\alpha B) \cap C &= (\mathrm{cl}_\alpha B \cap X \cap C) \cup (\mathrm{cl}_\alpha B \cap \rho_\alpha X \cap C) \\ &= (B \cap C) \cup (\mathrm{cl}_\alpha B \cap \rho_\alpha X \cap C) \subset \mathrm{cl}_\alpha(B \cap C).\end{aligned}$$

This completes the proof.

We recall that a closed set A is said to be *regular closed* if cl int $A = A$, and an open set B is said to be regular open if int cl $B = B$.

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LEMMA 6. *Let A be a regular closed subset of X. Then $\mathrm{cl}_\alpha A$ is a regular (closed) subset of αX.*

PROOF. Let $x_0 \notin \mathrm{int}_\alpha \mathrm{cl}_\alpha A$, and let V be an arbitrary neighbourhood of x_0. Then $V \cap (\alpha X - \mathrm{cl}_\alpha A) \neq \varnothing$, and hence $(X \cap V) \cap (\alpha X - A) = (X \cap V) \cap (\alpha X - \mathrm{cl}_\alpha A) = X \cap [V \cap (\alpha X - \mathrm{cl}_\alpha A)] \neq \varnothing$. This means that $x_0 \notin \mathrm{int}\, A$, and consequently $\mathrm{int}\, A \subset \mathrm{int}_\alpha \mathrm{cl}_\alpha A$. Taking into account the assumption $\mathrm{cl}\,\mathrm{int}\, A = A$, we obtain the inclusion $A = \mathrm{cl}\,\mathrm{int}\, A \subset \mathrm{cl}_\alpha \mathrm{int} A \subset \mathrm{cl}_\alpha \mathrm{int}_\alpha \mathrm{cl}_\alpha A$. From this follows the desired inclusion $\mathrm{cl}_\alpha A \subset \mathrm{cl}_\alpha \mathrm{int}_\alpha \mathrm{cl}_\alpha A$.

We recall that a *ring* of sets is a family which is closed under finite unions and finite intersections. We note that $\mathcal{C}_\alpha$ is a ring on αX.

Let $\mathfrak{F}$ be an arbitrary family of sets. Following [1] we denote by $\bigvee \mathfrak{F}$ the family of all finite unions of sets in $\mathfrak{F}$ and by $\bigwedge \mathfrak{F}$ the family of all finite intersections of sets in $\mathfrak{F}$. Then $\bigvee \bigwedge \mathfrak{F} = \bigwedge \bigvee \mathfrak{F}$ is the ring generated by $\mathfrak{F}$.

We write $\mathrm{cl}_\alpha \mathfrak{F}$ for the family $\{\mathrm{cl}_\alpha F \colon F \in \mathfrak{F}\}$, where $\mathfrak{F}$ is an arbitrary family of subsets of X.

LEMMA 7. *Let $\mathcal{G}$ be a ring of closed subsets of X contained in $\mathcal{C}_\alpha \cap X$. Then $\mathrm{cl}_\alpha \mathcal{G}$ is a ring of closed subsets of αX contained in $\mathcal{C}_\alpha$.*

PROOF. By Lemma 4 we have $\mathrm{cl}_\alpha \mathcal{G} \subset \mathcal{C}_\alpha$. Clearly $\mathrm{cl}_\alpha \mathcal{G}$ is closed under finite unions. Let $G_1, G_2 \in \mathcal{G}$, $G_1 = H_1 \cap X$, $G_2 = H_2 \cap X$, where $H_1, H_2 \in \mathcal{C}_\alpha$. Let $x_0 \in H_1 \cap H_2 \cap \rho_\alpha X$. As members of $\mathcal{C}_\alpha$ the sets H_1 and H_2 and hence also $H_1 \cap H_2$ are neighbourhoods of x_0. Let V be an arbitrary neighbourhood of x_0. Then $V \cap (G_1 \cap G_2) = (V \cap H_1 \cap H_2) \cap X \neq \varnothing$, and consequently $x_0 \in \mathrm{cl}_\alpha(G_1 \cap G_2)$. We conclude that $H_1 \cap H_2 = (H_1 \cap H_2 \cap X) \cup (H_1 \cap H_2 \cap \rho_\alpha X) \subset \mathrm{cl}_\alpha(G_1 \cap G_2)$, and hence $\mathrm{cl}_\alpha G_1 \cap \mathrm{cl}_\alpha G_2 = H_1 \cap H_2 = \mathrm{cl}_\alpha(G_1 \cap G_2)$. It follows that $\mathrm{cl}_\alpha \mathcal{G}$ is closed under finite intersections, and so is a ring. This completes the proof.

LEMMA 8. *Let X be rim-included in αX, and let $\mathcal{G}$ be a ring contained in $\mathcal{C}_\alpha \cap X$ which is a base for the closed sets on X. Let K be a compact subset of X, and let B be a closed subset of αX such that $K \cap B = \varnothing$. Then there exists an $F \in \mathrm{cl}_\alpha \mathcal{G}$ such that $B \subset F$ and $F \cap K = \varnothing$.*

PROOF. Let $x \in K$. By Lemma 2 there is a $G_x \in \mathcal{C}_\alpha$ such that $x \notin G_x$, $B \subset G_x$. Since $\mathcal{G}$ is a base for the closed sets on X, there is a $H_x \in \mathcal{C}_\alpha$ such that $H_x \cap X \in \mathcal{G}$, $x \notin H_x$, $H_x \cap X \supset G_x \cap X$. By Lemma 4, $\mathrm{cl}_\alpha(H_x \cap X) = H_x$, $\mathrm{cl}_\alpha(G_x \cap X) = G_x$, and hence $H_x \supset G_x$. Write $V_x = \alpha X - H_x$. The family

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$\{V_x: x \in K\}$ is an open covering of K. Let $\{V_{x_1}, \ldots, V_{x_n}\}$ be a finite sub-covering, and write $F = \alpha X - \bigcup_{i=1}^n V_{x_i} = \bigcap_{i=1}^n X_{x_i}$. Clearly $B \subset F$, and $F \cap K = \varnothing$. Furthermore $H_x \in \mathrm{cl}_\alpha \mathcal{G}$, and hence by Lemma 7 also $F \in \mathrm{cl}_\alpha \mathcal{G}$. This completes the proof.

An *s-ring* $\mathfrak{F}$ on X is a ring of regular closed sets which is separating in the sense that if $x \notin A$, A closed, then there are sets $F_1, F_2 \in \mathfrak{F}$ such that $x \in F_1$, $A \subset F_2$, $F_1 \cap F_2 = \varnothing$ (see [1, p. 126], [11, p. 297]). Since an s-ring evidently is a base for the closed sets, it follows that if an s-ring is contained in $\mathcal{C}_\alpha \cap X$, then X is rim-included in αX. A compact space which possesses an s-ring is called *regular Wallman*, and is a Wallman compactification of every dense subspace (see [11, p. 300]).

Let $\mathfrak{F}$ be an s-ring on S. We define the family $\mathfrak{F}_\alpha$ by

$$\mathfrak{F}_\alpha = \{F \cap C: F \in \mathfrak{F}, C \in \mathcal{C}_\alpha, F \cap \partial_\alpha C = \varnothing\}.$$

Clearly $\mathfrak{F}_\alpha$ contains $\mathfrak{F}$ (since $C = \alpha X \in \mathcal{C}_\alpha$), and is contained in $\mathcal{C}_\alpha \cap X$ if $\mathfrak{F}$ is contained in $\mathcal{C}_\alpha \cap X$. We also note that the elements of $\mathfrak{F}_\alpha$ are regular closed sets (since $F \cap C = F \cap \mathrm{int}_\alpha C$).

PROPOSITION 2. *Let $\mathfrak{F}$ be an s-ring contained in $\mathcal{C}_\alpha \cap X$. Then $\bigvee \mathfrak{F}_\alpha$ is an s-ring contained in $\mathcal{C}_\alpha \cap X$.*

PROOF. Since $\mathfrak{F}_\alpha$ consists of regular closed sets, the same is true for $\bigvee \mathfrak{F}_\alpha$. It is easily verified that $\mathfrak{F}_\alpha$ is closed under finite intersections, and consequently $\bigvee \mathfrak{F}_\alpha = \bigvee \bigwedge \mathfrak{F}_\alpha$ is a ring. The ring $\bigvee \mathfrak{F}_\alpha$ is separating since it contains the separating ring $\mathfrak{F}$. Furthermore $\mathfrak{F}_\alpha \subset \mathcal{C}_\alpha \cap X$, hence also $\bigvee \mathfrak{F}_\alpha \subset \mathcal{C}_\alpha \cap X$. This completes the proof.

3. Main results

We now give conditions under which the existence of an s-ring on X implies the existence of an s-ring on αX.

THEOREM 1. *Let $\rho_\alpha X$ be rim-excluded in αX, and let $\mathfrak{F}$ be an s-ring on X contained in $\mathcal{C}_\alpha \cap X$. Then $\mathrm{cl}_\alpha(\bigvee \mathfrak{F}_\alpha)$ is an s-ring on αX.*

PROOF. From Lemma 6, Lemma 7 and Proposition 2 we conclude that $\mathrm{cl}_\alpha(\bigvee \mathfrak{F}_\alpha)$ is a ring of regular closed sets on αX. It remains to show that $\mathrm{cl}_\alpha(\bigvee \mathfrak{F}_\alpha)$ is separating.

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We recall that since $\mathcal{C}_\alpha \cap X$ contains an s-ring, X is rim-included in αX. As $\rho_\alpha X$ is assumed to be rim-excluded in αX, $\mathcal{C}_\alpha$ is a base for the closed sets on αX (Corollary 1). Let $x_0 \in \alpha X$, A closed in αX, $x_0 \notin A$. Let V be a set in $\mathcal{O}_\alpha$ such that $x_0 \in V$, $A \cap \mathrm{cl}_\alpha V = \varnothing$. By Lemma 8 there is an $F \in \mathcal{F}$ such that $(\{x_0\} \cup A) \subset G$, $G \cap \partial_\alpha V = \varnothing$, where $G = \mathrm{cl}_\alpha F$. We observe that $\mathrm{cl}_\alpha V \in \mathcal{C}_\alpha$, $\mathrm{cl}_\alpha(\alpha X - \mathrm{cl}_\alpha V) \in \mathcal{C}_\alpha$, $\partial_\alpha \mathrm{cl}_\alpha V = \partial_\alpha \mathrm{cl}_\alpha(\alpha X - \mathrm{cl}_\alpha V) = \partial_\alpha V$, $F \cap \partial_\alpha V = \varnothing$. It follows that the sets $F_1 = F \cap \mathrm{cl}_\alpha V$ and $F_2 = F \cap \mathrm{cl}_\alpha(\alpha X - \mathrm{cl}_\alpha V)$ belong to $\mathcal{F}_\alpha$. By using Lemma 5 we get

$$x_0 \in G \cap \mathrm{cl}_\alpha V = \mathrm{cl}_\alpha F \cap \mathrm{cl}_\alpha V = \mathrm{cl}_\alpha(F \cap \mathrm{cl}_\alpha V) = \mathrm{cl}_\alpha F_1,$$

and

$$A \subset G \cap \mathrm{cl}_\alpha(\alpha X - \mathrm{cl}_\alpha V) = \mathrm{cl}_\alpha F \cap \mathrm{cl}_\alpha(\alpha X - \mathrm{cl}_\alpha V) = \mathrm{cl}_\alpha F_2.$$

Furthermore $(\mathrm{cl}_\alpha F_1 \cap \mathrm{cl}_\alpha F_2) \cap \rho_\alpha X = \varnothing$ and $F_1 \cap F_2 = \varnothing$ (since $F \cap \partial_\alpha V = \varnothing$). Thus $\mathrm{cl}_\alpha F_1 \cap \mathrm{cl}_\alpha F_2 = \varnothing$. Finally we evidently have $\mathrm{cl}_\alpha F_1, \mathrm{cl}_\alpha F_2 \in \mathrm{cl}_\alpha \mathcal{F} \subset \mathrm{cl}_\alpha(\bigvee \mathcal{F}_\alpha)$. This completes the proof.

We shall discuss some implications of Theorem 1 for the problem of deciding whether a compactification is regular Wallman.

For the sake of completeness we give a proof of the following easy result.

PROPOSITION 3. *If $\gamma X \leqslant \alpha X$, then $\mathcal{O}_\gamma \cap X \subset \mathcal{O}_\alpha \cap X$. Hence if X is rim-included in a compactification γX, then X is also rim-included in every larger compactification αX.*

PROOF. Let q be the quotient map of αX onto γX. For every $O \in \mathcal{O}_\gamma$, $U = q^{-1}(0)$ is open, and $U \cap X = O \cap X$. Let $y \in \partial_\alpha U$. Then $q(y) \in \partial_\gamma O \subset X$, hence $y = q(y) \in X$. Thus $\partial_\alpha U \subset X$, or $O \cap X = U \cap X \in \mathcal{O}_\alpha \cap X$. Consequently $\mathcal{O}_\gamma \cap X \subset \mathcal{O}_\alpha \cap X$, from which the result follows.

THEOREM 2. *Let $\mathcal{F}$ be an s-ring in $\mathcal{C}_\gamma \cap X$ for some compactification γX. Then for every larger compactification αX where $\rho_\alpha X$ is rim-excluded, $\mathrm{cl}_\alpha(\bigvee \mathcal{F}_\alpha)$ is an s-ring on αX.*

PROOF. Immediate by Proposition 3 and Theorem 1.

COROLLARY 2. *Assume that X has a compactification γX where $\mathcal{C}_\gamma$ contains an s-ring. Then every larger compactification αX for which $\rho_\alpha X$ is rim-excluded, is regular Wallman.*

Proof. Let $\mathcal{G}$ be an s-ring contained in $\mathcal{C}_\gamma$. It is easily verified that $\mathcal{G} \cap X$ is an s-ring on X. The result now follows from Theorem 2.

Remark 2. Theorem 2 (and Corollary 2) can only give a positive answer to the question whether a compactification αX is regular Wallman in cases where the space X is rim-compact and the remainder $\rho_\alpha X$ is zero-dimensionally embedded. For these conditions are clearly contained in the assumptions of the theorem.

We shall say that the space X is *totally rim-included* in the compactification αX if every point $x \in X$ has a neighbourhood V_x in X such that every regular open subset of V_x belongs to $\mathcal{O}_\alpha \cap X$. Every dense subspace of an extremally disconnected compact space K is totally rim-included in K. (In an extremally disconnected space the closure of every open set is open.) A locally compact space is obviously totally rim-included in every compactification.

We note that according to Proposition 3, if X is totally rim-included in some compactification, then X is also totally rim-included in every larger compactification.

Lemma 9. *Let X have a compactification γX such that X is totally rim-included in γX. If X possesses an s-ring, then X possesses an s-ring contained in $\mathcal{C}_\gamma \cap X$.*

Proof. Let $\mathcal{G}$ be an s-ring on X, and define $\mathcal{F} = \mathcal{G} \cap (\mathcal{C}_\gamma \cap X)$. Evidently $\mathcal{F}$ is a ring contained in $\mathcal{C}_\gamma \cap X$ consisting of regular closed sets. It remains to show that $\mathcal{F}$ is separating.

Let $x_0 \in X$, $x_0 \notin A$, A closed in X. There is a set V in $\mathcal{O}_\gamma$ such that $x_0 \in V$ and $\mathrm{cl}_\gamma V \cap A = \emptyset$, and such that every regular open subset of $V \cap X$ belongs to $\mathcal{O}_\gamma \cap X$. The set $B = \gamma X - V$ belongs to $\mathcal{C}_\gamma$. Since $\mathcal{G}$ is an s-ring, there are sets F_1 and F_2 in $\mathcal{G}$ such that $x_0 \in F_1$, $B \cap X \subset F_2$, $F_1 \cap F_2 = \emptyset$. Now $X - F_2$ is a regular open subset of $V \cap X$, hence $X - F_2 \in \mathcal{O}_\gamma \cap X$, and so $F_2 \in \mathcal{C}_\gamma \cap X$. Similarly $\mathrm{int}\, F_1$ is a regular open subset of $V \cap X$, hence $\mathrm{int}\, F_1 \in \mathcal{O}_\gamma \cap X$. Let $\mathrm{int}\, F_1 = O \cap X$, $O \in \mathcal{O}_\gamma$. Then $\mathrm{cl}_\gamma O \in \mathcal{C}_\gamma$, and $(\mathrm{cl}_\gamma O) \cap X = \mathrm{cl}(O \cap X)$ (since X is dense in γX). Thus $F_1 = \mathrm{cl}\,\mathrm{int}\, F_1 = (\mathrm{cl}_\gamma O) \cap X \in \mathcal{C}_\gamma \cap X$. It follows that $F_1, F_2 \in \mathcal{G} \cap (\mathcal{C}_\gamma \cap X) = \mathcal{F}$, which shows that $\mathcal{F}$ is separating.

This completes the proof.

From Lemma 9 and Theorem 2 we immediately obtain the following corollary.

Corollary 3. *Assume that X has a compactification γX such that X is totally rim-included in γX. If X possesses an s-ring, then every compactification αX where αX is larger than γX and $\rho_\alpha X$ is rim-excluded in αX, is regular Wallman.*

We shall here briefly discuss the situation for locally compact spaces. Let ωX denote the one-point compactification. Every compactification αX is larger than ωX. We have already noted that X is totally rim-included in ωX, in fact, in every compactification αX. If $\rho_\alpha X$ is zero-dimensional, then $\rho_\alpha X$ is also rim-excluded in αX. So Corollary 3 contains the following result of Baayen and van Mill ([1, Theorems 2 and 3]), the proof of which gave rise to some of the main ideas of this paper:

THEOREM 3. *Let X be a locally compact space which possesses an s-ring. Then every compactification αX where $\rho_\alpha X$ is zero-dimensional, is regular Wallman.*

4. Some applications

It may be of interest to discuss some intrinsic conditions on the space X which insure that X is totally rim-included in some compactification (or in some class of compactifications).

We recall that the Freudenthal compactification ϕX of a rim-compact space may be obtained as the Wallman compactification corresponding to the normal base $\mathfrak{R}$ consisting of all finite intersections of regular closed sets with compact boundaries (see, for example, [2, p. 108–112]). Every compactification with zero-dimensionally embedded remainder may be obtained as the Wallman compactification corresponding to a normal base $\mathcal{Q}$ contained in $\mathfrak{R}$ and having the property: $A \in \mathcal{Q}$ implies $X - \operatorname{int} A \in \mathcal{Q}$. Every normal base of this kind contained in $\mathfrak{R}$ (on a rim-compact space) gives rise to a Wallman compactification αX with zero-dimensionally embedded remainder. More precisely the basic sets A^* on αX corresponding to sets A in $\mathcal{Q}$ have their boundaries in X, which means that $\mathcal{Q} \subset \mathcal{C}_\alpha \cap X$, or $\tilde{\mathcal{Q}} = \{X - A: A \in \mathcal{Q}\} \subset \mathcal{O}_\alpha \cap X$. (See [3, p. 65–66].) Now in addition assume that $\mathcal{Q}$ contains a base $\mathfrak{B}$ such that every regular closed set which contains a set in $\mathfrak{B}$ belongs to $\mathfrak{B}$. Then the family $\tilde{\mathfrak{B}} = \{X - B: B \in \mathfrak{B}\}$ is a base for the open sets, and every regular open subset of a set in $\tilde{\mathfrak{B}}$ belongs to $\tilde{\mathfrak{B}}$. It follows from the remarks above that $\tilde{\mathfrak{B}} \subset \mathcal{O}_\alpha \cap X$, which means that X is totally rim-included in αX. From Corollary 3 and the fact that $\rho_\alpha X$ is zero-dimensionally embedded in αX, we may thus conclude:

PROPOSITION 4. *Let $\mathcal{Q}$ be a normal base on the rim-compact space X, contained in $\mathfrak{R}$. Assume that $\mathcal{Q}$ has the property: $A \in \mathcal{Q}$ implies $X - \operatorname{int} A \in \mathcal{Q}$. Further assume that $\mathcal{Q}$ contains a base $\mathfrak{B}$ such that every regular closed set containing a set in $\mathfrak{B}$ belongs to $\mathfrak{B}$. Then αX is regular Wallman if X possesses an s-ring.*

We shall say that X is *totally rim-compact* if for every $x \in X$ there is a neighbourhood V_x such that the boundary of every regular open subset of V_x is compact. Evidently every locally compact space and every extremally disconnected space is totally rim-compact.

If X is totally rim-compact, then the normal base $\mathfrak{R}$ clearly satisfies the conditions of Proposition 4. Thus we may state the following corollary.

COROLLARY 4. *If a totally rim-compact space has an s-ring, then the Freudenthal compactification ϕX is regular Wallman.*

We shall now discuss certain classes of totally rim-compact spaces in some detail.

We shall first describe a (rather special) type of space which is totally rim-compact but neither locally compact nor extremally disconnected. Let X be a space where all points are open, except for one point x_0. Assume that there is a set $A_0 \subset X - \{x_0\}$ such that neither $\{x_0\} \cup A_0$ nor $X - A_0$ is open. Further assume that for every neighbourhood V of x_0 there is a neighbourhood W of x_0 such that $V - W$ is infinite. In this situation we easily verify that X is totally rim-compact (every neighbourhood of x_0 has empty boundary), not locally compact (every neighbourhood V of x_0 contains a closed subset $V - W$ which is not compact), and not extremally disconnected (the closure of the open set A_0 is $\{x_0\} \cup A_0$, which is not open).

A space which satisfies the conditions described is the Appert space (see [9, p. 117–118]):

EXAMPLE 5. Let $X = N$ (the set of positive integers). For every $n \in N$ let $\nu(n, E)$ denote the number of integers in the set E which is less than or equal to n. Every integer 2, 3, 4, . . . is an open set, and the neighbourhoods of 1 are the sets E containing 1 for which $\lim_{n\to\infty} \nu(n, E)/n = 1$. This space is easily seen to have the properties required.

We shall next indicate a more general method for obtaining totally rim-compact spaces which are neither locally compact nor extremally disconnected.

Let D be a subset of a topological space X. We define the *D-discrete modification* X_D of X as the topological space having the same points as X, with all points of $X - D$ having their original neighbourhoods and all points of D being open. X_D is easily seen to be a completely regular Hausdorff space when X is.

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Proposition 5. *Let D be a subset of the space X such that $X - D$ is a closed nowhere dense subset which is locally compact as a subspace. Then X_D is totally rim-compact.*

If in the subspace D of X the set of non-isolated points is dense, then X_D is not locally compact.

If there is a point in $X - D$ which is a boundary point for an open set in X, then X_D is not extremally disconnected.

Proof. Let cl_D and ∂_D denote closure and boundary in the space X_D.

Let V be a neighbourhood of a point $x \in X - D$ such that $\mathrm{cl}\, V \cap (X - D)$ is compact. Let W be an arbitrary subset of V. Then clearly $\partial_D W \subset \mathrm{cl}\, V \cap (X - D)$, hence $\partial_D W$ is compact (in the topology induced from X_D, or from X). Thus X_D is totally rim-compact.

Assume that the set of non-isolated points of the subspace D of X is dense in D. Let $x \in X - D$, and let U be an arbitrary neighbourhood of x. Then $U \cap D \neq \emptyset$ since $X - D$ is nowhere dense in X. Let $y \in U$ be non-isolated in D, and let V be a neighbourhood of y such that $\mathrm{cl}\, V \subset D$. Clearly $\mathrm{cl}\, V$ is infinite, hence not compact in X_D. Thus every neighbourhood of x contains a closed non-compact subset. It follows that X_D is not locally compact.

Assume that the point $x_0 \in X - D$ is a boundary point of some open set G in X. Then x_0 is also a boundary point of G in X_D. It follows that $\mathrm{cl}_D\, G$ is not open, and so X_D is not extremally disconnected.

This completes the proof.

Example 6. Instances of the situation described above are easily obtained in the spaces R^n. Let for example: i) $X = R^1$, $X - D =$ the Cantor set C, or ii) $X = C \cup Q$ [Q denotes the rationals], $X - D = C$, or iii) $X = R^2$, $X - D =$ the diagonal $\Delta = \{(x, y) \in R^2 : x = y\}$, or: iv) $X = R^2$, $X - D = Z \times Z$ [Z denotes the integers], or: v) $X = Q \times Q$, $X - D = Z \times Z$, or vi) $X = (Q \times Q) \cup \Delta$, $X - D = \Delta$, or vii) $x = R^3$, $X - D = R^2 \times \{0\}$.

In these cases, all the conditions of Proposition 5 are easily seen to be satisfied.

We shall show that the Freudenthal compactification of some totally rim-compact spaces obtained by the procedure described above, is regular Wallman. Some of the spaces indicated in Example 6 are among those considered in the following proposition.

Proposition 6. *Let X be a locally compact metric space and let D be any subset such that $X - D$ is nowhere dense and the distance between two arbitrary points of*

$X - D$ is at least 1. Then the D-modified space X_D is totally rim-compact, and its Freudenthal compactification is regular Wallman.

PROOF. It follows from Proposition 5 that X_D is totally rim-compact.

Let $\mathfrak{F}$ be the family consisting of all finite subsets of D, of the complements of all such sets, of all sets $V - F$, where V is a closed disc in X with center at some point of $X - D$ and radius less than $\frac{1}{2}$, and F is a finite subset of D, and of the complements of all such sets. The family $\wedge\mathfrak{F}$ and hence also the family $\vee\wedge\mathfrak{F}$ is easily seen to consist of regular closed sets in X_D. (Actually the sets in $\vee\wedge\mathfrak{F}$ are open-closed.) Furthermore $\mathfrak{F}$ and hence $\vee\wedge\mathfrak{F}$ is clearly separating. Thus $\vee\wedge\mathfrak{F}$ is an s-ring on X_D. It follows from Corollary 4 that ϕX_D is regular Wallman.

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