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# THE PRODUCT OF INDEPENDENT RANDOM VARIABLES WITH SLOWLY VARYING TRUNCATED MOMENTS

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**Abstract**

The Mellin-Stieltjes convolution and related decomposition of distributions in $\boldsymbol{M}(\alpha)$ (the class of distributions $\mu$ on $[0, \infty)$ with slowly varying $\alpha$th truncated moments $\int_0^x t^\alpha \mu(dt)$) are investigated. Maller shows that if $X$ and $Y$ are independent non-negative random variables with distributions $\mu$ and $\nu$, respectively, and both $\mu$ and $\nu$ are in $\boldsymbol{D}_2$, the domain attraction of Gaussian distribution, then the distribution of the product $XY$ (that is, the Mellin- Stieltjes convolution $\mu \circ \nu$ of $\mu$ and $\nu$) also belongs to it. He conjectures that, conversely, if $\mu \circ \nu$ belongs to $\boldsymbol{D}_2$, then both $\mu$ and $\nu$ are in it. It is shown that this conjecture is not true: there exist distributions $\mu \in \boldsymbol{D}_2$ and $\nu \notin \boldsymbol{D}_2$ such that $\mu \circ \nu$ belongs to $\boldsymbol{D}_2$. Some subclasses of $\boldsymbol{D}_2$ are given with the property that if $\mu \circ \nu$ belongs to it, then both $\mu$ and $\nu$ are in $\boldsymbol{D}_2$.

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## 1. Introduction

Let $X$ and $Y$ be independent positive random variables with distributions $\mu$ and $\nu$, respectively. We denote the distribution of the product $XY$ by $\mu \circ \nu$ and call it the Mellin-Stieltjes convolution (MS- convolution) of $\mu$ and $\nu$. A distribution $\mu_1$ is said to be a *factor* of a distribution $\mu$, if $\mu = \mu_1 \circ \nu$ with some $\nu$. Let $\boldsymbol{M}(\alpha)$ $(\alpha > 0)$ be the class of distributions $\mu$ on $[0, \infty)$ whose $\alpha$th truncated moments $\int_0^x t^\alpha \mu(dt)$ are slowly varying. The purpose of this paper is to study properties of distributions in $\boldsymbol{M}(\alpha)$ related to MS-convolution. Let $\boldsymbol{D}_2$ be the domain of attraction of Gaussian distribution. Maller [5] shows that if $X$ and $Y$ are independent random variables both with distributions in $\boldsymbol{D}_2$, then the distribution of the product $XY$ also belongs to it. In the converse direction, he shows that if a distribution of product of two independent random variables belongs to $\boldsymbol{D}_2$ and one of them has finite variance, then the other is in $\boldsymbol{D}_2$. Furthermore, he conjectures that finite variance condition could be weakened to being in $\boldsymbol{D}_2$. Since $\boldsymbol{D}_2$ is identical with the class of distributions $\mu$ with slowly varying truncated variances $\int_{|t|<x} t^2\mu(dt)$, these facts mean that $\boldsymbol{M}(2)$ is closed under MS-convolution, and that, if one factor of a distribution in $\boldsymbol{D}_2$ has finite variance, then the other belongs to $\boldsymbol{D}_2$. We deal with this problem in detail. Considering the relation between the truncated moments of two distributions and that of their MS-convolution, we give some conditions for each factor of $\mu \circ \nu$ to belong to $\boldsymbol{M}(\alpha)$. The general results on the decomposition of non-decreasing slowly varying functions are applicable. In the end of this paper, we construct a counter-example for Maller's conjecture: there exists a distribution $\mu \notin \boldsymbol{D}_2$ such that the MS-convolution of $\mu$ and $\nu$ belongs to $\boldsymbol{D}_2$ for every $\nu$ in $\boldsymbol{D}_2$ with infinite variance.

## 2. Preliminaries

We prepare some notations and fundamental facts, which are in Bingham *et al.* [1], Feller [2], Gnedenko and Kolmogorov [3], Seneta [6] and Shimura [7, 8]. The totality of all probability measures on non-negative numbers $[0, \infty)$ is denoted by $\boldsymbol{P}$. Through this paper, we extend MS-convolution to the all distributions in $\boldsymbol{P}$ since the mass on 0 is not essential. A positive measurable function $f$ is said to be slowly varying (s.v.) if $\lim_{x\to\infty} f(kx)/f(x) = 1$ for each $k > 0$. If $f$ is monotone, this is equivalent to $\lim_{x\to\infty} f(2x)/f(x) = 1$. Slowly varying functions have the following representation: A function $f$ defined on $[A, \infty)$, $A > 0$, is s.v. if and only if there exists a positive number $B \geq A$ satisfying for all $x \geq B$ we have $f(x) = c(x)\exp\left(\int_B^x \varepsilon(t)t^{-1}dt\right)$, where $c(x)$ is a bounded positive measurable function on $[B, \infty)$ satisfying $\lim_{x\to\infty} c(x) = c$ $(0 < c < \infty)$, and $\varepsilon(t)$ is a continuous function on $[B, \infty)$ satisfying $\lim_{t\to\infty} \varepsilon(t) = 0$. This representation leads to the following lemma.

LEMMA 2.1. *If $l$ is s.v., then for some $B > 0$, $(l(x)x)/(l(y)y)$ is bounded with respect to $x$ and $y$ satisfying $x \leq y$ and $y \geq B$.*

We say that non-negative non-decreasing $f$ is *decomposed* into components $f_1$ and $f_2$, if both $f_1$ and $f_2$ are non-negative non-decreasing and $f = f_1 + f_2$. Concerning the decomposition of non-decreasing s.v. functions, the following are known. A non-negative non-decreasing function $f$ is said to be *dominatedly non-decreasing* if $\limsup_{x\to\infty}(f(2x) - f(x)) < \infty$. Then $f$ is s.v. and the class of dominatedly non-decreasing functions is closed under sum and decomposition. On the decomposition of non-decreasing s.v. functions, we recall the following in Shimura [7]. Related facts concerning monotone regularly varying functions are given in Shimura [8].

THEOREM 2.2. (1) *Every non-zero component of $f$ is s.v. if and only if $f$ is dominatedly non-decreasing. In this case, every non-zero component is dominatedly non-decreasing.*

(2) *A component $f_1$ of a non-decreasing s.v. $f$ satisfying* $\liminf_{\to\infty} f_1(x)/f(x) > 0$ *is s.v.*

(3) *For any unbounded non-decreasing s.v. function $f$, there exists a non-decreasing function $\tilde{f}$ that is asymptotically equal to $f$ but not dominatedly non-decreasing.*

(4) *For a dominatedly non-decreasing $f$,* $\limsup_{x\to\infty} f(x)/\log x < \infty$.

Let $\boldsymbol{F}(\alpha)$, $\boldsymbol{S}(\alpha)$, and $\boldsymbol{C}(\alpha)$ ($\alpha > 0$) denote the subclasses of $\boldsymbol{P}$ defined by the following conditions: $\mu$ is in $\boldsymbol{F}(\alpha)$ if $\mu$ has dominatedly non-decreasing $\alpha$th truncated moment, $\mu$ is in $\boldsymbol{S}(\alpha)$ if $\lim_{x\to\infty} \int_0^{x^2} t^\alpha \mu(dt) / \int_0^x t^\alpha \mu(dt) = 1$; $\mu$ is in $\boldsymbol{C}(\alpha)$ if $\limsup_{x\to\infty} \int_0^{x^2} t^\alpha \mu(dt) / \int_0^x t^\alpha \mu(dt) < \infty$. It is easy to see that $\mu$ is in $\boldsymbol{S}(\alpha)$ if and only if its truncated $\alpha$th moment is written as $\int_0^x t^\alpha \mu(dt) = l(\log x)$ for some non-decreasing s.v. function $l$. Similarly, $\mu$ is in $\boldsymbol{C}(\alpha)$ if and only if $\int_0^x t^\alpha \mu(dt) = \exp f(\log x)$ with some dominatedly non-decreasing function $f$. Although $\boldsymbol{S}(\alpha)$ is a subclass of $\boldsymbol{M}(\alpha)$ and $\boldsymbol{C}(\alpha)$, it is not a subclass of $\boldsymbol{F}(\alpha)$ (Theorem 2.2 (3)). $\boldsymbol{C}(\alpha)$ is not a subclass of $\boldsymbol{M}(\alpha)$ as we shall show in Section 4.

Let $X_1, X_2, \ldots$ be $\boldsymbol{R}^1$-valued i.i.d. (independent and identically distributed) random variables with distribution $\nu$. If, for suitably chosen constants $B_n > 0$ and $A_n \in \boldsymbol{R}^1$, the distribution of $B_n^{-1} \sum_{k=1}^n X_k - A_n$ converges to a distribution $\mu$ as $n \to \infty$, then we say that $\nu$ is *attracted* to $\mu$. The totality of distributions attracted to $\mu$ is called the *domain of attraction* of $\mu$. We denote the domain of attraction of Gaussian distribution by $\boldsymbol{D}_2$. If, for suitably chosen constants $B_n > 0$, the distribution of $B_n^{-1} \sum_{k=1}^n X_k$ converges to 1 in probability as $n \to \infty$, then we say that $\nu$ is *relatively stable*. Those classes are characterized by truncated moments as follows: $\nu$ belongs to $\boldsymbol{D}_2$ if and only if $\nu$ has s.v. truncated variance $\int_{|t|<x} t^2 \nu(dt)$. Under the assumption that $\nu$ is in $\boldsymbol{P}$, $\nu$ is relatively stable if and only if $\nu$ belongs to $\boldsymbol{M}(1)$.

## 3. Mellin-Stieltjes convolution of slow varying truncated moments

As we mentioned, Maller shows that $\boldsymbol{M}(2)$ is closed under MS-convolution. By change of variables, this implies that, for each $\alpha > 0$, $\boldsymbol{M}(\alpha)$ is closed. We give a new proof of this fact and investigate the relationship between the growth order of the truncated moment of MS-convolution and that of its factors.

THEOREM 3.1. *If $\mu$ is in $\boldsymbol{M}(\alpha)$, then*

$$\lim_{x\to\infty} \int_0^x t^\alpha \mu \circ \nu(dt) \Big/ \int_0^x t^\alpha \mu(dt) = \int_0^\infty t^\alpha \nu(dt). \tag{3.1}$$

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*If, moreover, $\nu$ has finite $\alpha$th moment, then $\mu \circ \nu$ belongs to $\boldsymbol{M}(\alpha)$.*

PROOF. Assume that $\int_0^\infty t^\alpha \nu(dt)$ is finite. Since $\int_0^x t^\alpha \mu \circ \nu(dt) = \int_0^\infty t^\alpha \nu(dt) \int_0^{x/t} s^\alpha \mu(ds)$, we have

$$\frac{\int_0^x t^\alpha \mu \circ \nu(dt)}{\int_0^x s^\alpha \mu(ds)} = \int_0^\infty t^\alpha \nu(dt) \frac{\int_0^{x/t} s^\alpha \mu(ds)}{\int_0^x s^\alpha \mu(ds)}$$
$$= \int_0^1 \nu(dt) \frac{x^\alpha \int_0^{x/t} s^\alpha \mu(ds)}{(x/t)^\alpha \int_0^x s^\alpha \mu(ds)} + \int_1^\infty t^\alpha \nu(dt) \frac{\int_0^{x/t} s^\alpha \mu(ds)}{\int_0^x s^\alpha \mu(ds)}.$$

Since $\sup_{x \geq B} \sup_{t \in (0,1]} \left( x^\alpha \int_0^{x/t} s^\alpha \mu(ds) \right) / \left( (x/t)^\alpha \int_0^x s^\alpha \mu(ds) \right) < \infty$ by Lemma 2.1, the first term goes to $\int_0^1 t^\alpha \nu(dt)$ as $x \to \infty$ by the bounded convergence theorem. The second term converges to $\int_1^\infty t^\alpha \nu(dt)$ because $\int_0^{x/t} s^\alpha \mu(ds) / \int_0^x s^\alpha \mu(ds) \leq 1$ on $t \in [1, \infty)$. If $\int_0^\infty t^\alpha \nu(dt)$ is infinite, then, by Fatou's lemma

$$\liminf_{x \to \infty} \int_0^x t^\alpha \mu \circ \nu(dt) / \int_0^x t^\alpha \mu(dt) \geq \int_0^\infty t^\alpha \nu(dt) = \infty.$$

THEOREM 3.2 (Maller [5, Theorem 1], if $\alpha = 2$). *$\boldsymbol{M}(\alpha)$ ($\alpha > 0$) is closed under* MS*-convolution.*

PROOF. Notice that

$$\frac{\int_x^{2x} t^\alpha \mu \circ \nu(dt)}{\int_0^x t^\alpha \mu \circ \nu(dt)} \leq \frac{\int_0^{\sqrt{2x}} t^\alpha \nu(dt) \int_{x/t}^{2x/t} s^\alpha \mu(ds)}{\int_0^{\sqrt{2x}} t^\alpha \nu(dt) \int_0^{x/t} s^\alpha \mu(ds)} + \frac{\int_0^{\sqrt{2x}} t^\alpha \mu(dt) \int_{x/t}^{2x/t} s^\alpha \nu(ds)}{\int_0^{\sqrt{2x}} t^\alpha \mu(dt) \int_0^{x/t} s^\alpha \nu(ds)}$$
$$\leq \sup_{t < \sqrt{2x}} \frac{\int_{x/t}^{2x/t} s^\alpha \mu(ds)}{\int_0^{x/t} s^\alpha \mu(ds)} + \sup_{t < \sqrt{2x}} \frac{\int_{x/t}^{2x/t} s^\alpha \nu(ds)}{\int_0^{x/t} s^\alpha \nu(ds)}.$$

Since $\mu$ belongs to $\boldsymbol{M}(\alpha)$, the first term tends to 0 as $x \to \infty$. Similarly, the second term goes to 0. Thus we get $\lim_{x \to \infty} \int_x^{2x} t^\alpha \mu \circ \nu(dt) / \int_0^x t^\alpha \mu \circ \nu(dt) = 0$, which means that $\mu \circ \nu$ belongs to $\boldsymbol{M}(\alpha)$.

By the above theorems, we obtain the following result.

COROLLARY 3.3. *If $\mu$ is in $\boldsymbol{M}(\alpha)$ and $\nu$ is in $\boldsymbol{M}(\beta)$ with $\alpha \leq \beta$, then $\mu \circ \nu$ belongs to $\boldsymbol{M}(\alpha)$.*

In the above corollary, if $\alpha < \beta$, then the growth order of the truncated moment of $\mu \circ \nu$ is given by (3.1). We will compare in the next section the truncated moment of $\mu \circ \nu$ and the product of the truncated moments of the two factors, including the case $\alpha = \beta$.

## 4. Decomposition problem of distributions in $M(\alpha)$

In this section, we investigate properties of factors of distributions in $\boldsymbol{M}(\alpha)$. One purpose is to give some conditions for every factor to belong to $\boldsymbol{M}(\alpha)$. In particular, $\boldsymbol{S}(\alpha)$ is closed under MS-convolution and decomposition. Another is to prove that Maller's conjecture is not true. Namely, we will prove that if $\nu$ is in $\boldsymbol{C}(\alpha)$, then $\mu \circ \nu$ belongs to $\boldsymbol{M}(\alpha)$ for every $\mu$ in $\boldsymbol{M}(\alpha)$ with infinite $\alpha$th moment. First, applying Theorem 2.2, we give a theorem on the decomposition problem.

THEOREM 4.1. *Every factor of distribution in $\boldsymbol{F}(\alpha)$ belongs to $\boldsymbol{F}(\alpha)$.*

PROOF. We assume that $\mu \circ \nu$ has dominatedly non-decreasing $\alpha$th truncated moment and $\nu(1, \infty) > 0$ without loss of generality. Notice that

$$\int_0^x t^\alpha \mu \circ \nu(dt) = \int_0^\infty t^\alpha \nu(dt) \int_0^{x/t} s^\alpha \mu(ds) = \sum_{k=1}^\infty \int_{k-1}^k t^\alpha \nu(dt) \int_0^{x/t} s^\alpha \mu(ds).$$

Set $v_k(x) = \int_{k-1}^k \int_0^{x/t} s^\alpha \mu(ds) t^\alpha \nu(dt)$. Then $\int_0^x t^\alpha \mu \circ \nu(dt) = \sum_{k=1}^\infty v_k(x)$. Choose and fix an integer $k \geq 2$ such that $v_k$ is not identically zero. By Theorem 2.2 (1), $v_k$ is dominatedly non-decreasing. Since

$$\text{(4.1)} \qquad \int_{k-1}^k t^\alpha \nu(dt) \int_0^{x/k} s^\alpha \mu(ds) \leq v_k(x) \leq \int_{k-1}^k t^\alpha \nu(dt) \int_0^{x/(k-1)} s^\alpha \mu(ds),$$

we have

$$\int_x^{2x} s^\alpha \mu(ds) \leq \left( \int_{k-1}^k t^\alpha \nu(dt) \right)^{-1} (v_k(2kx) - v_k((k-1)x)).$$

By the dominated non-decrease of $v_k$, we get $\limsup_{x\to\infty} \int_x^{2x} s^\alpha \mu(ds) < \infty$. Hence $\mu \in \boldsymbol{F}(\alpha)$. So is $\nu$.

REMARK. In this proof, we get the dominated non-decrease of the truncated moment of $\mu$ from the dominated non-decrease of $v_k$. Similarly, if it is shown that $v_k$ is s.v., then we can prove that $\mu$ belongs to $\boldsymbol{M}(\alpha)$ by (4.1). But, it is impossible to show that $v_k$ is s.v. under the assumption that $\int_0^x t^\alpha \mu \circ \nu(dt) = \sum_{k=1}^\infty v_k(x)$ is s.v., as will be shown in Section 4.

LEMMA 4.2. *If $\mu \circ \nu$ is in $\boldsymbol{M}(\alpha)$, then*

$$\limsup_{x\to\infty} \int_0^x t^\alpha \mu(dt) \Big/ \int_0^x t^\alpha \mu \circ \nu(dt) \leq \left( \int_0^\infty t^\alpha \nu(dt) \right)^{-1}.$$

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PROOF. For arbitrary $k > 0$,

$$\int_0^{kx} t^\alpha \mu \circ \nu(dt) \geq \int_0^k t^\alpha \nu(dt) \int_0^{kx/t} s^\alpha \mu(ds)$$
$$\geq \int_0^k t^\alpha \nu(dt) \int_0^x s^\alpha \mu(ds).$$

Since $\int_0^x t^\alpha \mu \circ \nu(dt)$ is s.v., we get

$$\limsup_{x\to\infty} \int_0^x t^\alpha \mu(dt) / \int_0^x t^\alpha \mu \circ \nu(dt) \leq \left(\int_0^k t^\alpha \nu(dt)\right)^{-1}.$$

Letting $k \to \infty$, we get the conclusion.

LEMMA 4.3. *If $\mu \circ \nu$ belongs to $\boldsymbol{M}(\alpha)$, then, for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\limsup_{x\to\infty} \frac{\int_0^\delta t^\alpha \nu(dt) \int_0^{x/t} s^\alpha \mu(ds)}{\int_0^x t^\alpha \mu \circ \nu(dt)} \leq \varepsilon.$$

PROOF. We can choose a positive constant $C$ such that $\int_0^x t^\alpha \mu(dt) / \int_0^x t^\alpha \mu \circ \nu(dt) < C$ for large $x$ by Lemma 4.2. Let $V(x) = \int_0^x t^\alpha \mu \circ \nu(dt)$, $U(x) = \int_0^\delta t^\alpha \nu(dt) \int_0^{x/t} s^\alpha \mu(ds)$, where $\delta$ is a positive constant satisfying $C \int_0^\delta t^\alpha \nu(dt) < \varepsilon$. We have

$$U(x) \leq C \int_0^\delta t^\alpha \nu(dt) \int_0^{x/t} s^\alpha \mu \circ \nu(ds). \tag{4.2}$$

On the other hand, by the representation theorem of s.v. function, we have $\int_0^x s^\alpha \mu \circ \nu(ds) = c(x) \exp\left(\int_B^x \varepsilon(u) u^{-1} du\right)$, where $\lim_{x\to\infty} c(x) = c$ $(0 < c < \infty)$ and $\lim_{u\to\infty} \varepsilon(u) = 0$. Since

$$\sup_{0<t\leq\delta} \int_x^{x/t} \frac{\varepsilon(u) - \alpha}{u} du = \int_x^{x/\delta} \frac{\varepsilon(u) - \alpha}{u} du$$

for sufficiently large $x$, we get

$$\lim_{x\to\infty} \sup_{0<t\leq\delta} \frac{\int_0^{x/t} s^\alpha \mu \circ \nu(ds)}{\int_0^x s^\alpha \mu \circ \nu(ds)} \left(\frac{t}{\delta}\right)^\alpha = 1.$$

Hence

$$\limsup_{x\to\infty} \int_0^\delta \frac{\int_0^{x/t} s^\alpha \mu \circ \nu(ds) t^\alpha}{V(x)} \nu(dt) \leq \int_0^\delta t^\alpha \nu(dt). \tag{4.3}$$

By (4.2) and (4.3), $\limsup_{x\to\infty} U(x)/V(x) \leq C \int_0^\delta t^\alpha \nu(dt) < \varepsilon$.

Using this lemma, we get the following propositions.

PROPOSITION 4.4. *If $\mu \circ \nu$ is in $\boldsymbol{M}(\alpha)$ and $\nu$ has finite $\alpha$th moment, then $\mu$ belongs to $\boldsymbol{M}(\alpha)$.*

PROOF. Let $U^c(x) = V(x) - U(x) = \int_\delta^\infty t^\alpha \nu(dt) \int_0^{x/t} s^\alpha \mu(ds)$ in the proof of Lemma 4.3. From this lemma, for $0 < \varepsilon < 1$, we can choose $\delta > 0$ such that $\liminf_{x\to\infty} U^c(x)/V(x) > 0$ and $\nu(\delta, \infty) > 0$. It follows from Theorem 2.2 (2) that $U^c(x)$ is s.v. If we choose a constant $B$ satisfying $\int_\delta^B t^\alpha \nu(dt) > 0$, then

$$\liminf_{x\to\infty} \frac{\int_\delta^B t^\alpha \nu(dt) \int_0^{x/t} s^\alpha \mu(ds)}{\int_B^\infty t^\alpha \nu(dt) \int_0^{x/t} s^\alpha \mu(ds)} \geq \frac{\int_\delta^B t^\alpha \nu(dt)}{\int_B^\infty t^\alpha \nu(dt)} > 0.$$

By Theorem 2.2 (2), $\int_\delta^B \int_0^{x/t} s^\alpha \mu(ds) t^\alpha \nu(dt)$ is an s.v. component of $U^c(x)$. Noticing that $\delta > 0$ and

$$\int_\delta^B \int_0^{\delta x/t} s^\alpha \mu(ds) t^\alpha \nu(dt) \leq \int_\delta^B t^\alpha \nu(dt) \int_0^x t^\alpha \mu(dt) \leq \int_\delta^B t^\alpha \nu(dt) \int_0^{Bx/t} t^\alpha \mu(dt),$$

we get $\int_0^x t^\alpha \mu(dt)$ is s.v.

PROPOSITION 4.5. *If $\mu \circ \nu$ belongs to $\boldsymbol{M}(\alpha)$, then*

$$\limsup_{x\to\infty} \int_0^x t^\alpha \mu \circ \nu(dt) \Big/ \left( \int_0^x t^\alpha \mu(dt) \int_0^x t^\alpha \nu(dt) \right) \leq 1. \tag{4.4}$$

PROOF. By Lemma 4.3, for arbitrary $\varepsilon > 0$, we can take $\delta > 0$ such that

$$\limsup_{x\to\infty} \frac{\int_0^\delta t^\alpha \nu(dt) \int_0^{x/t} s^\alpha \mu(ds)}{\int_0^x t^\alpha \mu \circ \nu(dt)} \leq \varepsilon \quad \text{and} \quad \limsup_{x\to\infty} \frac{\int_0^\delta t^\alpha \mu(dt) \int_0^{x/t} s^\alpha \nu(ds)}{\int_0^x t^\alpha \mu \circ \nu(dt)} \leq \varepsilon.$$

Therefore we have

$$\limsup_{x\to\infty} \frac{\int_0^x t^\alpha \mu \circ \nu(dt)}{\int_\delta^\infty t^\alpha \nu(dt) \int_\delta^{x/t} s^\alpha \mu(ds)} \leq \frac{1}{1-2\varepsilon},$$

$$\begin{aligned}
&\limsup_{x\to\infty} \int_0^x t^\alpha \mu \circ \nu(dt) \Big/ \left( \int_0^x t^\alpha \mu(dt) \int_0^x t^\alpha \nu(dt) \right) \\
&\quad = \limsup_{x\to\infty} \int_0^x t^\alpha \mu \circ \nu(dt) \Big/ \left( \int_0^{x/\delta} t^\alpha \mu(dt) \int_0^{x/\delta} t^\alpha \nu(dt) \right) \\
&\quad \leq \limsup_{x\to\infty} \int_0^x t^\alpha \mu \circ \nu(dt) \Big/ \left( \int_\delta^\infty t^\alpha \mu(dt) \int_\delta^{x/t} s^\alpha \nu(ds) \right)
\end{aligned}$$

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$$\leq \frac{1}{1-2\varepsilon}.$$

Letting $\varepsilon \to 0$, we have completed the proof.

The following proposition gives another condition for every factor to belong to $\boldsymbol{M}(\alpha)$.

PROPOSITION 4.6. *$\mu \circ \nu$ belongs to $\boldsymbol{S}(\alpha)$ if and only if both $\mu$ and $\nu$ are in $\boldsymbol{S}(\alpha)$. In this case,*

$$\lim_{x\to\infty} \int_0^x t^\alpha \mu \circ \nu(dt) \Big/ \left( \int_0^x t^\alpha \mu(dt) \int_0^x t^\alpha \nu(dt) \right) = 1. \tag{4.5}$$

PROOF. Since $\boldsymbol{S}(\alpha)$ is a subclass of $\boldsymbol{M}(\alpha)$ and $\boldsymbol{M}(\alpha)$ is closed under MS-convolution, $\mu \circ \nu$ is in $\boldsymbol{M}(\alpha)$. Hence it follows from Proposition 4.5 and the assumption that

$$\limsup_{x\to\infty} \int_0^{x^2} t^\alpha \mu \circ \nu(dt) \Big/ \left( \int_0^x t^\alpha \mu(dt) \int_0^x t^\alpha \nu(dt) \right) \leq 1.$$

Since

$$\int_0^{x^2} t^\alpha \mu \circ \nu(dt) \Big/ \left( \int_0^x t^\alpha \mu(dt) \int_0^x t^\alpha \nu(dt) \right) \geq 1 \tag{4.6}$$

for any distributions $\mu$ and $\nu$ in $\boldsymbol{P}$, the left-hand side of (4.5) is not less than 1.

REMARK. Though (4.4) and (4.6) give the relation between the truncated moments of MS-convolution and the product of those of their factors, the asymptotic orders of the following three can be different from each other:

$$\int_0^x t^\alpha \mu \circ \nu(dt), \qquad \int_0^x t^\alpha \mu(dt) \int_0^x t^\alpha \nu(dt), \qquad \int_0^{x^2} t^\alpha \mu \circ \nu(dt).$$

Hence if $\mu$ in $\boldsymbol{F}(\alpha)$ satisfies $\lim_{x\to\infty} \int_0^x t^\alpha \mu(dt)/\log x = 1$, then $\lim_{x\to\infty} \int_0^x t^\alpha \mu \circ \mu(dt)/\log x = \infty$ by (4.6) (or Lemma 4.2). Therefore, $\mu \circ \mu \notin \boldsymbol{F}(\alpha)$ by Theorem 2.2 (4) and we see that $\boldsymbol{F}(\alpha)$ is not closed under MS-convolution.

The following theorem shows that Maller's conjecture is not true.

THEOREM 4.7. *If $\mu$ is in $\boldsymbol{M}(\alpha)$ with infinite $\alpha$th moment and $\nu$ is in $\boldsymbol{C}(\alpha)$, then $\mu \circ \nu$ belongs to $\boldsymbol{M}(\alpha)$.*

PROOF. Let $C$ be a constant such that $\int_0^{x^2} s^\alpha \nu(ds)/\int_0^x s^\alpha \nu(ds) < C$ for large $x$. In a similar way to the proof of Theorem 3.2,

$$\frac{\int_x^{2x} t^\alpha \mu\circ\nu(dt)}{\int_0^x t^\alpha \mu\circ\nu(dt)} \le \sup_{t<\sqrt{2x}} \frac{\int_{x/t}^{2x/t} s^\alpha \mu(ds)}{\int_0^{x/t} s^\alpha \mu(ds)} + \frac{\int_0^{\sqrt{2x}} t^\alpha \mu(dt) \int_{x/t}^{2x/t} s^\alpha \nu(ds)}{\int_0^x t^\alpha \mu\circ\nu(dt)}.$$

The first term tends to 0 as $x \to \infty$ by $\mu \in \boldsymbol{M}(\alpha)$. Since $\int_0^x t^\alpha \mu\circ\nu(dt) \ge \int_0^{\sqrt{x/2}} t^\alpha \mu(dt) \int_0^{\sqrt{2x}} t^\alpha \nu(dt)$, it is sufficient to prove that

$$\lim_{x\to\infty} \frac{\int_0^{\sqrt{2x}} t^\alpha \mu(dt) \int_{x/t}^{2x/t} s^\alpha \nu(ds)}{\int_0^{\sqrt{x/2}} t^\alpha \mu(dt) \int_0^{\sqrt{2x}} t^\alpha \nu(dt)} = 0. \tag{4.7}$$

We split the numerator into three parts and estimate each term:

$$\begin{aligned}
&\int_0^{\sqrt{2x}} t^\alpha \mu(dt) \int_{x/t}^{2x/t} s^\alpha \nu(ds) \\
&\quad = \int_0^1 t^\alpha \mu(dt) \int_{x/t}^{2x/t} s^\alpha \nu(ds) + \int_1^{\sqrt{2x}} t^\alpha \mu(dt) \int_{x/t}^{2x/t} s^\alpha \nu(ds) \\
&\quad = \int_{2x}^\infty s^\alpha \nu(ds) \int_{x/s}^{2x/s} t^\alpha \mu(dt) + \int_x^{2x} s^\alpha \nu(ds) \int_{x/s}^1 t^\alpha \mu(dt) \\
&\qquad + \int_1^{\sqrt{2x}} t^\alpha \mu(dt) \int_{x/t}^{2x/t} s^\alpha \nu(ds);
\end{aligned}$$

$$\begin{aligned}
\int_{2x}^\infty s^\alpha \nu(ds) \int_{x/s}^{2x/s} t^\alpha \mu(dt) &= \sum_{k=1}^\infty \int_{(2x)^{2^{k-1}}}^{(2x)^{2^k}} s^\alpha \nu(ds) \int_{x/s}^{2x/s} t^\alpha \mu(dt) \\
&\le \sum_{k=1}^\infty \int_{(2x)^{2^{k-1}}}^{(2x)^{2^k}} s^\alpha \nu(ds) \int_{1/2(2x)^{2^k-1}}^{1/(2x)^{2^{k-1}-1}} t^\alpha \mu(dt) \\
&\le \sum_{k=1}^\infty C^k \int_0^{2x} s^\alpha \nu(ds) \int_{1/2(2x)^{2^k-1}}^{1/(2x)^{2^{k-1}-1}} t^\alpha \mu(dt) \\
&\le \sum_{k=1}^\infty \frac{C^k}{(2x)^{(2^{k-1}-1)\alpha}} \int_0^{2x} s^\alpha \nu(ds);
\end{aligned}$$

$$\int_x^{2x} s^\alpha \nu(ds) \int_{x/s}^1 t^\alpha \mu(dt) \le \int_0^1 t^\alpha \mu(dt) \int_x^{2x} s^\alpha \nu(ds) \le \int_0^1 t^\alpha \mu(dt) \int_0^{2x} s^\alpha \nu(ds).$$

The last term is the most important and estimated as follows. Define $n = n(x) \in \boldsymbol{N}$ as $2^{n-1} \le \sqrt{2x} < 2^n$. Then

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$$\begin{aligned}\int_1^{\sqrt{2x}} t^\alpha \mu(dt) \int_{x/t}^{2x/t} s^\alpha \nu(ds) &\le \sum_{k=1}^{n} \int_{2^{k-1}}^{2^k} t^\alpha \mu(dt) \int_{x/t}^{2x/t} s^\alpha \nu(ds) \\ &\le \sum_{k=1}^{n} \int_{2^{k-1}}^{2^k} t^\alpha \mu(dt) \int_{x/2^k}^{4x/2^k} s^\alpha \nu(ds) \\ &\le \sup_{1\le i\le n} \int_{2^{i-1}}^{2^i} t^\alpha \mu(dt) \sum_{k=1}^{n} \int_{x/2^k}^{4x/2^k} s^\alpha \nu(ds) \\ &\le 2 \sup_{1\le i\le n} \int_{2^{i-1}}^{2^i} t^\alpha \mu(dt) \int_{x/2^n}^{2x} s^\alpha \nu(ds) \\ &\le 2 \sup_{1\le i\le n} \int_{2^{i-1}}^{2^i} t^\alpha \mu(dt) \int_{0}^{2x} s^\alpha \nu(ds).\end{aligned}$$

Thus we have

$$\begin{aligned}&\int_0^{\sqrt{2x}} t^\alpha \mu(dt) \int_{x/t}^{2x/t} s^\alpha \nu(ds) \\ &\quad\le \left(2 \sup_{1\le i\le n} \int_{2^{i-1}}^{2^i} t^\alpha \mu(dt) + \sum_{k=1}^{\infty} \frac{C^k}{(2x)^{(2^{k-1}-1)\alpha}} + \int_0^1 t^\alpha \mu(dt)\right) \int_0^{2x} s^\alpha \nu(ds) \\ &\quad\le C\left(2 \sup_{1\le i\le n} \int_{2^{i-1}}^{2^i} t^\alpha \mu(dt) + \sum_{k=1}^{\infty} \frac{C^k}{(2x)^{(2^{k-1}-1)\alpha}} + \int_0^1 t^\alpha \mu(dt)\right) \int_0^{\sqrt{2x}} s^\alpha \nu(ds).\end{aligned}$$

Since $\mu$ belongs to $\boldsymbol{M}(\alpha)$ and has infinite $\alpha$th moment, we get $\lim_{x\to\infty} \sup_{1\le i\le n} \int_{2^{i-1}}^{2^i} t^\alpha \mu(dt)/\int_0^{\sqrt{x/2}} t^\alpha \mu(dt) = 0$. It is easy to show that $\lim_{x\to\infty} \sum_{k=1}^{\infty} C^k/(2x)^{(2^{k-1}-1)\alpha} = 0$. Using these facts, we get (4.7).

REMARK. By Theorem 4.7, it can occur that $\mu \circ \nu$ belongs to $\boldsymbol{M}(\alpha)$ for $\mu \in \boldsymbol{M}(\alpha)$ and $\nu \notin \boldsymbol{M}(\alpha)$ even if $\lim_{x\to\infty} \int_0^x t^\alpha \mu(dt)/\int_0^x t^\alpha \nu(dt) = 0$.

We construct a distribution in $\boldsymbol{C}(\alpha)\backslash\boldsymbol{M}(\alpha)$ to show that it is not empty.

EXAMPLE. Let $f(x) = \log x$ and $r = e$. Define a discrete probability measure $p$ as follows: $p\left(\left\{e^{e^k}\right\}\right) = ce^{-2e^k+k}$, where $c$ is a normalized constant and $k = 0, 1, \ldots$. Then $V(x)$, the truncated second moment of $p$, is

$$V(x) = c\sum_{k=0}^{n} e^k = c\frac{e^{n+1}-1}{e-1} \quad \text{for} \quad e^{e^n} \le x < e^{e^{n+1}}.$$

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If $x < e^{e^{n+1}}$, then $x^2 < e^{e^{n+2}}$. Therefore we have $V(x^2)/V(x) \leq (e^{n+2}-1)/(e^{n+1}-1)$. Thus we conclude that

$$\limsup_{x\to\infty} \frac{V(2x)}{V(x)} = \limsup_{x\to\infty} \frac{V(x^2)}{V(x)} = e.$$

REMARK. It is still open whether there exists a distribution in $\boldsymbol{D}_2$ that can be decomposed into two factors neither of which belongs to $\boldsymbol{D}_2$.

We add a general result to this problem. We say that $\nu$ *belongs to the domain of partial attraction* of a distribution $\mu$ if, for i.i.d. random variables $X_n$ with disribution $\nu$, there is an increasing sequence $m_n$ of positive integers such that, for some constants $A_n \in \boldsymbol{R}^1$ and $B_n > 0$, the distribution of $B_n^{-1}\sum_{k=1}^{m_n} X_k - A_n$ converges to $\mu$ as $n \to \infty$.

PROPOSITION 4.8. *Every factor of a distribution in $\boldsymbol{D}_2$ belongs to the domain of the partial attraction of Gaussian distribution.*

PROOF. Since $\mu \circ \nu$ belongs to $\boldsymbol{D}_2$, $\mu \circ \nu$ has finite absolute $\alpha$th moment for every $\alpha \in (0, 2)$, which is equivalent to that both $\mu$ and $\nu$ have finite absolute $\alpha$th moments for each $\alpha \in (0, 2)$. Maller [4] shows that this implies that both $\mu$ and $\nu$ belong to the domain of partial attraction of Gaussian distribution.

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