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Surfaces of section for geodesic flows of closed surfaces

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Abstract. We prove several results concerning the existence of surfaces of section for the geodesic flows of closed orientable Riemannian surfaces. The surfaces of section Σ that we construct are either Birkhoff sections, which means that they intersect every sufficiently long orbit segment of the geodesic flow, or at least they have some hyperbolic components in $\partial\Sigma$ as limit sets of the orbits of the geodesic flow that do not return to Σ. In order to prove these theorems, we provide a study of configurations of simple closed geodesics of closed orientable Riemannian surfaces, which may have independent interest. Our arguments are based on the curve shortening flow.

Key words: surfaces of section, Birkhoff sections, geodesic flows, simple closed geodesics, waists, curve shortening flow
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Contents

1. *Introduction*

Let N be a closed connected 3-manifold and let X be a nowhere vanishing vector field on N with flow $\phi_t : N \to N$. A *surface of section* for X is a (not necessarily connected) immersed compact surface $\Sigma \looparrowright N$ whose boundary $\partial\Sigma$ is tangent to X, and whose interior $\mathrm{int}(\Sigma)$ is embedded in $N \setminus \partial\Sigma$ and transverse to X. A surface of section allows us to read part of the dynamics of X as discrete dynamics of the first return map to Σ. For this purpose, we consider the first return time function

$$\tau : \mathrm{int}(\Sigma) \to (0, +\infty], \quad \tau(z) := \inf\{t > 0 \mid \phi_t(z) \in \Sigma\}.$$

Here, as usual, we set $\inf \varnothing = +\infty$. The function τ is continuous on the whole $\mathrm{int}(\Sigma)$ and smooth on the open subsets where it is finite. The surface of section Σ is called a *Birkhoff section* when there exists a finite constant $\ell > 0$ such that, for each $z \in N$, the orbit segment $\phi_{[0,\ell]}(z)$ intersects Σ. Under this condition, outside $\partial\Sigma$ the dynamical system defined by X is the suspension of the discrete dynamical system defined by the first return map

$$\psi : \mathrm{int}(\Sigma) \to \mathrm{int}(\Sigma), \quad \psi(z) = \phi_{\tau(z)}(z).$$

Notice that a Birkhoff section Σ is not necessarily connected, but any connected component of Σ is also a Birkhoff section. By means of a Birkhoff section, statements concerning two-dimensional discrete dynamics can be translated into corresponding statements for the dynamics of vector fields in dimension three.

The quest for Birkhoff sections has been a problem of major interest in dynamical systems since the seminal work of Poincaré [**Poi12**] and Birkhoff [**Bir17**]. In the context of Reeb vector fields on closed 3-manifolds, by a spectacular application of holomorphic curves techniques, Hofer, Wysocki and Zehnder [**HWZ98**] established the existence of Birkhoff sections for all positively curved 3-spheres in the four-dimensional symplectic vector space, equipped with the canonical contact form. A result of the first and third author [**CM22**] proved the existence of Birkhoff sections for all closed contact 3-manifolds satisfying the Kupka–Smale condition: non-degeneracy of the closed Reeb orbits, and transversality of the stable and unstable manifolds of the hyperbolic closed Reeb orbits. In particular, the existence of Birkhoff sections holds for the Reeb vector field of a C^∞ generic contact form on any closed 3-manifold and for the geodesic vector field of a C^∞ generic Riemannian metric on any closed surface.

An independent work of Colin *et al* [**CDHR24**] established the existence of Birkhoff sections for those Reeb vector fields on any closed 3-manifold whose closed orbits are non-degenerate and equidistributed (that is, the contact volume form can be approximated, as a measure, by closed orbits). It is not known whether the equidistribution of the closed

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orbits holds for the geodesic vector field of a C^∞ generic Riemannian metric on a closed surface; indeed, even the fact that, for such a C^∞ generic Riemannian metric, the lifts of the closed geodesics form a dense subset of the unit tangent bundle is an open problem. Nevertheless, Irie's equidistribution theorem [**Iri21**] implies that the equidistribution of the closed orbits holds for the Reeb vector field of a C^∞ generic contact form on any closed 3-manifold, and together with [**CDHR24**] this provides an alternative proof of the C^∞ generic existence of Birkhoff sections for closed contact 3-manifolds. We stress that, on a closed 3-manifold diffeomorphic to the unit tangent bundle of a closed surface, the Reeb flow of a C^∞ generic contact form is not necessarily conjugate (or even orbitally equivalent) to the geodesic flow of a Riemannian metric.

In this paper, we focus on geodesic flows of closed orientable Riemannian surfaces (M, g); hereafter, the Riemannian metrics are assumed to be C^∞. The unit tangent bundle $SM = \{v \in TM \mid \|v\|_g = 1\}$ is equipped with the Liouville contact form $\lambda_v = g(v, d\pi(v)\,\cdot\,)$, where $\pi : SM \to M$ is the base projection. The geodesic vector field X on SM is the associated Reeb vector field, which means that $\lambda(X) \equiv 1$ and $d\lambda(X, \cdot) \equiv 0$. The geodesic flow $\phi_t : SM \to SM$ is the associated Reeb flow. Its orbits have the form $\phi_t(\dot\gamma(0)) = \dot\gamma(t)$, where $\gamma : \mathbb{R} \looparrowright M$ is a geodesic parameterized with unit speed $\|\dot\gamma\|_g \equiv 1$. The closed geodesics of (M, g) are precisely the base projections of the closed orbits of the geodesic flow. A closed geodesic is called simple when it is an embedded circle in M.

In this setting, surfaces of section have had important dynamical applications, among which we mention the following: they played a crucial role in Bangert, Franks and Hingston's proof [**Ban93**, **Fra92**, **Hin93**] of the existence of infinitely many closed geodesics on every Riemannian 2-sphere; and they were employed by Contreras and Oliveira [**CO04**] to establish the existence of an elliptic closed geodesic on a C^2-generic Riemannian 2-sphere, by Contreras and Mazzucchelli [**CM24**] to prove the C^2-structural stability conjecture for Riemannian geodesic flows of closed surfaces and by Knieper and Schulz [**KS24**] to characterize Anosov Riemannian geodesic flows of closed surfaces as the C^2-stably transitive ones.

Although the above-mentioned result in [**CM22**] implies the existence of a Birkhoff section for 'most' geodesic flows of closed Riemannian surfaces, it does not provide any information concerning the topology of such a Birkhoff section. In contrast, the celebrated work of Birkhoff [**Bir17**] provides explicit Birkhoff sections for two classes of geodesic flows: every positively curved Riemannian 2-sphere admits an embedded Birkhoff section diffeomorphic to an annulus, and any negatively curved Riemannian surface admits a Birkhoff section of genus one (see also [**Fri83**, §3]). This paper is largely inspired by the work of Birkhoff.

Our first main result provides specific Birkhoff sections for geodesic flows of closed surfaces that do not admit contractible simple closed geodesics without conjugate points. We refer the reader to §2.3 for the background on the classical notion of conjugate points. A simple closed geodesic γ is called a *waist* when it is a local minimizer of the length functional over the free loop space. (In the literature, sometimes 'waist' refers to a closed geodesic that is a local length minimizer in the free loop space, but is not necessary a simple closed geodesic.) We recall that a non-degenerate simple closed geodesic is a waist if and only if it does not have conjugate points.

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THEOREM A. *Let (M, g) be a closed connected orientable Riemannian surface of genus $G \geq 1$ that does not have any contractible simple closed geodesics without conjugate points. Its geodesic vector field admits a Birkhoff section $\Sigma \looparrowright SM$, where Σ is a compact connected surface of genus one and $8G - 4$ boundary components that cover waists.*

Theorem A applies, in particular, to Riemannian surfaces without contractible simple closed geodesics, and we obtain the following corollary.

COROLLARY B. *The geodesic vector field of any closed connected orientable Riemannian surface (M, g) of genus $G \geq 1$ admits a Birkhoff section of genus one and $8G - 4$ boundary components that cover waists, provided at least one of the following two conditions is satisfied.*

- *The Riemannian surface (M, g) has no conjugate points.*
- *The Gaussian curvature is bounded from above as*

$$\max(K_g) \leq \frac{2\pi}{\operatorname{area}(M, g)}.$$

Proof. If (M, g) has no conjugate points, it does not have contractible closed geodesics (see, e.g., [**GM24**, Theorem 3.28]), and Theorem A applies. If, instead, (M, g) contains a contractible simple closed geodesic γ, then we denote by $B \subsetneq M$ the open disk such that $\partial B = \gamma$, and the Gauss–Bonnet theorem implies that

$$2\pi = \int_B K_g \, dm_g \leq \max(K_g) \operatorname{area}(B, g),$$

where m_g denotes the Riemannian measure. In particular, K_g attains positive values on B. This proves that, under the curvature bound $\max(K_g) \leq 2\pi/\operatorname{area}(M, g)$, there are no contractible simple closed geodesics, and Theorem A provides a Birkhoff section, as claimed. □

Any oriented simple closed geodesic γ on an oriented Riemannian surface (M, g) defines two surfaces of section $A(\dot{\gamma})$, $A(-\dot{\gamma}) \subset SM$ diffeomorphic to annuli, which are referred to as the *Birkhoff annuli* of γ (see §4.1). Theorem A was inspired by the following statement for Riemannian 2-spheres due to Bangert [**Ban93**, Theorem 2], which, in turn, generalizes the above-mentioned result of Birkhoff for positively curved Riemannian 2-spheres. Our methods will provide a slightly simpler proof.

THEOREM C. (Bangert) *Let (S^2, g) be a Riemannian 2-sphere and let γ be a simple closed geodesic with conjugate points whose complement $S^2 \setminus \gamma$ does not contain simple closed geodesics without conjugate points. Then both Birkhoff annuli of γ are Birkhoff sections.*

Remark 1.1. Every Riemannian 2-sphere has at least two simple closed geodesics with conjugate points. This follows from an addendum to the celebrated theorem of Lusternik and Schnirelmann [**LS29**], see [**DPMMS22**, Theorem 1.3(iii) and Proposition 4.2]).

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Our third theorem is an improvement of the above-mentioned result [CM22] for the case of geodesic flows. If K is a hyperbolic invariant subset for the geodesic flow, then we denote, as usual, by $W^s(K)$ and $W^u(K)$ its stable and unstable manifolds, respectively.

THEOREM D. *Let (M, g) be a closed connected orientable Riemannian surface satisfying the following two conditions.*

(i) *All contractible simple closed geodesics without conjugate points are non-degenerate.*
(ii) *Any pair of not necessarily distinct contractible waists γ_1, γ_2 (if they exist) satisfies the transversality condition $W^u(\dot\gamma_1) \pitchfork W^s(\dot\gamma_2)$.*

Then the geodesic vector field of (M, g) admits a Birkhoff section.

The Birkhoff section provided by Theorem D is constructed by applying a surgery procedure due to Fried (see §4.1) to the Birkhoff annuli of a suitable finite collection of closed geodesics. The topology of such a Birkhoff section depends on the configuration of the closed geodesics in the collection, which, in turn, depends on the geometry of the Riemannian surface.

When an orientable Riemannian surface does not satisfy the assumptions of Theorems A, C and D, but still satisfies a mild non-degeneracy condition, we are at least able to construct a surface of section Σ whose escaping orbits are asymptotic to certain hyperbolic components of $\partial\Sigma$.

THEOREM E. *Let (M, g) be a closed connected orientable Riemannian surface all of whose contractible simple closed geodesics without conjugate points are non-degenerate. Its geodesic vector field admits a surface of section $\Sigma \looparrowright SM$ satisfying the following properties.*

(i) Topology: *If $M = S^2$, then Σ is the disjoint union of the Birkhoff annuli of some simple closed geodesics. If M has genus $G \geq 1$, then Σ is the disjoint union of the Birkhoff annuli of some simple closed geodesics and of a compact connected surface of genus one and $8G - 4$ boundary components, all covering non-contractible waists.*
(ii) Completeness: *Σ intersects any orbit $\phi_{(-\infty,\infty)}(z)$ of the geodesic flow.*
(iii) Escape set: *There exists a possibly empty union of connected components $K \subset \partial\Sigma$, whose base projection $\pi(K)$ is the union of hyperbolic contractible waists, such that the complements $W^s(K) \setminus K$ and $W^u(K) \setminus K$ are given by*

$$W^s(K) \setminus K = \{z \in SM \mid \phi_{[t,\infty)}(z) \cap \Sigma = \varnothing \text{ for some } t \in \mathbb{R}\},$$
$$W^u(K) \setminus K = \{z \in SM \mid \phi_{(-\infty,t]}(z) \cap \Sigma = \varnothing \text{ for some } t \in \mathbb{R}\}.$$

(iv) Return time: *There exists $\ell > 0$ such that, for each $z \in SM$ sufficiently close to $\partial\Sigma \setminus K$, we have $\phi_{(0,\ell]}(z) \cap \Sigma \neq \varnothing$.*

For general non-degenerate Reeb flows on closed 3-manifolds, surfaces of section satisfying properties analogous to (ii), (iii) and (iv) of Theorem E were constructed by Colin, Dehornoy and Rechtman [CDR23] by applying surgery to certain holomorphic curves provided by Hutchings' embedded contact homology [Hut14]. Colin, Dehornoy and Rechtman further employed such surfaces of section to produce a so-called broken

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book decomposition of the 3-manifold, which is a generalization of the classical notions of open book decomposition and of Hofer, Wysocki and Zehnder's finite energy foliations [**HWZ03**].

The mentioned works [**CDHR24**, **CDR23**, **CM22**, **HWZ03**, **HWZ98**] are ultimately based on holomorphic curve techniques [**HWZ02**], and the last three even on embedded contact homology [**Hut14**]. In this paper, we do not need any of these techniques and instead employ the curve shortening flow [**Gag90**, **Gra89**]. Our approach allows us to obtain sharper results for geodesic flows. On the one hand, our theorems require only the non-existence or the non-degeneracy of contractible simple closed geodesics without conjugate points, but no conditions on the other closed geodesics. On the other hand, in all the statements except Theorem D, we obtain surfaces of section all of whose components have genus at most one, which may be important for future applications. Incidentally, our arguments require a study of maximal families of pairwise disjoint simple closed geodesics of closed orientable Riemannian surfaces (Theorems 3.4 and 3.5), which may have independent interest.

Finally, we remark that, even though we only considered Riemannian geodesic flows, our results are also valid for geodesic flows of reversible Finsler metrics, by using the generalization of the curve shortening flow developed by Oaks [**Oak94**] and further investigated by Angenent [**Ang08**] and De Philippis *et al* [**DPMMS22**].

1.1. *Organization of the paper.* In §2, we recall the main properties of the curve shortening flow, we provide the details of some applications to the existence of simple closed geodesics that cannot be found in the literature and we recall some features of conjugate points. In §3, we study configurations of simple closed geodesics on closed Riemannian surfaces, whose properties will be an essential ingredient for our main theorems. Finally, in §4, we state and prove Theorems A, C, D and E.

2. *Preliminaries*

2.1. *The curve shortening flow.* It is well known that closed geodesics are critical points of the length and energy functionals, and therefore their existence can be investigated by means of critical point theory [**Kli78**]. In this paper, we will be interested in contractible closed geodesics on surfaces; their critical point theory requires the curve shortening flow [**Gra89**], whose main properties we now recall.

Let (M, g) be a closed oriented Riemannian surface. For any smooth embedded circle $\gamma : S^1 \hookrightarrow M$, we denote by ν_γ its positively oriented normal vector field and by k_γ the signed geodesic curvature of γ. Here, $S^1 = \mathbb{R}/\mathbb{Z}$. We denote by $\mathrm{Emb}(S^1, M)$ the space of smooth embedded circles in M endowed with the C^∞ topology. The length functional

$$L(\gamma) = \int_{S^1} \|\dot\gamma(t)\|_g \, dt \tag{2.1}$$

is continuous over this space. Indeed, it is even differentiable, and its critical points are the simple closed geodesics (that is, the closed geodesics in $\mathrm{Emb}(S^1, M)$) with arbitrary time

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reparameterization. The *curve shortening flow* is a continuous map

$$\mathcal{U} \to \mathrm{Emb}(S^1, M), \quad (s, \gamma_0) \mapsto \Psi_s(\gamma_0) := \gamma_s,$$

defined on a maximal open neighborhood $\mathcal{U} \subset [0, \infty) \times \mathrm{Emb}(S^1, M)$ of $\{0\} \times \mathrm{Emb}(S^1, M)$ by means of the partial differential equation

$$\partial_s \gamma_s = k_{\gamma_s} \nu_{\gamma_s}.$$

Its main properties are the following.

(i) $\Psi_0 = \mathrm{id}$, and $\Psi_{s_2} \circ \Psi_{s_1} = \Psi_{s_2+s_1}$ for all $s_1, s_2 \geq 0$.
(ii) $\Psi_s(\gamma \circ \theta) = \Psi_s(\gamma) \circ \theta$ for all $(s, \gamma) \in \mathcal{U}$ and $\theta \in \mathrm{Diff}(S^1)$.
(iii) $(d/ds)L(\Psi_s(\gamma)) \leq 0$ for all $(s, \gamma) \in \mathcal{U}$, and the equality holds if and only if γ is a simple closed geodesic (not necessarily parameterized with constant speed).
(iv) For each $\gamma \in \mathrm{Emb}(S^1, M)$, if $s_\gamma \in (0, \infty]$ denotes the supremum of the times $s > 0$ for which $(s, \gamma) \in \mathcal{U}$, then s_γ is finite if and only if $\Psi_s(\gamma)$ converges to a constant as $s \to s_\gamma$.

We refer the reader to [**DPMMS22**, **Gra89**] and to the references therein for the proofs of these facts. For each $\ell > 0$ and $\epsilon > 0$, we consider the open subsets

$$\mathrm{Emb}(S^1, M)^{<\ell} := \{\gamma \in \mathrm{Emb}(S^1, M) \mid L(\gamma) < \ell\},$$
$$\mathcal{W}(\ell, \epsilon) := \{\gamma \in \mathrm{Emb}(S^1, M) \mid |L(\gamma) - \ell| < \epsilon^2,\ \|k_\gamma\|_{L^\infty} < \epsilon\}.$$

The intersection

$$\mathcal{K}_\ell := \bigcap_{\epsilon > 0} \mathcal{W}(\ell, \epsilon)$$

is precisely the subspace of those embedded circles that are reparameterizations of simple closed geodesics of length $\ell > 0$. Moreover, we have $\mathcal{K}_\ell = \varnothing$ if and only if $\mathcal{W}(\ell, \epsilon) = \varnothing$ for $\epsilon > 0$ sufficiently small.

In order to employ the curve shortening flow in the critical point theory of the length functional, the following property is crucial. Its proof can be extracted from Grayson's [**Gra89**], and the details can also be found in [**DPMMS22**, Theorem 1.2(iv)].

(v) For each $\ell > 0$ and $\epsilon > 0$, there exists $\delta \in (0, \ell)$ and a continuous function $\tau : \mathrm{Emb}(S^1, M)^{<\ell+\delta} \to [0, \infty)$ such that, for each $\gamma \in \mathrm{Emb}(S^1, M)^{<\ell+\delta}$, we have $\tau(\gamma) < s_\gamma$ and

$$\Psi_s(\gamma) \in \mathrm{Emb}(S^1, M)^{<\ell-\delta} \cup \mathcal{W}(\ell, \epsilon) \quad \text{for all } s \in [\tau(\gamma), s_\gamma).$$

A path-connected subset $U \subseteq M$ is *weakly convex* when, for any pair of distinct points $x, y \in U$ that can be joined by an absolutely continuous curve contained in U of length strictly less than the injectivity radius $\mathrm{inj}(M, g)$, the shortest geodesic segment joining x and y is entirely contained in U. One of the crucial properties of the curve shortening flow is that it preserves the embeddedness of loops. This was first proved by Gage [**Gag90**, §3] as a consequence of a suitable maximum principle. The same arguments actually imply the following analogous property.

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(vi) The curve shortening flow preserves any weakly convex open subset $U \subseteq M$: that is, for any smooth embedded circle γ in U and for any $s \in (0, s_\gamma)$, the embedded circle $\Psi_s(\gamma)$ is still contained in U.

The following is a rather straightforward consequence of the properties of the curve shortening flow.

LEMMA 2.1. *Let $U \subseteq M$ be a weakly convex open subset that is not simply connected and let $\mathcal{C} \subset \mathrm{Emb}(S^1, U)$ be a connected component containing loops that are non-contractible in U. Then, there exists a sequence $\gamma_n \in \mathcal{C}$ converging in the C^2-topology to a simple closed geodesic γ contained in $\overline{U}$ of length*

$$L(\gamma) = \inf_{\zeta \in \mathcal{C}} L(\zeta) > 0.$$

Remark 2.2. The simple closed geodesic γ may not be a waist (see the definition in §2.2) if it intersects the boundary ∂U.

Proof of Lemma 2.1. We claim that

$$\ell := \inf_{\zeta \in \mathcal{C}} L(\zeta) \geq 2\,\mathrm{inj}(M, g) > 0.$$

Indeed, assume by contradiction that $L(\zeta) < 2\,\mathrm{inj}(M, g)$ for some $\zeta \in \mathcal{C}$. This implies that ζ can be written as

$$\zeta(t) = \exp_{\zeta(0)}(V(t))$$

for some smooth function $V : S^1 \to T_{\zeta(0)}M$. We define the smooth homotopy

$$\zeta_r : S^1 \to M,\ \zeta_r(t) := \exp_{\zeta(0)}(rV(t)), \quad r \in [0, 1],$$

which satisfies $\zeta_0 \equiv \zeta(0)$ and $\zeta_1 = \zeta$. Since the open subset U is weakly convex, each ζ_r is contained in U. Therefore, ζ is contractible in U, which is a contradiction.

We claim that

$$\mathcal{W}(\ell, \epsilon) \cap \mathcal{C} \neq \varnothing \quad \text{for all } \epsilon > 0.$$

Indeed, assume that $\mathcal{W}(\ell, \epsilon) \cap \mathcal{C} = \varnothing$ for some $\epsilon > 0$. Consider the constant $\delta > 0$ and the continuous function $\tau : \mathrm{Emb}(S^1, M)^{<\ell+\delta} \to (0, \infty)$ provided by property (v). For each $\zeta_0 \in \mathcal{C}$ of length $L(\zeta_0) < \ell + \delta$, property (vi) guarantees that $\zeta_s := \Psi_s(\zeta_0) \in \mathcal{C}$ for all $s > 0$ for which it is well defined. Property (v) then implies that $L(\zeta_{\tau(\zeta)}) < \ell - \delta$, which contradicts the definition of ℓ.

We choose an arbitrary $\gamma_n \in \mathcal{W}(\ell, 1/n)$ parameterized with constant speed. Up to extracting a subsequence, we have that $\gamma_n(0) \to x$ and $\dot{\gamma}_n(0) \to v$ for some $(x, v) \in TM$ with $\|v\| = \ell$. The curve $\gamma : S^1 \to M$, $\gamma(t) := \exp_x(tv)$ is a closed geodesic of length ℓ, and $\gamma_n \to \gamma$ in the C^1 topology. Since $\|k_{\gamma_n} - k_\gamma\|_{L^\infty} \leq 1/n$, actually, $\gamma_n \to \gamma$ in the C^2 topology. Finally, since the closed geodesic γ is the C^2 limit of embedded circles on an orientable surface, γ is a simple closed geodesic. □

We denote by $SM = \{v \in TM \mid \|v\|_g = 1\}$ the unit tangent bundle, by $\pi : SM \to M$ the base projection and by $\phi_t : SM \to SM$ the geodesic flow, which is defined by

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$\phi_t(\dot\gamma(0)) = \dot\gamma(t)$, where $\gamma : \mathbb{R} \looparrowright M$ is a geodesic parameterized with unit speed $\|\dot\gamma\|_g \equiv 1$. In §3, we will need the following property of weakly convex sets.

LEMMA 2.3. *If $U \subseteq M$ is a weakly convex subset and $K \subset SM$ is a subset invariant under the geodesic flow (that is, $\phi_t(K) = K$ for all $t \in \mathbb{R}$), then any path-connected component of $U \setminus \pi(K)$ is weakly convex.*

Proof. Let $V \subset U \setminus \pi(K)$ be a path-connected component and consider two arbitrary distinct points $x_1, x_2 \in V$ that can be joined by an absolutely continuous curve ζ contained in V of length strictly less than the injectivity radius $\mathrm{inj}(M, g)$. Since U is weakly convex, the shortest geodesic segment γ joining x_1 and x_2 is contained in U. We assume, by contradiction, that $\gamma \cap \pi(K) \neq \varnothing$, and choose a point $v \in K$ such that $x := \pi(v) \in \gamma$. Let $B \subset M$ be the Riemannian open ball of radius $\mathrm{inj}(M, g)$ centered at x and let $\eta \subset \pi(K) \cap B$ be the maximal geodesic segment passing through x tangent to v. Notice that η separates B and intersects γ only in x. In particular, x_1 and x_2 lie in distinct connected components of $B \setminus \eta$. This is not possible, since the curve ζ joining x_1 and x_2 is contained in $B \setminus \pi(K) \subset B \setminus \eta$; indeed, any absolutely continuous curve joining x_1 and x_2 and not entirely contained in B must have length larger than or equal to $\mathrm{inj}(M, g)$. □

2.2. *Types of closed geodesics.* Let γ be a closed geodesic of length $\ell > 0$ in the closed oriented Riemannian surface (M, g). We parameterize γ with unit speed, so that it is a curve of the form

$$\gamma : \mathbb{R}/\ell\mathbb{Z} \to M, \quad \gamma(t) = \pi \circ \phi_t(v),$$

where $v \in SM$ and the length ℓ is the minimal period of γ. The Floquet multipliers of γ are the eigenvalues of the linearized Poincaré map $d\phi_\ell(v)|_V$, where V is the vector subspace

$$V := \{w \in T_vSM \mid g(v, d\pi(v)w) = 0\}.$$

Since the linearized Poincaré map preserves a symplectic structure on the plane V, the Floquet multipliers are of the form $\sigma, \sigma^{-1} \in \mathbb{U} \cup \mathbb{R} \setminus \{0\}$, where $\mathbb{U}$ denotes the unit circle in the complex plane. The closed geodesic γ is called *non-degenerate* when $\sigma \neq 1$ and *hyperbolic* when $\sigma \in \mathbb{R} \setminus \{1, -1\}$. (A closed geodesic γ of length ℓ and Floquet multiplier $\sigma = e^{i2\pi/k}$ for some integer $k \geq 2$ is non-degenerate. However, the kth iterate of γ is degenerate. In this paper, we will not need to consider iterates of closed geodesics.)

We recall that a simple closed geodesic $\gamma : S^1 \hookrightarrow M$ is called a *waist* when any absolutely continuous curve $\zeta : S^1 \to M$ that is sufficiently C^0-close to γ satisfies $L(\zeta) \geq L(\gamma)$; here L is the length functional (2.1).

Remark 2.4. Let $\gamma : S^1 \hookrightarrow M$ be a waist, let $W \subset M$ be a sufficiently small open neighborhood of γ and let $\mathcal{W}$ be the space of absolutely continuous curves $\zeta : S^1 \to W$ homotopic to γ within W. One can easily prove that any $\zeta \in \mathcal{W}$ has length $L(\zeta) \geq L(\gamma)$. Moreover, the equality $L(\zeta) = L(\gamma)$ holds if and only if ζ becomes itself a waist after being reparameterized with constant speed $\|\dot\zeta\|_g \equiv L(\zeta)$. If γ is a non-degenerate waist,

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then any $\zeta \in \mathcal{W}$ satisfying the equality $L(\zeta) = L(\gamma)$ must be geometrically equivalent to γ: that is, $\zeta \circ \theta = \gamma$ for some homeomorphism $\theta : S^1 \to S^1$.

Later, we employ waists to infer the existence of other simple closed geodesics, according to the following lemma. For closed geodesics that are possibly self-intersecting, the analogous lemma is well known (see, e.g., [**Ban93**]).

LEMMA 2.5. *Let (M, g) be an oriented Riemannian surface.*

(i) *If $A \subset M$ is a compact annulus bounded by two waists, then* $\mathrm{int}(A)$ *contains a non-contractible simple closed geodesic.*

(ii) *If D is a compact disk bounded by a waist, then* $\mathrm{int}(D)$ *contains a simple closed geodesic.*

Proof. Let $A \subset M$ be a compact annulus bounded by the waists γ_0, γ_1, with $L(\gamma_1) \leq L(\gamma_0)$. Let $W \subset A \setminus \gamma_1$ be a sufficiently small open neighborhood of γ_0. If $W \setminus \gamma_0$ contains a simple closed geodesic, such a closed geodesic must be C^0-close to γ_0 and in, particular, be non-contractible in A, and the proof is complete. Now, assume that $W \setminus \gamma_0$ does not contain any simple closed geodesic. In particular, every absolutely continuous curve $\gamma : S^1 \to W$ homotopic to γ_0 within W and not geometrically equivalent to γ_0 satisfies the strict inequality $L(\gamma) > L(\gamma_0)$ (see Remark 2.4). Up to shrinking W, there exists $\rho > 0$ such that any smooth curve $\gamma : S^1 \hookrightarrow \overline{W}$ homotopic to γ_0 within $\overline{W}$ and that intersects $\partial W \setminus \gamma_0$ has length $L(\gamma) \geq L(\gamma_0) + \rho$. We now detect a simple closed geodesic in $\mathrm{int}(A)$ by means of a minmax procedure, as follows. Let

$$\ell := \inf_{\mathcal{F}} \max_{r \in [0,1]} L(\zeta_r), \tag{2.2}$$

where the infimum ranges over the family $\mathcal{F}$ of continuous homotopies of smooth embedded loops $\zeta_r : S^1 \hookrightarrow A$, $r \in [0, 1]$ such that $\zeta_0 = \gamma_0$, $\zeta_1 = \gamma_1$, and $\zeta_r \subset \mathrm{int}(A)$ for all $r \in (0, 1)$. Notice that, for any such homotopy $(\zeta_r)_{r \in [0,1]}$, there exists a minimal $r_0 \in (0, 1]$ such that the curve ζ_r intersects $\partial W \setminus \gamma_0$; therefore,

$$L(\zeta_{r_0}) \geq L(\gamma_0) + \rho.$$

This readily implies that

$$\ell \geq L(\gamma_0) + \rho > L(\gamma_0).$$

We claim that ℓ is the length of a non-contractible simple closed geodesic in A, which must actually be contained in $\mathrm{int}(A)$ since $L(\gamma_1) \leq L(\gamma_0) < \ell - \rho$. Let us assume that this is not the case. In particular, for $\epsilon > 0$ small enough, no embedded loop $\gamma \in \mathcal{W}(\ell, \epsilon)$ is entirely contained in A and non-contractible therein. We apply property (v) of the curve shortening flow, which provides $\delta > 0$ and a suitable continuous function $\tau : \mathrm{Emb}(S^1, M)^{<\ell+\delta} \to (0, \infty)$. Let $(\zeta_r)_{r \in [0,1]}$ be a homotopy in $\mathcal{F}$ that is optimal up to δ, which means that

$$\max_{r \in [0,1]} L(\zeta_r) < \ell + \delta.$$

We push the homotopy by means of the curve shortening flow, defining

$$\eta_r := \Psi_{\tau(\zeta_r)}(\zeta_r), \ r \in [0, 1].$$

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Property (v) implies that $\eta_r \in \mathrm{Emb}(S^1, M)^{<\ell-\delta} \cup \mathcal{W}(\ell, \epsilon)$ for each $r \in [0, 1]$. Since the compact annulus A is bounded by simple closed geodesics, $\mathrm{int}(A)$ is weakly convex. Property (vi) implies that $\eta_r \subset \mathrm{int}(A)$ for each $r \in (0, 1)$. Since $\zeta_0 = \gamma_0$ and $\zeta_1 = \gamma_1$ are simple closed geodesics, we have $\eta_0 = \gamma_0$ and $\eta_1 = \gamma_1$. Therefore, the homotopy $(\eta_r)_{r\in[0,1]}$ belongs to $\mathcal{F}$. Since no curve $\gamma \in \mathcal{W}(\ell, \epsilon)$ is entirely contained in A and non-contractible therein, we conclude that $\eta_r \in \mathrm{Emb}(S^1, M)^{<\ell-\delta}$ for each $r \in [0, 1]$, which contradicts the definition of the minmax value ℓ.

The case of a compact disk $D \subset M$ bounded by a waist γ is analogous, except that, in the definition of the minmax (2.2), the infimum ranges over the family of continuous homotopies of smooth embedded loops $\zeta_r : S^1 \hookrightarrow D$, $r \in [0, 1]$, such that $\zeta_0 = \gamma$, $\zeta_r \subset \mathrm{int}(D)$ for all $r \in (0, 1]$ and $L(\zeta_1) < L(\gamma)$. □

2.3. *Conjugate points.* Let us recall the classical notion of conjugate points. If $\gamma : \mathbb{R} \to M$ is an (open or closed) geodesic parameterized with unit speed, the points $\gamma(t_1), \gamma(t_2)$ are conjugate along $\gamma|_{[t_1,t_2]}$ when

$$\ker d(\pi \circ \phi_{t_2-t_1}(\dot\gamma(t_1)))|_{\ker(d\pi(\dot\gamma(t_1)))} \neq \{0\}.$$

On orientable Riemannian surfaces, simple closed geodesics with conjugate points are not waists, and they actually satisfy the following lemma due to Bangert [**Ban93**, Lemma 2].

LEMMA 2.6. *Let (M, g) be an orientable Riemannian surface and let $\gamma : S^1 \hookrightarrow M$ be a simple closed geodesic with conjugate points. Then, for any open neighborhood $U \subset M$ of γ, any connected component $V \subset U \setminus \gamma$ contains a smooth embedded circle $\zeta : S^1 \hookrightarrow V$ homotopic to γ within U such that $L(\zeta) < L(\gamma)$.*

On an orientable Riemannian surface, a non-degenerate simple closed geodesic is a waist if and only if it does not have conjugate points (see, e.g., [**DPMMS22**, Lemma 4.1(iii) and Proposition 4.2(iii, vii)]). Moreover, a non-degenerate waist γ is hyperbolic (see, e.g., [**Kli95**, Theorem 3.4.2]), and the corresponding orbit $\dot\gamma$ of the geodesic flow ϕ_t has a stable manifold

$$W^s(\dot\gamma) = \{z \in SM \mid \omega(z) = \dot\gamma\},$$

which is an injectively immersed surface in SM. Here, $\omega(z)$ denotes the ω-limit of z: that is,

$$\omega(z) = \bigcap_{t>0} \overline{\phi_{[t,\infty)}(z)}.$$

LEMMA 2.7. *Let (M, g) be an orientable closed Riemannian surface and let γ be a non-degenerate waist. For any sufficiently small neighborhood $V \subset SM$ of $\dot\gamma$, if $W \subset V \cap W^s(\dot\gamma)$ is the path-connected component containing $\dot\gamma$, then the base projection $\pi|_W : W \to M$ is a diffeomorphism onto a neighborhood of γ. In particular, for each $z \in W^s(\dot\gamma) \setminus \dot\gamma$ with associated geodesic $\zeta(t) := \pi \circ \phi_t(z)$, there exists $t \in \mathbb{R}$ such that $\zeta|_{[t,\infty)}$ does not intersect γ.*

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Remark 2.8. Since $W^u(\dot\gamma) = -W^s(-\dot\gamma)$, the analogous statement holds for the unstable manifold.

Proof of Lemma 2.7. Since γ is hyperbolic, the closed orbit $\dot\gamma$ has a Floquet multiplier $\sigma \in (-1, 1)$. We parameterize γ with unit speed, and denote by $\ell > 0$ its length, so that $\dot\gamma(t) = \dot\gamma(t + \ell)$. We consider the stable line bundle E^s over $\dot\gamma$, which is given by

$$E^s(\dot\gamma(t)) = \ker(d\phi_\ell(\dot\gamma(t)) - \sigma\,\mathrm{id}).$$

The stable bundle is invariant under the linearized geodesic flow: that is,

$$d\phi_t(\dot\gamma(0))E^s(\dot\gamma(0)) = E^s(\dot\gamma(t)).$$

If we had $E^s(\dot\gamma(t)) = \ker(d\pi(\dot\gamma(t)))$ for some $t \in \mathbb{R}/\ell\mathbb{Z}$, the endpoints of the geodesic segment $\gamma|_{[t,t+\ell]}$ would be conjugate. Therefore, since γ has no conjugate points, we infer that

$$E^s(\dot\gamma(t)) \cap \ker(d\pi(\dot\gamma(t))) = \{0\} \quad \text{for all } t \in \mathbb{R}/\ell\mathbb{Z}.$$

This, together with the fact that

$$T_{\dot\gamma(t)}W^s(\dot\gamma) = \mathrm{span}\{\dot\gamma(t)\} \oplus E^s(\dot\gamma(t)) \quad \text{for all } t \in \mathbb{R}/\ell\mathbb{Z},$$

readily implies that the local stable manifold of $\dot\gamma$ is a graph over the base manifold M. Namely, for any sufficiently small neighborhood $V \subset SM$ of the closed orbit $\dot\gamma$, if $W \subset V \cap W^s(\dot\gamma)$ is the path-connected component containing $\dot\gamma$, then the base projection $\pi|_W : W \to M$ is a diffeomorphism onto a neighborhood of γ. For each $z \in W^s(\dot\gamma) \setminus \dot\gamma$, for all $t > 0$ large enough, we have $\phi_t(z) \in W \setminus \dot\gamma$, and therefore $\pi \circ \phi_t(z) \in \pi(W) \setminus \gamma$. □

We recall the following elementary property of geodesics with conjugate points on surfaces (see, e.g., [**DPMMS22**, Lemma 5.9] for a proof).

LEMMA 2.9. *Let (M, g) be a Riemannian surface and let $\gamma : [-T, T] \to M$ be a geodesic arc such that, for some $[t_1, t_2] \subset (-T, T)$, the points $\gamma(t_1)$ and $\gamma(t_2)$ are conjugate along $\gamma|_{[t_1,t_2]}$. There exists an open neighborhood $V \subset SM$ of $\dot\gamma(0)$ such that, for each $z \in V$, the geodesic $\zeta(t) := \pi \circ \phi_t(z)$ intersects γ for some $t \in [-T, T]$.*

This lemma has the following immediate consequence for simple closed geodesics with conjugate points.

COROLLARY 2.10. *Let (M, g) be an orientable Riemannian surface and let γ be a simple closed geodesic with conjugate points.*

(i) *There exists $T > 0$ and an open neighborhood $V \subset SM$ of the lift $\dot\gamma$ such that, for each $z \in V$, the geodesic $\zeta(t) := \pi \circ \phi_t(z)$ intersects γ for some positive time $t \in (0, T]$ and for some negative time $t \in [-T, 0)$.*

(ii) *There exists $T > 0$ and an open neighborhood $U \subset M$ of γ such that, for each $z \in SU$, the geodesic $\zeta(t) := \pi \circ \phi_t(z)$ intersects γ for some $t \in [-T, T]$.*

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3. *Contractible simple closed geodesics on surfaces*

Let (M, g) be a closed Riemannian surface with geodesic flow $\phi_t : SM \to SM$. If $K \subset SM$ is a hyperbolic compact invariant subset for the geodesic flow, then its stable manifold is defined by

$$W^s(K) = \{z \in SM \mid \omega(z) \subseteq K\},$$

where $\omega(z)$ denotes the ω-limit of z. The unstable manifold $W^u(K)$ is defined analogously by employing the α-limit instead of the ω-limit, or, equivalently,

$$W^u(K) = -W^s(-K).$$

In this paper, we consider invariant compact sets K of the form

$$K = \bigcup_{i=1}^{n} (\dot{\gamma}_i \cup -\dot{\gamma}_i),$$

where $\gamma_1, \ldots, \gamma_n$ are hyperbolic closed geodesics of (M, g). In this case, the stable and unstable manifolds of K decompose as the disjoint unions of the stable and unstable manifolds of the closed orbits $\pm\dot{\gamma}_i$: that is,

$$W^s(K) = \bigcup_{i=1}^{n} (W^s(\dot{\gamma}_i) \cup W^s(-\dot{\gamma}_i)), \quad W^u(K) = \bigcup_{i=1}^{n} (W^u(\dot{\gamma}_i) \cup W^u(-\dot{\gamma}_i)).$$

Let $V \subset SM$ be an open subset. We define the *forward trapped set* $\mathrm{trap}_+(V)$ and the *backward trapped set* $\mathrm{trap}_-(V)$ by

$$\mathrm{trap}_\pm(V) = \{z \in SM \mid \phi_{\pm t}(z) \in V \text{ for all } t > 0 \text{ large enough}\}.$$

Notice that

$$\mathrm{trap}_-(V) = -\mathrm{trap}_+(-V). \tag{3.1}$$

In §2.1, we introduced the notion of weak convexity for an open subset of a closed orientable Riemannian surface (M, g). In this section, we consider open disks $B \subset M$ satisfying the following, stronger, convexity assumption.

Assumption 3.1. The open disk $B \subset M$ is weakly convex, and there exist $\delta > 0$, $T > 0$ and an open neighborhood $N \subset \overline{B}$ of ∂B such that every smooth curve $\gamma : [-T, T] \to M$ parameterized with unit speed $\|\dot{\gamma}\|_g \equiv 1$, with curvature bound $\|k_\gamma\|_{L^\infty} \leq \delta$ and such that $\gamma(0) \in N$, is not entirely contained in B.

Example 3.2. Assumption 3.1 is satisfied when $B \subset M$ is a *convex geodesic polygon*, which means an open disk whose boundary ∂B, seen as a piecewise smooth immersed submanifold of M, is a piecewise geodesic circle with at least one corner and all the inner angles at the corners of ∂B are less than π. This definition allows B to be a fundamental domain of M defined as the complement of a suitable collection of finitely many simple closed geodesics (Figure 1).

Example 3.3. Corollary 2.10(ii) readily implies that Assumption 3.1 is satisfied when $B \subset M$ is an open disk whose boundary is a simple closed geodesic with conjugate points.

The following two theorems are the main results of this section.

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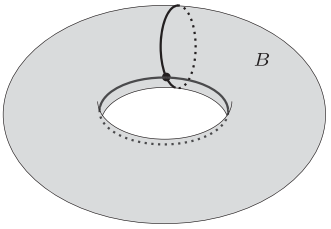

FIGURE 1. An open convex geodesic polygon B that is the complement of two simple closed geodesics of a 2-torus of revolution.

THEOREM 3.4. *Let (M, g) be a closed orientable Riemannian surface and let $B \subset M$ be an open disk satisfying Assumption 3.1. If B does not contain any simple closed geodesic without conjugate points, then* $\mathrm{trap}_+(SB) = \mathrm{trap}_-(SB) = \varnothing$.

Proof. Equation (3.1) implies that $\mathrm{trap}_+(SB) = -\mathrm{trap}_-(SB)$. Let us assume that $\mathrm{trap}_+(SB) \neq \varnothing$ and consider an arbitrary point $z \in \mathrm{trap}_+(SB)$. Assumption 3.1 implies that

$$\pi(\omega(z)) \subset B \setminus N \quad \text{for all } z \in \mathrm{trap}_+(SB).$$

By Lemma 2.3, the connected component $U \subset B \setminus \pi(\omega(z))$ containing $N \setminus \partial B$ is weakly convex. Let $\mathcal{C} \subset \mathrm{Emb}(S^1, U)$ be a connected component containing loops that are non-contractible in U. By Lemma 2.1, there exists a sequence $\gamma_n \in \mathcal{C}$ that converges in the C^2 topology to a simple closed geodesic $\gamma \subset \overline{U}$ of length

$$L(\gamma) = \inf_{\zeta \in \mathcal{C}} L(\zeta). \tag{3.2}$$

Notice that $\gamma_n \subset U \setminus N$ for all n large enough, as, otherwise, γ_n would intersect ∂B according to Assumption 3.1. Therefore, γ is contained in B. This, together with Lemma 2.6 and equation (3.2), implies that γ has no conjugate points. □

THEOREM 3.5. *Let (M, g) be a closed orientable Riemannian surface and let $B \subset M$ be an open disk satisfying Assumption 3.1 that contains at least one closed geodesic but no degenerate simple closed geodesics without conjugate points. Then there exists a finite even number of pairwise disjoint simple closed geodesics $\gamma_1, \ldots, \gamma_{2k} \subset B$ satisfying the following properties.*

(i) *$\gamma_{i+1} \subset B_{\gamma_i}$, where $B_{\gamma_i} \subset B$ are the open disks with boundary $\partial B_{\gamma_i} = \gamma_i$.*
(ii) *For each i odd, γ_i is a waist.*
(iii) *For each i even, γ_i has conjugate points.*
(iv) *Let $U := B \setminus (\gamma_1 \cup \cdots \cup \gamma_{2k})$. The trapped sets of SU are given by*

$$\mathrm{trap}_+(SU) = W^s(K) \setminus K, \quad \mathrm{trap}_-(SU) = W^u(K) \setminus K,$$

where

$$K := \bigcup_{i \text{ odd}} (\dot{\gamma}_i \cup -\dot{\gamma}_i).$$

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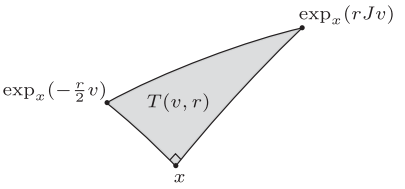

FIGURE 2. The geodesic triangle $T(v, r)$.

(v) *No complete geodesic (that is, a geodesic parameterized with constant speed and defined for all times) is entirely contained in U.*

We prove Theorem 3.5 after some preliminaries.

LEMMA 3.6. *Let (M, g) be a closed oriented Riemannian surface. There exists a constant $a_0 > 0$ such that, for each embedded compact annulus $A \subset M$ with* $\operatorname{area}(A, g) \le a_0$ *and whose boundary ∂A is the union of two simple closed geodesics γ_1, γ_2,*

$$\tfrac{2}{9}L(\gamma_1) \le L(\gamma_2) \le \tfrac{9}{2}L(\gamma_1).$$

Proof. We denote by J the complex structure of the oriented Riemannian surface (M, g): that is, for all tangent vectors $v \in S_xM$, we have that v, Jv is an oriented orthonormal basis of T_xM. For each $r \in (0, \operatorname{inj}(M, g)/2)$, $x \in M$ and $v \in S_xM$, we denote by $T(v, r)$ the unique geodesic triangle with vertices $x, \exp_x(rJv), \exp_x(-(r/2)v)$ (see Figure 2). We consider the continuous function

$$a : (0, \operatorname{inj}(M, g)/4) \to (0, \infty), \quad a(r) = \min_{v \in SM} \operatorname{area}(T(v, r), g).$$

We fix $r \in (0, \operatorname{inj}(M, g)/4)$ to be small enough so that, for each $x \in M$ and $v \in S_xM$ with corresponding geodesic $\gamma_v(t) := \exp_x(tv)$, the smooth map

$$[0, r] \times [0, r] \to M, \quad (t, s) \mapsto \exp_{\gamma_v(t)}(sJ\dot{\gamma}_v(t)) \tag{3.3}$$

is an embedding. The positive constant of the lemma will be $a_0 := a(r)$.

Consider two simple closed geodesics γ_1, γ_2 bounding a compact annulus $A \subset M$ such that $\operatorname{area}(A, g) \le a(r)$. We choose two points $x_0 \in \gamma_1$ and $y_0 \in \gamma_2$ such that

$$d_g(x_0, y_0) = \min\{d_g(x, y) \mid x \in \gamma_1,\ y \in \gamma_2\}. \tag{3.4}$$

We set $\ell_1 := L(\gamma_1)$, $\ell_2 := L(\gamma_2)$, and we consider the unit-speed parameterizations

$$\gamma_1 : \mathbb{R}/\ell_1\mathbb{Z} \hookrightarrow M, \quad \gamma_2 : \mathbb{R}/\ell_2\mathbb{Z} \hookrightarrow M$$

such that $\gamma_1(0) = x_0$, $\gamma_2(0) = y_0$ and $J\dot{\gamma}_1(0)$ points inside the annulus A while $J\dot{\gamma}_2(0)$ points outside the annulus A. Notice that

$$d_g(x_0, y_0) < r.$$

Otherwise, we would have $T(\dot{\gamma}_1(0), r) \subset A$, which would give the contradiction

$$a(r) \ge \operatorname{area}(A, g) > \operatorname{area}(T(\dot{\gamma}_1(0), r)) \ge a(r).$$

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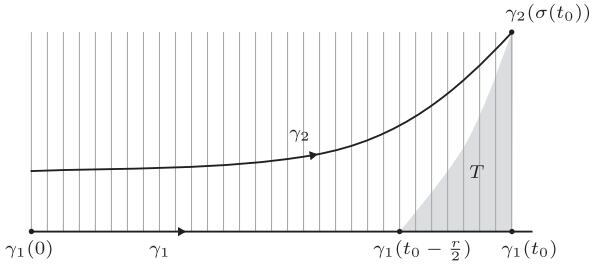

FIGURE 3. The geodesic triangle $T = T(\dot{\gamma}_1(t_0), r)$ contained in A.

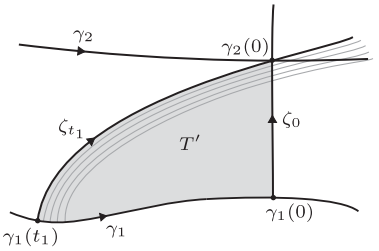

FIGURE 4. The geodesic triangle T'.

For each $t \in \mathbb{R}/\ell_1\mathbb{Z}$, we consider the geodesic arc

$$\zeta_t : [0, r] \to M, \quad \zeta_t(s) = \exp_{\gamma_1(t)}(s\, J\dot{\gamma}_1(t)).$$

The geodesic γ_2 intersects $\zeta_0((0, r))$ at $y_0 = \gamma_2(0)$. Notice that all intersections between γ_2 and the geodesic arcs ζ_t must be transverse (otherwise, some ζ_t would be contained in γ_2, which would contradict the fact that γ_2 and γ_1 are disjoint). Therefore, by the implicit function theorem, there exists a maximal time $t_0 \in (0, \ell_1]$ and a smooth monotone increasing function $\sigma : [0, t_0] \to \mathbb{R}$ such that $\sigma(0) = 0$ and, for all $t \in [0, t_0]$, γ_2 intersects $\zeta_t((0, r])$ transversely at $\gamma_2(\sigma(t))$. We denote by $\rho : [0, t_0] \to (0, r]$ the smooth function such that $\zeta_t(\rho(t)) = \gamma_2(\sigma(t))$ for all $t \in [0, t_0]$. Notice that

$$\zeta_t|_{[0,\rho(t))} \cap \gamma_2 = \varnothing \quad \text{for all } t \in [0, t_0]. \tag{3.5}$$

We claim that $t_0 = \ell_1$. Indeed, if $t_0 < \ell_1$, we would have $\gamma_2(\sigma(t_0)) = \zeta_{t_0}(r)$. Therefore $T(\dot{\gamma}_1(t_0), r) \subset A$ (Figure 3), which would give the contradiction

$$\text{area}(A, g) > \text{area}(T(\dot{\gamma}_1(t_0), r)) \geq a(r) \geq \text{area}(A, g).$$

Next, we claim that $\sigma(\ell_1) \leq \ell_2$. Indeed, assume by contradiction that $\sigma(\ell_1) > \ell_2$, so that there exists $t_1 \in (0, \ell_1)$ such that $\sigma(t_1) = \ell_2$. Suitable segments of the geodesics γ_1, ζ_{t_1} and ζ_0 bound a triangle $T' \subset A$, as in Figure 4. By (3.5), γ_2 does not intersect the interior of T'. Since the maps (3.3) are embeddings, the geodesic segments ζ_t and $\zeta_{t'}$ are disjoint whenever $|t - t'| < r$. Therefore, for each $t \in (t_1, \ell_1)$, the geodesic segment ζ_t must intersect $\zeta_{\ell_1} = \zeta_0$, but, when $t > \ell_1 - r$, this contradicts the embeddedness of the maps (3.3).

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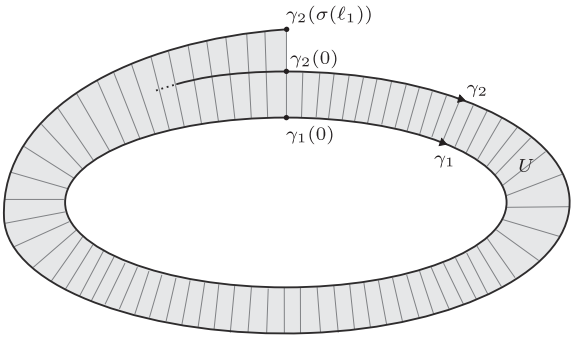

FIGURE 5. The point $\gamma_2(0)$ intersecting the geodesic arc $\zeta_{\ell_1}|_{[0,\rho(\ell_1))}$.

Finally, we claim that $\sigma(\ell_1) = \ell_2$. Indeed, assume by contradiction that the strict inequality $\sigma(\ell_1) < \ell_2$ holds, so that we have distinct values $\rho(0) \neq \rho(\ell_1)$. Since $\gamma_2(0) = \zeta_0(\rho(0))$ and $\gamma_2(\sigma(\ell_1)) = \zeta_0(\rho(\ell_1))$, equation (3.4) implies that $\rho(0) < \rho(\ell_1)$. Namely, $\gamma_2(0)$ lies in $\zeta_0|_{[0,\rho(\ell_1))} = \zeta_{\ell_1}|_{[0,\rho(\ell_1))}$ (Figure 5), which contradicts (3.5).

We can now estimate the length of γ_2 as follows. Notice that

$$\ell_1 \geq 2\,\mathrm{inj}(M, g) > 8r.$$

Let $k \geq 9$ be the integer such that

$$\frac{\ell_1}{k} < r \leq \frac{\ell_1}{k-1}.$$

Notice that each restriction $\gamma_2|_{[\sigma(t),\sigma(t+\ell_1/k)]}$ is contained in the open Riemannian ball $B_g(\gamma_1(t), 2r)$, which has diameter less than $\mathrm{inj}(M, g)$. Therefore,

$$L(\gamma_2|_{[\sigma(t),\sigma(t+\ell_1/k)]}) = d_g(\gamma_2(\sigma(t)), \gamma_2(\sigma(t+\ell_1/k))) < 4r \quad \text{for all } t \in [0, \ell_1].$$

We infer that

$$\ell_2 = \sum_{h=0}^{k-1} d_g\left(\gamma_2\left(\sigma\left(\ell_1 \frac{h}{k}\right)\right), \gamma_2\left(\sigma\left(\ell_1 \frac{h+1}{k}\right)\right)\right) < 4rk < 4\ell_1 \frac{k}{k-1} \leq \frac{9}{2}\ell_1.$$

Switching the roles of γ_1 and γ_2, we obtain the other inequality $\ell_1 \leq \frac{9}{2}\ell_2$. □

We now consider an open disk $B \subset M$ satisfying Assumption 3.1. For any smooth embedded circle $\gamma \subset B$, we denote by $B_\gamma \subset B$ its filling, namely, the open disk with boundary $\partial B_\gamma = \gamma$.

LEMMA 3.7. *If B contains closed geodesics, then it contains a simple closed geodesic γ without conjugate points and whose filling B_γ contains any other closed geodesic in B.*

Proof. We consider the compact subset

$$K := \overline{\bigcup_{\gamma} \dot{\gamma}},$$

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where the union ranges over all closed geodesics γ contained in B. The subset K is invariant under the geodesic flow (that is, $\phi_t(K) = K$ for all $t \in \mathbb{R}$), and therefore its projection $\pi(K)$ is contained in the compact subset $B \setminus N$, where $N \subset \overline{B}$ is the open subset of ∂B given by Assumption 3.1. By Lemma 2.3, the connected component $U \subset B \setminus \pi(K)$ intersecting N is weakly convex. Since U is not simply connected, there exists a connected component $\mathcal{C} \subset \mathrm{Emb}(S^1, U)$ consisting of loops that are non-contractible in U and whose fillings contain $\pi(K)$. By Lemma 2.1, there exists a sequence $\gamma_n \in \mathcal{C}$ that converges in the C^2 topology to a simple closed geodesic $\gamma \subseteq \overline{U}$ of length

$$L(\gamma) = \inf_{\zeta \in \mathcal{C}} L(\zeta) \tag{3.6}$$

and whose filling B_γ contains $\pi(K) \setminus \gamma$. We now conclude as in the proof of Theorem 3.4. We must have $\gamma_n \subset U \setminus N$ for all n large enough, as, otherwise, γ_n would intersect ∂B according to Assumption 3.1. Therefore, γ is contained in B. This, together with Lemma 2.6 and equation (3.6), implies that γ has no conjugate points. □

From now on, we assume that B contains closed geodesics and that all simple closed geodesics without conjugate points that are entirely contained in B are non-degenerate (and therefore they are hyperbolic waists). The open ball B may contain infinitely many simple closed geodesics. Nevertheless, we have the following statement.

LEMMA 3.8. *Any collection of pairwise disjoint simple closed geodesics contained in B is finite.*

Proof. Let $\mathcal{G}$ be a collection of pairwise disjoint simple closed geodesics contained in the open ball B. Let $\mathcal{G}' \subseteq \mathcal{G}$ be a maximal subcollection of simple closed geodesics such that $B_{\gamma_1} \cap B_{\gamma_2} = \varnothing$ for all distinct $\gamma_1, \gamma_2 \in \mathcal{G}'$: namely, for each $\zeta \in \mathcal{G} \setminus \mathcal{G}'$, we have $B_\zeta \cap B_\gamma \neq \varnothing$ for some $\gamma \in \mathcal{G}'$. The Gauss–Bonnet formula implies that

$$2\pi = \int_{B_\gamma} K_g \, dm_g \leq \max(K_g)\,\mathrm{area}(B_\gamma, g) \quad \text{for all } \gamma \in \mathcal{G},$$

where m_g is the Riemannian measure and K_g the Gaussian curvature of (M, g). Therefore, $\mathcal{G}'$ is a finite collection of cardinality

$$k := \#\mathcal{G}' \leq \frac{\max(K_g)\,\mathrm{area}(B, g)}{2\pi}.$$

We define $U \subset B$ to be the complement of the simple closed geodesics in $\mathcal{G}'$, that is,

$$U := B \setminus \bigcup_{\gamma \in \mathcal{G}'} \gamma.$$

We consider the family $\pi_0(\mathrm{Emb}(S^1, U))$ of path-connected components of the space of embedded loops in U. Notice that $\pi_0(\mathrm{Emb}(S^1, U))$ is infinite when $k \geq 3$. Nevertheless, since $\mathcal{G}$ is a collection of pairwise disjoint simple closed geodesics, there are only finitely many homotopy classes $h \in \pi_0(\mathrm{Emb}(S^1, U))$ containing elements of $\mathcal{G} \setminus \mathcal{G}'$. For every such homotopy class h, let $\mathcal{G}_h$ be the subcollection of those $\gamma \in \mathcal{G} \setminus \mathcal{G}'$ contained in h. For each pair of distinct $\gamma_1, \gamma_2 \in \mathcal{G}_h$, we denote by $A_{\gamma_1,\gamma_2} \subset B$ the compact annulus

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with boundary $\partial A_{\gamma_1,\gamma_2} = \gamma_1 \cup \gamma_2$. Consider the constant $a_0 > 0$ provided by Lemma 3.6. Since $\text{area}(U, g) < \infty$, there exist finitely many $\gamma_{h,1}, \ldots, \gamma_{h,n_h} \in \mathcal{G}_h$ such that, for each $\gamma \in \mathcal{G}_h \setminus \{\gamma_{h,1}, \ldots, \gamma_{h,n_h}\}$, we have $\text{area}(A_{\gamma,\gamma_{h,i}}, g) \le a_0$ for some i. We consider the finite subcollection $\mathcal{G}'' \subset \mathcal{G}$ that comprises $\mathcal{G}'$ and all the elements $\gamma_{h,j}$. Summing up, we have shown that, for each $\gamma \in \mathcal{G} \setminus \mathcal{G}''$, there exists $\zeta \in \mathcal{G}''$ such that $\text{area}(A_{\gamma,\zeta}) \le a_0$. This, together with Lemma 3.6, implies that

$$\ell := \sup_{\gamma \in \mathcal{G}} L(\gamma) \le \max_{\gamma \in \mathcal{G}''} \frac{9}{2} L(\gamma) < \infty.$$

Let $\mathcal{K} \subset W^{1,2}(S^1, M)$ be the subspace of closed geodesics $\gamma : S^1 \to M$ of length $L(\gamma) \le \ell$ parameterized with constant speed. As a consequence of the Palais–Smale condition for the geodesic energy functional [**Kli78**, Theorem 1.4.7], $\mathcal{K}$ is compact in the $W^{1,2}$ topology.

Assume, by contradiction, that there exist an infinite sequence $\gamma_n \in \mathcal{G} \subset \mathcal{K}$. Up to extracting a subsequence, γ_n converges to a simple closed geodesic γ in the $W^{1,2}$ topology. *A priori*, γ is contained in the closure $\overline{B}$, but Lemma 3.7 guarantees that γ is actually contained in the open ball B. Since the γ_n are pairwise disjoint, they are also disjoint from their limit γ. This, together with Corollary 2.10(ii), implies that γ is without conjugate points, and therefore it must be non-degenerate by our assumption. This gives a contradiction, since a non-degenerate closed geodesic γ cannot be the $W^{1,2}$-limit of closed geodesics that are disjoint from γ. □

Proof of Theorem 3.5. Let $\gamma_1, \ldots, \gamma_n$ be a maximal sequence of pairwise disjoint simple closed geodesics contained in the open ball B such that $\gamma_{i+1} \subset B_{\gamma_i}$ for each $i = 1, \ldots, n-1$. Lemma 3.8 implies that such a sequence must be finite. The maximality of this sequence implies that every other simple closed geodesic γ contained in B and not intersecting any γ_i must have filling B_γ contained in the complement $U := B \setminus (\gamma_1 \cup \cdots \cup \gamma_n)$. Notice that the first simple closed geodesic γ_1 must be the waist provided by Lemma 3.7, whose filling B_{γ_1} contains any other closed geodesic contained in B. We recall that, by our assumption on B, any simple closed geodesic contained in B that is not a waist must have conjugate points.

For each $i \in \{1, \ldots, n-1\}$, γ_i or γ_{i+1} must be a waist. Indeed, if none of them were waists, Lemma 2.6 would imply that the open annulus $A_i := B_{\gamma_i} \setminus \overline{B_{\gamma_{i+1}}}$ contains a non-contractible embedded circle ζ_0 with length $L(\zeta_0) < \min\{L(\gamma_i), L(\gamma_{i+1})\}$, and Lemma 2.1 would imply that there exists a non-contractible closed geodesic $\zeta \subset \overline{A_i}$ such that $L(\zeta) \le L(\zeta_0)$; these inequalities would imply that ζ is contained in A_i, which would contradict the maximality of the sequence $\gamma_1, \ldots, \gamma_n$. Moreover, γ_i and γ_{i+1} cannot both be waists. Otherwise Lemma 2.5(i) would imply that the open annulus A_i contains a non-contractible simple closed geodesic ζ, which would contradict the maximality of the sequence $\gamma_1, \ldots, \gamma_n$. Finally, the last simple closed geodesic γ_n of the sequence cannot be a waist, as, otherwise, Lemma 2.5(ii) would imply that its filling B_γ contains another simple closed geodesic, which would again contradict the maximality of the sequence $\gamma_1, \ldots, \gamma_n$. Summing up, we have proved that n is even, and that γ_i is a waist if i is odd, whereas γ_i has conjugate points if i is even.

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We denote by $\mathcal{K} = \{\gamma_1, \gamma_3, \dots, \gamma_{n-1}\}$ the collection of waists in the sequence $\gamma_1, \dots, \gamma_n$, and we set

$$K := \bigcup_{\gamma \in \mathcal{K}} (\dot{\gamma} \cup -\dot{\gamma}).$$

Lemma 2.7 implies that

$$W^s(K) \setminus K \subseteq \mathrm{trap}_+(SU).$$

Conversely, consider an arbitrary point $z \in \mathrm{trap}_+(SU)$, so that $\phi_t(z) \in SU$ for all $t > 0$ large enough. We shall prove that the ω-limit $\omega(z)$ is either $\dot{\gamma}$ or $-\dot{\gamma}$ for some $\gamma \in \mathcal{K}$, which will imply the opposite inclusion

$$\mathrm{trap}_+(SU) \subseteq W^s(K) \setminus K.$$

Let $\pi : SM \to M$ be the base projection. Since $z \in \mathrm{trap}_+(SU)$, there exists a connected component $W \subset U$ such that $\phi_t(z) \in SW$ for all $t > 0$ large enough, and therefore $\pi(\omega(z)) \subset \overline{W}$. We have three possible cases.

Case 1: $W = B_{\gamma_n}$. Since γ_n has conjugate points, Corollary 2.10(ii) implies that the geodesic $\pi \circ \phi_t(z)$ does not enter a sufficiently small neighborhood of ∂B_{γ_n} for all $t > 0$ large enough. Therefore, $\pi(\omega(z))$ is contained in the open disk B_{γ_n}. Since $\omega(z)$ is compact, so is its projection $\pi(\omega(z))$. Lemma 2.3 implies that any path-connected component of the open set $B_{\gamma_n} \setminus \pi(\omega(z))$ is weakly convex, and Lemma 2.6 implies that $B_{\gamma_n} \setminus \pi(\omega(z))$ contains a non-contractible embedded loop ζ_0 with length $L(\zeta_0) < L(\gamma_n)$. We can now apply Lemma 2.1, which provides a sequence of non-contractible smooth embedded circles in $B_{\gamma_n} \setminus \pi(\omega(z))$ converging in the C^2 topology to a simple closed geodesic $\zeta \subset \overline{B_{\gamma_n} \setminus \pi(\omega(z))}$ with length $L(\zeta) \leq L(\zeta_0) < L(\gamma_n)$. These inequalities readily imply that ζ is contained in the open disk B_{γ_n}, which contradicts the maximality of the sequence $\gamma_1, \dots, \gamma_n$. This proves that case 1 cannot occur.

Case 2: $W = B \setminus \overline{B_{\gamma_1}}$. Since B satisfies Assumption 3.1, the geodesic $\pi \circ \phi_t(z)$ does not enter a sufficiently small neighborhood of ∂B for all $t > 0$ large enough, as, otherwise, it would intersect ∂B for $t > 0$ arbitrarily large. Therefore, the compact set $\pi(\omega(z))$ is contained in $\overline{W} \setminus \partial B$. Let $V \subset W \setminus \pi(\omega(z))$ be the connected component whose boundary ∂V contains ∂B. By Lemma 2.3, V is weakly convex. Lemma 2.1 provides a sequence of non-contractible smooth embedded loops $\zeta_k \subset V$ converging in the C^2 topology to a simple closed geodesic $\zeta \subset \overline{V}$. Notice that ζ cannot intersect ∂B, as, otherwise, ζ_k would intersect ∂B as well for k large enough, according to Assumption 3.1. Therefore, Lemma 3.7 implies that $\zeta = \gamma_1$, and we have $V = B \setminus \overline{B_{\gamma_1}}$ and $\pi(\omega(z)) = \gamma_1$. Since $\omega(z)$ is connected and invariant under the geodesic flow, we infer that either $\omega(z) = \dot{\gamma}_1$ or $\omega(z) = -\dot{\gamma}_1$.

Case 3: $W = B_{\gamma_i} \setminus \overline{B_{\gamma_{i+1}}}$ for some $i \in \{1, \dots, n-1\}$. Let us assume that i is odd, so that γ_i is a waist and γ_{i+1} has conjugate points (the case of i even is analogous). Since γ_{i+1} has conjugate points, Corollary 2.10(ii) implies that there exists a neighborhood of γ_{i+1} that does not intersect $\pi(\omega(z))$. If $\pi(\omega(z))$ intersects the interior W, then we can apply Lemma 2.1, as in case 1, and infer the existence of a simple closed geodesic $\zeta \subset W$, which contradicts the maximality of the sequence $\gamma_1, \dots, \gamma_n$. This proves that

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$\pi(\omega(z)) \subseteq \gamma_i$. Since $\omega(z)$ is connected and invariant under the geodesic flow, we infer that either $\omega(z) = \dot{\gamma}_i$ or $\omega(z) = -\dot{\gamma}_i$.

Summing up, we have proved that

$$\mathrm{trap}_+(SU) = W^s(K) \setminus K.$$

Since $\mathrm{trap}_-(SU) = -\mathrm{trap}_+(SU)$ and $W^u(K) = -W^s(K)$, we also have

$$\mathrm{trap}_-(SU) = W^u(K) \setminus K.$$

It remains to show that no complete geodesic is entirely contained in U. Assume, by contradiction, that this does not hold, so that there exists a point $z \in SU$ such that $\phi_t(z) \in SU$ for all $t \in \mathbb{R}$. In particular, $z \in \mathrm{trap}_+(SU) \cap \mathrm{trap}_-(SU)$, and therefore $z \in W^u(\dot{\gamma} \cup -\dot{\gamma}) \cap W^s(\dot{\gamma} \cup -\dot{\gamma}) \setminus (\dot{\gamma} \cup -\dot{\gamma})$ for some waist $\gamma \in \mathcal{K}$. Let $W \subset U$ be the connected component such that $\phi_t(z) \in SW$ for all $t \in \mathbb{R}$. The waist γ is a connected component of ∂W, and therefore we must have either $W = B \setminus \overline{B_{\gamma_1}}$ or $W = B_{\gamma_i} \setminus \overline{B_{\gamma_{i+1}}}$ for some $i \in \{1, \ldots, n-1\}$. The compact subset

$$C := \overline{\bigcup_{t \in \mathbb{R}} \phi_t(z)} \subseteq \dot{\gamma} \cup -\dot{\gamma} \cup \bigcup_{t \in \mathbb{R}} \phi_t(z)$$

is invariant under the geodesic flow, and therefore any path-connected component of $W \setminus \pi(C)$ is weakly convex according to Lemma 2.3. The end of the argument is entirely analogous to the one in cases 2 and 3 above. There exists a neighborhood of $\partial W \setminus \gamma$ that does not intersect $\pi(C)$. Since $\pi(C)$ intersects the open set W, with a suitable application of Lemma 2.1, we infer the existence of a simple closed geodesic in W, which contradicts the maximality of the sequence $\gamma_1, \ldots, \gamma_n$. □

4. *Construction of surfaces of section*

4.1. *Fried surgery of Birkhoff annuli.*

The surfaces of section provided by our main theorems will be obtained by performing surgeries à la Fried [**Fri83**] on a suitable collection of Birkhoff annuli of closed geodesics. In this subsection, we briefly recall this procedure.

Let (M, g) be an oriented Riemannian surface and let J be its associated complex structure: that is, v, Jv is an oriented orthonormal basis of T_xM for all tangent vectors $v \in S_xM$. Let $\gamma : \mathbb{R}/\ell\mathbb{Z} \to M$ be a closed geodesic parameterized with unit speed $\|\dot{\gamma}\|_g \equiv 1$. The *Birkhoff annulus* of γ is the immersed compact annulus in SM given by

$$A(\dot{\gamma}) := \{v \in S_{\gamma(t)}M \mid t \in \mathbb{R}/\ell\mathbb{Z},\ g(J\dot{\gamma}(t), v) \geq 0\}.$$

Its boundary $\partial A(\dot{\gamma}) = \dot{\gamma} \cup -\dot{\gamma}$ is embedded, whereas its interior $\mathrm{int}(A(\dot{\gamma}))$ is immersed and transverse to the geodesic vector field X on SM.

Now assume that γ is simple. In this case, $A(\dot{\gamma})$ is embedded; therefore, it is a surface of section for X. By considering the opposite orientation on γ, we obtain a second Birkhoff annulus $A(-\dot{\gamma})$ such that

$$\mathrm{int}(A(\dot{\gamma})) \cap \mathrm{int}(A(-\dot{\gamma})) = \varnothing, \quad \partial A(\dot{\gamma}) = \partial A(-\dot{\gamma}) = \dot{\gamma} \cup -\dot{\gamma}.$$

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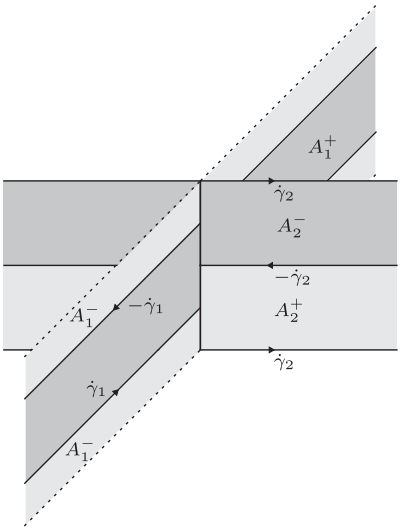

FIGURE 6. The four Birkhoff annuli A_1^+, A_1^-, A_2^+, A_2^- of the simple closed geodesics γ_1, γ_2. The central vertical line of double points is the fiber S_xM of a point $x \in \gamma_1 \cap \gamma_2$. The geodesic vector field should be thought as horizontal and makes a full turn as we move through the line of double points.

For any pair of distinct simple closed geodesics γ_1, γ_2 of (M, g) that intersect each other, the Birkhoff annuli $A_i^{\pm} := A(\pm\dot\gamma_i)$ intersect as in Figure 6. We see their union

$$\Upsilon := A_1^+ \cup A_1^- \cup A_2^+ \cup A_2^-$$

as a non-injectively immersed surface with boundary

$$\partial\Upsilon = \dot\gamma_1 \cup -\dot\gamma_1 \cup \dot\gamma_2 \cup -\dot\gamma_2.$$

Notice that, for any $\ell > \max\{L(\gamma_1), L(\gamma_2)\}$, there exists an open neighborhood $N \subset \Upsilon$ of $\partial\Upsilon$ such that, for each $z \in N$, the orbit segment $\phi_{(0,\ell]}(z)$ intersects Υ.

We apply the following surgery procedures, due to Fried, in order to resolve the self-intersections with interior points of Υ, and we produce a surface of section $\Sigma \subset SM$ with the same boundary $\partial\Sigma = \partial\Upsilon$. Away from an arbitrarily small neighborhood U of the subspace of self-intersections of $\mathrm{int}(\Upsilon)$ with Υ, we set $\Sigma \cap (SM \setminus U) := \Upsilon \cap (SM \setminus U)$. Along the lines of double points in $\mathrm{int}(\Upsilon)$, we resolve the self-intersections and obtain Σ as in Figure 7. Finally, near the intersections $\mathrm{int}(\Upsilon) \cap \partial\Upsilon$, the surface Σ is obtained as in Figure 8. Up to choosing the neighborhood U where the surgery takes place to be small enough, for each $z \in \Upsilon$, the orbit segment $\phi_{(-1,1)}(z)$ intersects Σ; analogously, for each $z \in \Sigma$, the orbit segment $\phi_{(-1,1)}(z)$ intersects Υ.

If a closed geodesic γ is not simple, one can apply Fried surgeries to resolve the self-intersections of its Birkhoff annulus $A(\dot\gamma)$ and produce a surface of section. More generally, one can apply Fried surgeries to produce a surface of section out of the Birkhoff annuli of any finite collection of closed geodesics.

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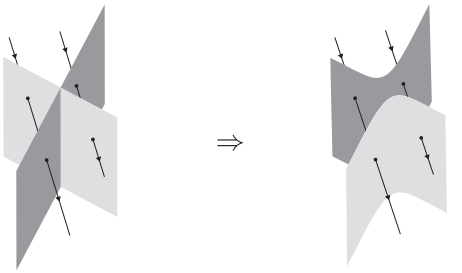

FIGURE 7. Fried surgery to resolve the double points in int(Υ).

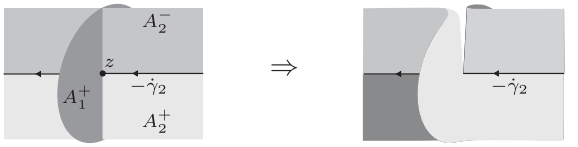

FIGURE 8. Fried surgery to resolve the lines of double points near the boundary point $z \in \partial\Upsilon$.

4.2. *Complete system of closed geodesics.* Let (M, g) be a closed connected oriented Riemannian surface with unit tangent bundle $\pi : SM \to M$ and geodesic flow $\phi_t : SM \to SM$. We introduce the following notion, which we will employ in the next subsection.

Definition 4.1. A *complete system of closed geodesics* $\mathcal{G}$ is a non-empty finite collection of closed geodesics of (M, g), together with a (possibly empty) subcollection $\mathcal{K} \subset \mathcal{G}$, satisfying the following properties. We set

$$K := \bigcup_{\gamma \in \mathcal{K}} (\dot{\gamma} \cup -\dot{\gamma}), \quad R := \bigcup_{\gamma \in \mathcal{G} \setminus \mathcal{K}} (\dot{\gamma} \cup -\dot{\gamma}), \quad U := M \setminus \bigcup_{\gamma \in \mathcal{G}} \gamma.$$

(i) Every $\gamma \in \mathcal{K}$ is a non-degenerate contractible waist disjoint from all the other closed geodesics in $\mathcal{G} \setminus \{\gamma\}$. In particular, K is a hyperbolic invariant subset for the geodesic flow.

(ii) No complete orbit $\phi_{(-\infty,\infty)}(z)$ is entirely contained in SU, and the trapped sets of SU are given by

$$\mathrm{trap}_+(SU) = W^s(K) \setminus K, \quad \mathrm{trap}_-(SU) = W^u(K) \setminus K.$$

For this reason, we briefly call $\mathcal{K}$ the *limit subcollection* of $\mathcal{G}$ and K the *limit set*.

(iii) The invariant subset R admits an open neighborhood $N \subset SM$ and a positive number $\ell > 0$ such that, for each $z \in N$, the orbit segment $\phi_{(0,\ell]}(z)$ is not contained in SU.

We stress that the closed geodesics in $\mathcal{G} \setminus \mathcal{K}$ are not necessarily simple and can have mutual intersections as well. A complete system of closed geodesics with empty limit subcollection will produce a Birkhoff section. For this purpose, we need the following lemma.

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LEMMA 4.2. *If the limit subcollection $\mathcal{K}$ is empty, then there is a constant $\ell' > 0$ such that every geodesic segment of length ℓ' intersects some geodesic in $\mathcal{G}$.*

Proof. We denote by Υ the union of the Birkhoff annuli of the closed geodesics in $\mathcal{G}$, that is,

$$\Upsilon := \bigcup_{\gamma \in \mathcal{G}} (A(\dot{\gamma}) \cup A(-\dot{\gamma})).$$

Since $\mathcal{K} = \varnothing$, we have empty trapped sets $\mathrm{trap}_{\pm}(SU) = \varnothing$. Therefore, for each $z \in SM \setminus \Upsilon$, there exists $\ell_z > 1$ such that the orbit segment $\phi_{[1,\ell_z-1]}(z)$ intersects $\mathrm{int}(\Upsilon)$ transversely. The transversality guarantees that there exists an open neighborhood $N_z \subset SM$ of z such that, for each $z' \in N_z$, the orbit segment $\phi_{[0,\ell_z]}(z')$ intersects $\mathrm{int}(\Upsilon)$. Now consider the constant $\ell > 0$ and the open neighborhood N of R given by property (iii) above. Since SM is compact, there exist finitely many points $z_1, \ldots, z_n \in SM$ such that $N \cup N_{z_1} \cup \cdots \cup N_{z_n} = SM$. For $\ell' := \max\{\ell, \ell_{z_1}, \ldots, \ell_{z_n}\}$, we conclude that, for each $z \in SM$, the orbit segment $\phi_{[0,\ell']}(z)$ intersects Υ. Namely, every geodesic segment of length ℓ' intersects some geodesic in $\mathcal{G}$. □

The arguments of Contreras and Mazzucchelli's [**CM22**, §4] allow us to produce, out of a suitable complete system of closed geodesics with non-empty limit subcollection, a new such complete system with strictly smaller limit subcollection. The setting of [**CM22**, §4] employs Colin, Dehornoy and Rechtman's broken book decompositions [**CDR23**], but it turns out that the arguments go through in our simpler setting as well. We include the details in the rest of this section, for the reader's convenience. We begin with the following preliminary lemma due to Colin, Dehornoy and Rechtman [**CDR23**, Lemma 4.9], which is based on an argument originally due to Hofer, Wysocki and Zehnder [**HWZ03**, Proposition 7.5].

LEMMA 4.3. *If the limit set K is non-empty and satisfies the transversality condition*

$$W^u(K) \pitchfork W^s(K),$$

then there exists $\gamma \in \mathcal{K}$ such that $W \cap W^s(\dot{\gamma}) \neq \varnothing$ for each path-connected component $W \subseteq W^u(\dot{\gamma}) \setminus \dot{\gamma}$. Namely, the closed orbit $\dot{\gamma}$ has homoclinics in all path-connected components of $W^u(\dot{\gamma}) \setminus \dot{\gamma}$.

Proof. We first show that, for each $\gamma \in \mathcal{K}$ and for each connected component $W \subseteq W^u(\dot{\gamma}) \setminus \dot{\gamma}$, there exists a heteroclinic

$$W \cap W^s(K) \neq \varnothing. \tag{4.1}$$

We prove this by contradiction, assuming that $W \cap W^s(K) = \varnothing$.

The path-connected component W is an immersed cylinder in SM with one end equal to $\dot{\gamma}$. Let $S \subset W$ be an embedded essential circle that is C^1-close to $\dot{\gamma}$ and transverse to the geodesic vector field, so that its base projection $\pi(S)$ does not intersect γ (Lemma 2.7) nor any other closed geodesic in the collection $\mathcal{G}$. We consider the union of the Birkhoff

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annuli

$$\Upsilon' := \bigcup_{\zeta \in \mathcal{G}\setminus\{\gamma\}} (A(\dot\zeta) \cup A(-\dot\zeta))$$

and an open neighborhood $V \subset SM$ of Υ' such that $V \cap \phi_{(-\infty,0]}(S) = \varnothing$. We apply Fried surgery, as explained in §4.1, to resolve the self-intersection points of Υ' with its interior (if there are any) and produce surfaces of section $\Sigma' \subset V$ such that

$$\phi_{(-1,1)}(z) \cap \Sigma' \neq \varnothing \quad \text{for all } z \in \Upsilon'. \tag{4.2}$$

We set

$$\Upsilon = \Upsilon' \cup A(\dot\gamma) \cup A(-\dot\gamma), \quad \Sigma = \Sigma' \cup A(\dot\gamma) \cup A(-\dot\gamma).$$

For each $z \in S$, we have $\phi_{(-\infty,0]}(z) \subset SU$, whereas $\phi_{(-\infty,\infty)}(z)$ is not entirely contained in SU and thus intersects Υ. This, together with (4.2), implies that there exists a minimal $t_z > 0$ such that the orbit $t \mapsto \phi_t(z)$ intersects Σ transversely for $t = t_z$. This transversality, together with the compactness of the circle S, implies that the function $z \mapsto t_z$ is smooth on S, and we obtain an embedded circle

$$S_0 := \{\phi_{t_z}(z) \mid z \in S\} \subset W \cap \Sigma.$$

We consider the first return time

$$\tau : \operatorname{int}(\Sigma) \to (0, +\infty], \quad \tau(z) := \inf\{t > 0 \mid \phi_t(z) \in \Sigma\},$$

which is smooth on the open subset $\Sigma_0 := \tau^{-1}(0, \infty)$. The first return map

$$\psi : \Sigma_0 \to \operatorname{int}(\Sigma), \quad \psi(z) = \phi_{\tau(z)}(z)$$

is a diffeomorphism onto its image that preserves the area form $d\lambda|_{\operatorname{int}(\Sigma)}$, where λ is the Liouville contact form of SM.

Since $S_0 \cap W^s(K) = \varnothing$, we have $S_0 \cap \operatorname{trap}_+(SU) = \varnothing$, and therefore $\psi^n(S_0) \subset \Sigma_0$ for all $n \geq 0$. We obtained an infinite sequence $S_n := \psi^n(S_0)$ of pairwise disjoint embedded circles in the interior of the surface of section Σ. Since the unstable manifold $W^u(\dot\gamma)$ is an immersed surface tangent to the geodesic vector field, the 2-form $d\lambda|_W$ vanishes identically. If S_n bounds a disk $B_n \subset \operatorname{int}(\Sigma)$, then we denote by $A_n \subset W$ the compact annulus with boundary $\partial A_n = \dot\gamma \cup S_n$, and Stokes' theorem implies that

$$\operatorname{area}(B_n, d\lambda) = \int_{B_n} d\lambda = \int_{A_n \cup B_n} d\lambda = \int_{\dot\gamma} \lambda = L(\gamma),$$

where $L(\gamma) > 0$ is the length of the simple closed geodesic γ. In particular, if S_{n_1}, S_{n_2} bound disks $B_{n_1}, B_{n_2} \subset \operatorname{int}(\Sigma)$ for some distinct $n_1, n_2 \geq 0$, we have $B_{n_1} \cap B_{n_2} = \varnothing$. This, together with the finiteness of the area

$$\operatorname{area}(\operatorname{int}(\Sigma), d\lambda) = \int_{\Sigma} d\lambda = \int_{\partial\Sigma} \lambda,$$

readily implies that there exist at most finitely many $n \geq 0$ such that S_n is contractible in $\operatorname{int}(\Sigma)$. Therefore, since the embedded circles S_n are pairwise disjoint, there exist distinct $n_1, n_2 \geq 0$ and an embedded compact annulus $A \subset \operatorname{int}(\Sigma)$ with boundary

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$\partial A = S_{n_1} \cup S_{n_2}$. Let $A' \subset W$ be the embedded compact annulus with boundary $\partial A' = S_{n_1} \cup S_{n_2}$, so that the union $A \cup A'$ is a piecewise smooth embedded torus in SM. By Stokes theorem,

$$\int_A d\lambda = \int_{A \cup A'} d\lambda = 0,$$

which contradicts the fact that $d\lambda|_A$ is an area form. This proves the existence of heteroclinics (4.1).

Our transversality assumption on the heteroclinics, together with the shadowing lemma from hyperbolic dynamics [**FH19**, Theorem 5.3.3], implies that, for all $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}$ and for each path-connected component $W \subset W^u(\dot\gamma_1) \setminus \dot\gamma_1$, if there are heteroclinics $W \cap W^s(\dot\gamma_2) \neq \varnothing$ and $W^u(\dot\gamma_2) \cap W^s(\dot\gamma_3) \neq \varnothing$, then there are also heteroclinics $W \cap W^s(\dot\gamma_3) \neq \varnothing$.

For each $\gamma \in \mathcal{K}$, we fix arbitrary path-connected components $W_{\pm\dot\gamma} \subset W^u(\pm\dot\gamma) \setminus \pm\dot\gamma$. We have already proved that every such path-connected component must contain a heteroclinic to K. Therefore, there exists a sequence of oriented waists $\gamma_i \in \mathcal{K}$ with heteroclinics $W_{\dot\gamma_i} \cap W^s(\dot\gamma_{i+1}) \neq \varnothing$. The same waist may appear at different times with opposite orientations in the sequence. Nevertheless, since the collection $\mathcal{G}$ is finite, there exists $n \leq 2\,\#\mathcal{G} + 1$ such that $\gamma_1 = \gamma_n$ as oriented waists. This implies that $W_{\dot\gamma_1} \cap W^s(\dot\gamma_1) \neq \varnothing$. □

LEMMA 4.4. *If the limit set K is non-empty and satisfies the transversality condition*

$$W^u(K) \pitchfork W^s(K),$$

then there exists $\gamma \in \mathcal{K}$ and another complete system of closed geodesics with limit subcollection contained in $\mathcal{K} \setminus \{\gamma\}$.

Proof. By Lemma 4.3, there exists $\gamma \in \mathcal{K}$ whose associated periodic orbit $\dot\gamma$ has homoclinics in all path-connected components of $W^u(\dot\gamma) \setminus \dot\gamma$. Since γ is a non-degenerate waist, Lemma 2.7 implies that there exists a tubular neighborhood $A \subset M$ of γ and an open neighborhood $V \subset SM$ of $\dot\gamma$ such that, if we denote by $W \subset V \cap W^u(\dot\gamma)$ the path-connected component containing $\dot\gamma$, the restriction of the base projection $\pi|_W : W \to A$ is a diffeomorphism. An analogous statement holds for the stable manifold $W^s(\dot\gamma)$. Therefore, for any homoclinic point $z \in W^u(\dot\gamma) \cap W^s(\dot\gamma) \setminus \dot\gamma$, the corresponding geodesic $\zeta(t) := \pi \circ \phi_t(z)$ is contained in $A \setminus \gamma$ provided $|t|$ is large enough.

The complement $A \setminus \gamma$ is the disjoint union of two open annuli A_1 and A_2, and therefore $W \setminus \dot\gamma$ is the disjoint union of the open annuli $W_1 := \pi|_W^{-1}(A_1)$ and $W_2 := \pi|_W^{-1}(A_2)$. Our assumption on γ implies that both W_1 and W_2 intersect the stable manifold $W^s(\dot\gamma)$, and we fix homoclinic points $z_i \in W_i \cap W^s(\dot\gamma)$ such that, if we denote by $\zeta_i(t) := \pi \circ \phi_t(z)$ the associated geodesics, we have $\zeta_i(t) \in A_i$ for all $t \leq 0$. We have two possible cases.

Case 1: There exists $i \in \{1, 2\}$ such that $\zeta_i(t) \in A_{3-i}$ for all $t > 0$ large enough. For each $\delta > 0$, there exist arbitrarily large $a, b > 0$ such that the points $\phi_{-a}(z_i)$ and $\phi_b(z_i)$ are δ-close (where the distance on SM is the one induced by the Riemannian metric g).

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We consider the orbit segment

$$\Gamma_\delta : [-a, b] \to SM, \quad \Gamma_\delta(t) = \phi_t(z_i),$$

and we extend it as a discontinuous periodic curve of period $\tau := a + b$.

Case 2: For all $i \in \{1, 2\}$, we have $\zeta_i(t) \in A_i$ for all $t > 0$ large enough. For each $\delta > 0$, there exist arbitrarily large $a_1, b_1, a_2, b_2 > 0$ such that the points $\phi_{-a_i}(z_i)$ and $\phi_{b_i}(z_i)$ are δ-close. We consider the orbit segments

$$\Gamma_{\delta,i} : [-a_i, b_i] \to SM, \quad \Gamma_{\delta,i}(t) = \phi_t(z_i),$$

and we define Γ_δ to be the discontinuous periodic curve of period $\tau := a_1 + b_1 + a_2 + b_2$ obtained by extending periodically the concatenation of $\Gamma_{\delta,1}$ and $\Gamma_{\delta,2}$.

In both cases, for each $\delta > 0$, we obtained a periodic δ-pseudo orbit Γ_δ of the geodesic flow with arbitrarily large minimal period. Moreover, the projection $\pi \circ \Gamma_\delta$ makes a δ-jump from A_1 to A_2. The shadowing lemma from hyperbolic dynamics [**FH19**, Theorem 5.3.3] implies that, for each $\epsilon > 0$, there exist $\delta > 0$ and $z \in SM$ such that the orbit $t \mapsto \phi_t(z)$ is periodic and pointwise ϵ-close to Γ_δ up to a time reparameterization with Lipschitz constant ϵ. Up to choosing $\epsilon > 0$ small enough, the corresponding closed geodesic $\zeta(t) := \pi \circ \phi_t(z)$ must intersect γ transversely.

Let $\mathcal{K}'$ be the subcollection of those simple closed geodesics $\eta \in \mathcal{K}$ that intersect ζ, which is non-empty since it contains γ. The collection $\mathcal{G} \cup \{\zeta\}$ is a complete system of closed geodesics with limit subcollection $\mathcal{K} \setminus \mathcal{K}'$. □

4.3. *Proof of Theorems A, C, D and E.* We first single out a suitable family of non-contractible waists, which is always available in every closed orientable Riemannian surface of positive genus.

LEMMA 4.5. *On any closed orientable Riemannian surface (M, g) of genus $G \geq 1$, there exist waists $\gamma_1, \ldots, \gamma_{2G}$ such that:*

(i) *$\gamma_i \cap \gamma_{i+1}$ is a singleton for all $i \in \{1, \ldots, 2G - 1\}$;*
(ii) *$\gamma_i \cap \gamma_j = \varnothing$ if $|i - j| \geq 2$;*
(iii) *$M \setminus (\gamma_1 \cup \cdots \cup \gamma_{2G})$ is simply connected;*
(iv) *every γ_i is a waist.*

Proof. We first consider non-contractible embedded loops $\zeta_1, \ldots, \zeta_{2G} \subset M$ such that $\zeta_i \cap \zeta_{i+1}$ is a singleton for all $i \in \{1, \ldots, 2G - 1\}$, $\zeta_i \cap \zeta_j = \varnothing$ if $|i - j| \geq 2$, and $M \setminus (\zeta_1 \cup \cdots \cup \zeta_{2G})$ is simply connected (Figure 9). We denote by $\mathcal{C}_i \subset \mathrm{Emb}(S^1, M)$ the connected component containing ζ_i. Notice that the $\mathcal{C}_i$ are pairwise distinct and the embedded loops $\zeta_1, \ldots, \zeta_{2G}$ are in minimal position: that is,

$$\#(\eta_i \cap \eta_j) \geq \#(\zeta_i \cap \zeta_j) \quad \text{for all } i < j,\ \eta_i \in \mathcal{C}_i,\ \eta_j \in \mathcal{C}_j.$$

By Lemma 2.1, for each $i = 1, \ldots, 2G$, there exists a waist $\gamma_i \in \mathcal{C}_i$ such that

$$L(\gamma_i) = \min_{\gamma \in \mathcal{C}_i} L(\gamma). \tag{4.3}$$

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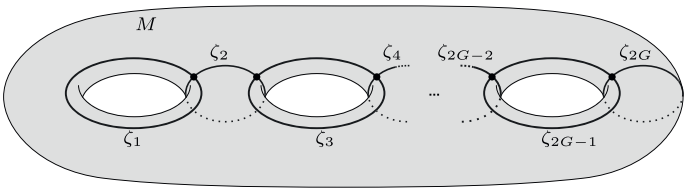

FIGURE 9. The embedded loops $\zeta_1, \dots, \zeta_{2G}$.

We recall that a geodesic bigon is a compact disk $D \subset M$ whose boundary is the union of two geodesic arcs. For each $i < j$, there is no geodesic bigon bounded by an arc in $\alpha_i \subset \gamma_i$ and an arc $\alpha_j \subset \gamma_j$. Indeed, if there were such a geodesic bigon with $L(\alpha_i) \leq L(\alpha_j)$, we could further shrink the simple closed geodesics γ_j and obtain a shorter embedded loop in the same homotopy class $\mathcal{C}_j$, which would contradict (4.3). By the bigon criterium [**FM12**, Proposition 1.7], the waists $\gamma_1, \dots, \gamma_{2G}$ are in minimal position. Therefore,

$$\#(\gamma_i \cap \gamma_j) = \#(\zeta_i \cap \zeta_j) \quad \text{for all } i < j.$$

To prove that $B := M \setminus (\gamma_1 \cup \cdots \cup \gamma_{2G})$ is simply connected, we cut open the surface M at the simple closed geodesics $\gamma_2, \gamma_4, \dots, \gamma_{2G}$, and we thus obtain a compact surface M' of genus zero with $2G$ boundary components. Each waist γ_{2i}, with $i = 1, \dots, G$, corresponds to two boundary components $\gamma_{2i}^+, \gamma_{2i}^- \subset \partial M'$. Each waist γ_{2i+1}, with $i = 1, \dots, G-1$, splits into two embedded arcs $\gamma_{2i+1}^+, \gamma_{2i+1}^-$; the arc $\gamma_{2i+1}^{\pm}$ joins the boundary components $\gamma_{2i}^{\pm}$ and $\gamma_{2i+2}^{\pm}$. Finally, the waist γ_1 corresponds to an embedded arc γ_1^+ joining the boundary components γ_2^+ and γ_2^-. We now cut open M' along the embedded arcs $\gamma_{2i+1}^{\pm}$ and obtain a compact disk M''. The interior $\mathrm{int}(M'')$ is diffeomorphic to B. □

By means of Fried surgery, we produce a connected surface of section out of the union of the Birkhoff annuli of the waists given by Lemma 4.5.

LEMMA 4.6. *Let $\gamma_1, \dots, \gamma_{2G}$ be the waists provided by Lemma 4.5 and consider the union of the associated Birkhoff annuli*

$$\Upsilon := \bigcup_{i=1}^{2G} (A(\dot{\gamma}_i) \cup A(-\dot{\gamma}_i)). \tag{4.4}$$

For each open neighborhood $V \subset SM$ of Υ, there exists a surface of section

$$\iota : \Sigma \looparrowright V$$

with the following properties.

(i) *Σ is a compact connected surface of genus one and $8G - 4$ boundary components.*

(ii) *The restriction*

$$\iota|_{\partial\Sigma} : \partial\Sigma \to \bigcup_{i=1}^{2G} (\dot{\gamma}_i \cup -\dot{\gamma}_i)$$

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is a double cover. The preimages $\iota^{-1}(\dot\gamma_i)$ *and* $\iota^{-1}(-\dot\gamma_i)$ *are connected if* $i \in \{1, 2G\}$, *and they have two connected components each if* $G > 1$ *and* $2 \le i \le 2G-1$.

(iii) *There exist* $\ell > 0$ *such that, for each* $z \in SM$ *sufficiently close to* $\partial\Sigma$, *we have* $\phi_{[-\ell,0)}(z) \cap \Sigma \neq \varnothing$ *and* $\phi_{(0,\ell]}(z) \cap \Sigma \neq \varnothing$.

(iv) *For each* $z \in \Upsilon$, *we have* $\phi_{(-1,1)}(z) \cap \Sigma \neq \varnothing$, *and, for each* $z' \in \Sigma$, *we have* $\phi_{(-1,1)}(z') \cap \Upsilon \neq \varnothing$.

Proof. We see Υ as a non-injectively immersed surface in SM with boundary and interior

$$\partial\Upsilon = \bigcup_{i=1}^{2G} (\dot\gamma_i \cup -\dot\gamma_i), \quad \operatorname{int}(\Upsilon) = \bigcup_{i=1}^{2G} (\operatorname{int}(A_i^+) \cup \operatorname{int}(A_i^-)),$$

where $A_i^\pm := A(\pm\dot\gamma_i)$. We set $\ell_0 := \max\{L(\gamma_1), \dots, L(\gamma_{2G})\}$. Since γ_i intersects γ_{i-1} and γ_{i+1} for all $2 \le i \le 2G-1$, we have $\phi_{[2,\ell_0+2]}(z) \cap \operatorname{int}(\Upsilon) \neq \varnothing$ for all $z \in \partial\Upsilon$. Therefore, there exists an open neighborhood $N \subset SM$ of $\partial\Upsilon$ such that

$$\phi_{[1,\ell_0+3]}(z) \cap \operatorname{int}(\Upsilon) \neq \varnothing \quad \text{for all } z \in N. \tag{4.5}$$

Let $V \subset SM$ be an open neighborhood of Υ. We apply Fried surgeries in an arbitrarily small neighborhood of the self-intersection points of Υ, as explained in §4.1, and obtain a surface of section $\iota : \Sigma \looparrowright V$ with the same boundary $\partial\Sigma = \partial\Upsilon$ and satisfying point (iv) in the statement of the lemma. This, together with (4.5), implies point (iii) for $\ell := \ell_0 + 4$.

We claim that the interior $\operatorname{int}(\Sigma)$ is path connected. Indeed, near the point in $\operatorname{int}(A_1^+) \cap \partial A_2^+ \cap \partial A_2^-$, Fried surgery glues together the intersecting annuli A_1^+, A_2^- and A_2^+ in the same connected component of $\operatorname{int}(\Sigma)$, as is clear from Figures 6 and 8; analogously, near the point in $\operatorname{int}(A_1^-) \cap \partial A_2^+ \cap \partial A_2^-$, Fried surgery glues together the intersecting annuli A_1^-, A_2^- and A_2^+. Therefore, the surgery sends the four annuli A_1^+, A_1^-, A_2^+ and A_2^- to the same path-connected component of $\operatorname{int}(\Sigma)$. Assume, by induction, that Fried surgery sends the annuli $A_1^+, A_1^-, \dots, A_i^+, A_i^-$ to the same path-connected component $W \subset \operatorname{int}(\Sigma)$. Near the point in $A_i^+ \cap \partial A_{i+1}^+ \cap \partial A_{i+1}^-$, Fried surgery glues together the intersecting annuli A_i^+, A_{i+1}^- and A_{i+1}^+. Therefore, the surgery sends A_{i+1}^- and A_{i+1}^+ to the connected component W as well, and we conclude that $W = \operatorname{int}(\Sigma)$.

We now determine the number of connected components of the boundary $\partial\Sigma$ and how they cover the closed orbits $\pm\dot\gamma_i$. The simple closed geodesic γ_1 intersects γ_2 in one point, but does not intersect any of the other simple closed geodesics $\gamma_3, \dots, \gamma_{2G}$; therefore, $\dot\gamma_1$ intersects the interior of the Birkhoff annulus $\operatorname{int}(A_2^+)$ in one point, but does not intersect any of the other Birkhoff annuli $A_i^\pm$. Analogously, $-\dot\gamma_1$ intersects $\operatorname{int}(A_2^-)$ in one point but none of the other $A_i^\pm$. Therefore, there is a unique connected component of $\partial\Sigma$ that covers $\dot\gamma_1$, and such a connected component winds around $\dot\gamma_1$ twice; analogously, there is a unique connected component of $\partial\Sigma$ that covers $-\dot\gamma_1$, and such a connected component winds around $-\dot\gamma_1$ twice (Figure 10). The same conclusions hold for γ_{2G}. For each $i \in \{2, \dots, 2G-1\}$, γ_i intersects γ_{i-1} and γ_{i+1} in one point each, but does not intersect any other γ_j; therefore, $\dot\gamma_i$ intersects the interiors of the Birkhoff annuli $\operatorname{int}(A_{i-1}^-)$ and $\operatorname{int}(A_{i+1}^+)$ in one point each, but does not intersect any of the other Birkhoff annuli $A_j^\pm$; analogously, $-\dot\gamma_i$ intersects A_{i-1}^+ and A_{i+1}^- in one point each, but does not intersect

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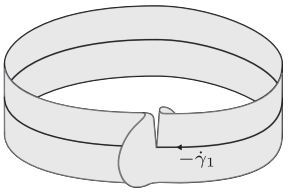

FIGURE 10. The surface Σ near the boundary component $-\dot{\gamma}_1$.

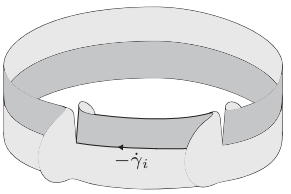

FIGURE 11. The surface Σ near the boundary component $-\dot{\gamma}_i$, for $2 \leq i \leq 2G-1$. Notice that there are two open annuli in $\mathrm{int}(\Sigma)$ having boundary on $-\dot{\gamma}_i$.

any other $A_j^{\pm}$. Therefore, there are two connected components of $\partial\Sigma$ that are mapped diffeomorphically to $\dot{\gamma}_i$ and two other connected components of $\partial\Sigma$ that are mapped diffeomorphically to $-\dot{\gamma}_i$ (Figure 11). All together, $\partial\Sigma$ has $8G-4$ boundary components.

Since $\mathrm{int}(\Sigma)$ is transverse to the geodesic vector field X, Σ is orientable. Indeed, the Liouville contact form λ of SM induces an area form $d\lambda|_{\mathrm{int}(\Sigma)}$. To compute the genus of Σ, we first triangulate every Birkhoff annulus $A_i^{\pm}$ with 6 vertices, 14 edges and 8 faces. We choose such triangulations so that, along the intersections $A_i^+ \cap A_i^-$, $A_i^{\pm} \cap A_{i+1}^{\pm}$ and $A_i^{\pm} \cap A_{i+1}^{\mp}$, the vertices and edges of $A_i^{\pm}$ and $A_{i+1}^{\pm}$ match as in Figure 12. All together, Υ is triangulated with $24\,G$ vertices, $56\,G$ edges and $32\,G$ faces (once again, we stress that Υ must be seen as an immersed compact surface with boundary, and therefore distinct Birkhoff annuli do not share vertices, edges or faces). The same number of edges and faces can be used to triangulate Σ as well; however, the triangulation of Υ provides one extra vertex for each of the points x_i, y_i, w_i, z_i depicted in Figure 12, where $i = 1, \ldots, 2G-1$, and therefore we need to throw away $8\,G-4$ vertices. All together, Σ is triangulated with $16\,G+4$ vertices, $56\,G$ edges and $32\,G$ faces, and therefore has Euler characteristic

$$\chi(\Sigma) = 16\,G + 4 - 56\,G + 32\,G = -8\,G + 4.$$

Since

$$-8\,G + 4 = \chi(\Sigma) = 2 - 2\,\mathrm{genus}(\Sigma) - \#\pi_0(\partial\Sigma) = 2 - 2\,\mathrm{genus}(\Sigma) - 8\,G + 4,$$

we conclude that Σ has genus one. □

In the following, we shall employ the notion of a complete system of closed geodesics, which we introduced in Definition 4.1. Theorem A is a consequence of the following statement.

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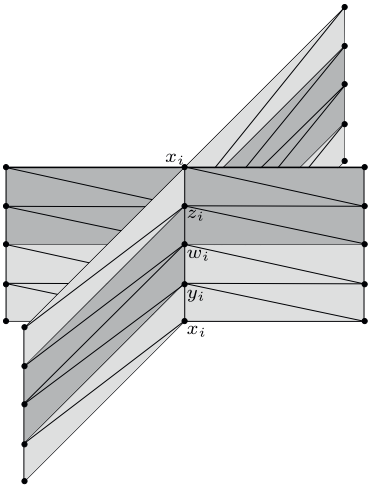

FIGURE 12. Triangulation of $A_i^+ \cup A_i^- \cup A_{i+1}^+ \cup A_{i+1}^-$.

THEOREM 4.7. *Let (M, g) be a closed oriented Riemannian surface of positive genus, let $\gamma_1, \ldots, \gamma_{2G}$ be the waists provided by Lemma 4.5 and let Σ be the surface of section provided by Lemma 4.6. If the open disk $M \setminus (\gamma_1 \cup \cdots \cup \gamma_{2G})$ does not contain any simple closed geodesic without conjugate points, then Σ is a Birkhoff section.*

Proof. With the terminology of Example 3.2, the open disk $B := M \setminus (\gamma_1 \cup \cdots \cup \gamma_{2G})$ is a convex geodesic polygon and, in particular, satisfies Assumption 3.1. Since B does not contain any simple closed geodesic without conjugate points, Theorem 3.4 implies that we have empty trapped sets $\mathrm{trap}_+(SB) = \mathrm{trap}_-(SB) = \varnothing$. Therefore, the collection of waists $\{\gamma_1, \ldots, \gamma_{2G}\}$ is a complete system of closed geodesics with empty limit subcollection. Let Υ be the union of the Birkhoff annuli of the waists $\gamma_1, \ldots, \gamma_{2G}$, as in (4.4). By Lemma 4.2, there exists $\ell > 0$ such that, for each $z \in SM$, the orbit segment $\phi_{[0,\ell]}(z)$ intersects Υ. This, together with Lemma 4.6(iv), implies that, for each $z \in SM$, the orbit segment $\phi_{(0,\ell+2)}(z)$ intersects Σ. Therefore, Σ is a Birkhoff section. □

The proof of Theorem C is analogous. We rewrite the statement for the reader's convenience.

THEOREM C. *Let (S^2, g) be a Riemannian 2-sphere and let γ be a simple closed geodesic with conjugate points whose complement $S^2 \setminus \gamma$ does not contain simple closed geodesics without conjugate points. Then both Birkhoff annuli $A(\dot\gamma)$ and $A(-\dot\gamma)$ are Birkhoff sections.*

Proof. Both connected components B_1 and B_2 of the complement $S^2 \setminus \gamma$ satisfy the convexity Assumption 3.1. Since B_1 and B_2 do not contain any simple closed geodesic without conjugate points, Theorem 3.4 implies that we have empty trapped sets $\mathrm{trap}_\pm(S(B_1 \cup B_2)) = \varnothing$. Therefore, $\{\gamma\}$ is a complete system of closed geodesics with empty limit subcollection. By Lemma 4.2, there exists $\ell > 0$ such that any geodesic segment of length

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ℓ intersects γ. This implies that both Birkhoff annuli $A(\dot\gamma)$ and $A(-\dot\gamma)$ are Birkhoff sections. □

We recall that a consequence of the Lusternik–Schnirelmann theorem is that every Riemannian 2-sphere admits a simple closed geodesic with conjugate points (see Remark 1.1).

LEMMA 4.8. *Let (M, g) be a closed connected oriented Riemannian surface all of whose contractible simple closed geodesics without conjugate points are non-degenerate. If $M \neq S^2$, we consider the collection of waists $\mathcal{G}' = \{\gamma_1, \dots, \gamma_{2G}\}$ given by Lemma 4.5; if, instead, $M = S^2$, we set $\mathcal{G}' = \{\gamma\}$, where γ is any simple closed geodesic with conjugate points. There exists a possibly empty finite collection $\mathcal{G}''$ of contractible simple closed geodesics that are pairwise disjoint and disjoint from all the closed geodesics in $\mathcal{G}'$ such that $\mathcal{G} := \mathcal{G}' \cup \mathcal{G}''$ is a complete system of closed geodesics whose limit subcollection $\mathcal{K}$ is given by the waists in $\mathcal{G}''$.*

Proof. First, assume that M has genus $G \geq 1$, so that $\mathcal{G}' = \{\gamma_1, \dots, \gamma_{2G}\}$. The complement $B \setminus (\gamma_1 \cup \cdots \cup \gamma_{2G})$ satisfies the convexity Assumption 3.1. Since every closed geodesic $\gamma \in \mathcal{G}'$ intersects some other closed geodesic in $\mathcal{G}'$, if we fix $\ell > L(\gamma)$, for every $z \in SM$ sufficiently close to $\dot\gamma \cup -\dot\gamma$, then the orbit segment $\phi_{(0,\ell]}(z)$ is not contained in SB. If B does not contain simple closed geodesics, then Theorem 3.4 implies that $\mathrm{trap}_{\pm}(SB) = \varnothing$, and therefore $\mathcal{G}'$ is a complete system of closed geodesics with empty limit subcollection $\mathcal{K} = \varnothing$. Now assume that B contains at least a closed geodesic. By Theorem 3.5, B contains a finite collection of pairwise disjoint simple closed geodesics $\mathcal{G}''$ such that:

- no complete geodesic is entirely contained in
$$U := B \setminus \bigcup_{\zeta \in \mathcal{G}''} \zeta;$$
- if we denote by $\mathcal{K}$ the subcollection of the waists in $\mathcal{G}''$, and we set
$$K := \bigcup_{\gamma \in \mathcal{K}} (\dot\gamma \cup -\dot\gamma),$$
the trapped sets of SU are given by
$$\mathrm{trap}_+(SU) = W^s(K) \setminus K, \quad \mathrm{trap}_-(SU) = W^u(K) \setminus K.$$

Since every $\gamma \in \mathcal{G}'' \setminus \mathcal{K}$ has conjugate points, Corollary 2.10 implies that there exists $\ell > 0$ such that, for every $z \in SM$ sufficiently close to $\dot\gamma \cup -\dot\gamma$, the orbit segment $\phi_{(0,\ell]}(z)$ is not contained in SU. This proves that the union $\mathcal{G} := \mathcal{G}' \cup \mathcal{G}''$ is a complete system of closed geodesics with limit subcollection $\mathcal{K}$.

Now assume that $M = S^2$ so that the collection $\mathcal{G}'$ consists in just one simple closed geodesic γ with conjugate points. Let B_1 and B_2 be the connected components of the complement $S^2 \setminus \gamma$. Since γ has conjugate points, each B_i satisfies the convexity Assumption 3.1. Corollary 2.10 implies that there exists $\ell > 0$ such that, for every $z \in SM$ sufficiently close to $\dot\gamma \cup -\dot\gamma$, the orbit segment $\phi_{(0,\ell]}(z)$ is not contained in $S(B_1 \cup B_2)$. If $S^2 \setminus \gamma$ does not contain any simple closed geodesic, then Theorem 3.4 implies

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that $\mathrm{trap}_{\pm}(sb:1) \cup \mathrm{trap}_{\pm}(sb:2) = \varnothing$, and therefore $\mathcal{G}'$ is a complete system of closed geodesics with empty limit subcollection $\mathcal{K} = \varnothing$. If $S^2 \setminus \gamma$ contains closed geodesics, then we argue as in the previous paragraph in both B_1 and B_2, and we find another collection $\mathcal{G}''$ of pairwise disjoint simple closed geodesics in $B_1 \cup B_2$ that, together with $\mathcal{G}'$, form a complete system of closed geodesics $\mathcal{G} := \mathcal{G}' \cup \mathcal{G}''$ whose limit subcollection $\mathcal{K}$ is given by the waists in $\mathcal{G}''$. □

We now have all the ingredients to prove Theorem D, which we restate for the reader's convenience.

THEOREM D. *Let (M, g) be a closed connected orientable Riemannian surface satisfying the following two conditions.*

(i) *All contractible simple closed geodesics without conjugate points are non-degenerate.*
(ii) *Any pair of not necessarily distinct contractible waists γ_1, γ_2 (if it exists) satisfies the transversality condition $W^u(\dot\gamma_1) \pitchfork W^s(\dot\gamma_2)$.*

Then the geodesic vector field of (M, g) admits a Birkhoff section.

Proof. Lemma 4.8 provides a complete system of closed geodesics. If its limit subcollection is non-empty, by applying Lemma 4.4 a finite number of times, we end up with another complete system of closed geodesics $\mathcal{G}$ with empty limit subcollection. We denote by Υ the union of the Birkhoff annuli of the closed geodesics in $\mathcal{G}$, that is,

$$\Upsilon := \bigcup_{\gamma \in \mathcal{G}} (A(\dot\gamma) \cup A(-\dot\gamma)).$$

By Lemma 4.2, there exists $\ell > 0$ such that, for each $z \in SM$, the orbit segment $\phi_{(0,\ell]}(z)$ intersects Υ. We apply Fried surgery to resolve the self-intersection of Υ (if there is any), and we end up with a surface of section $\Sigma \pitchfork SM$ such that, for each $z \in \Upsilon$, the orbit segment $\phi_{(-1,1)}(z)$ intersects Σ. Therefore, for each $z \in SM$, the orbit segment $\phi_{(0,\ell+2)}(z)$ intersects Σ, and we conclude that Σ is a Birkhoff section. □

Finally, Theorem E is a consequence of the following statement.

THEOREM 4.9. *Let (M, g) be a closed connected orientable surface. If M has genus $G \geq 1$, then we consider the collection of waists $\mathcal{G}' = \{\gamma_1, \ldots, \gamma_{2G}\}$ given by Lemma 4.5 and the surface of section $\Sigma' \pitchfork SM$ provided by Lemma 4.6, which has genus one and $8G - 4$ boundary components, all covering the closed geodesics in $\mathcal{G}'$; if, instead, $M = S^2$, then we set $\mathcal{G}' = \{\gamma_0\}$, where γ_0 is any simple closed geodesic with conjugate points, and we set $\Sigma' := A(\dot\gamma_0) \cup A(-\dot\gamma_0)$. Assume that the complement*

$$B := M \setminus \bigcup_{\gamma \in \mathcal{G}'} \gamma$$

contains at least a closed geodesic and no degenerate simple closed geodesics without conjugate points. Then there exists a finite collection $\mathcal{G}''$ of simple closed geodesics that are pairwise disjoint and disjoint from the closed geodesics in $\mathcal{G}'$ with the following properties.

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(i) *Every orbit of the geodesic flow intersects* $\Sigma' \cup \Sigma''$*, where*

$$\Sigma'' := \bigcup_{\gamma \in \mathcal{G}''} (A(\dot\gamma) \cup A(-\dot\gamma)) \subset SM \setminus \Sigma'.$$

(ii) *We denote by* $\mathcal{K}$ *the subcollection of waists in* $\mathcal{G}'$*, and*

$$K := \bigcup_{\gamma \in \mathcal{K}} (\dot\gamma \cup -\dot\gamma).$$

The trapped sets of $SM \setminus (\Sigma' \cup \Sigma'')$ *are given by*

$$\mathrm{trap}_+(SM \setminus (\Sigma' \cup \Sigma'')) = W^s(K) \setminus K,$$
$$\mathrm{trap}_-(SM \setminus (\Sigma' \cup \Sigma'')) = W^u(K) \setminus K.$$

(iii) *There exists* $\ell > 0$ *such that, for each* $z \in SM$ *sufficiently close to the boundary components* $\partial\Sigma' \cup \partial\Sigma'' \setminus K$*, we have* $\phi_{(0,\ell]}(z) \cap (\Sigma' \cup \Sigma'') \neq \varnothing$.

Proof. Lemma 4.8 provides a (possibly empty) finite collection $\mathcal{G}''$ of contractible simple closed geodesics that are pairwise disjoint and disjoint from all the closed geodesics in $\mathcal{G}'$ such that $\mathcal{G} := \mathcal{G}' \cup \mathcal{G}''$ is a complete system of closed geodesics whose limit subcollection $\mathcal{K}$ is given by the waists in $\mathcal{G}''$. We set

$$\Upsilon' := \bigcup_{\gamma \in \mathcal{G}'} (A(\dot\gamma) \cup A(-\dot\gamma)), \quad \Sigma'' := \bigcup_{\gamma \in \mathcal{G}''} (A(\dot\gamma) \cup A(-\dot\gamma)).$$

If $M = S^2$, then we have set $\Sigma' := \Upsilon'$. If, instead, $M \neq S^2$, then we require the surface of section Σ' provided by Lemma 4.5 to be contained in a sufficiently small neighborhood of Υ' so that $\Sigma' \cap \Sigma'' = \varnothing$, and by Lemma 4.6(iv),

$$\phi_{(-1,1)}(z) \cap \Sigma' \neq \varnothing \quad \text{for all } z \in \Upsilon', \quad \phi_{(-1,1)}(z') \cap \Upsilon' \neq \varnothing \quad \text{for all } z' \in \Sigma'.$$

We set

$$K := \bigcup_{\gamma \in \mathcal{K}} (\dot\gamma \cup -\dot\gamma), \quad R := \bigcup_{\gamma \in \mathcal{G}' \cup \mathcal{G}'' \setminus \mathcal{K}} (\dot\gamma \cup -\dot\gamma) = \partial\Sigma' \cup \partial\Sigma'' \setminus K.$$

By properties (ii) in Definition 4.1, every complete orbit $\phi_{(-\infty,\infty)}(z)$ intersects $\Sigma' \cup \Sigma''$, and

$$\mathrm{trap}_+(SM \setminus (\Sigma' \cup \Sigma'')) = \mathrm{trap}_+(SM \setminus (\Upsilon' \cup \Sigma'')) = W^s(K) \setminus K,$$
$$\mathrm{trap}_-(SM \setminus (\Sigma' \cup \Sigma'')) = \mathrm{trap}_-(SM \setminus (\Upsilon' \cup \Sigma'')) = W^u(K) \setminus K.$$

Moreover, by properties (iii) in Definition 4.1, there exist $\ell > 0$ and an open neighborhood $N \subset SM$ of R such that

$$\phi_{(0,\ell]}(z) \cap (\Upsilon' \cup \Sigma'') \neq \varnothing \quad \text{for all } z \in N.$$

If we take a sufficiently small open neighborhood $N' \subset N$ of R, then we also have

$$\phi_{(0,\ell+1]}(z) \cap (\Sigma' \cup \Sigma'') \neq \varnothing \quad \text{for all } z \in N'.$$

□

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