CONVERGENCE OF CONTINUED FRACTIONS

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1. Introduction. Let $\{s_n(z)\}$ be a given sequence of linear fractional transformations (or simply l.f.t.'s) of the form

$$s_n(z) = \frac{a_n}{b_n + z}, \qquad a_n \neq 0, n \geqq 1, \tag{1.1}$$

and let

$$S_1(z) = s_1(z); \quad S_n(z) = S_{n-1}(s_n(z)), \qquad n \geqq 2. \tag{1.2}$$

The sequence of l.f.t.'s $\{S_n(z)\}$ is called a *continued fraction generating sequence* (or simply a c.f.g. sequence). Sequences of l.f.t.'s $\{S_n(z)\}$ which are c.f.g. sequences are characterized by the property

$$S_n(\infty) = S_{n-1}(0), \qquad n \geqq 2. \tag{1.3}$$

A *continued fraction* is a sequence of constants $\{S_n(0)\}$ obtained from a c.f.g. sequence $\{S_n(z)\}$. We shall subsequently extend this definition to allow for the degenerate case $a_n = 0$ for certain values of n. The value $S_n(0)$ is called the nth *approximant* and the numbers a_n and b_n are referred to as the *elements* of the continued fraction. To exhibit the elements explicitly we write

$$S_n(0) = \mathop{\mathrm{K}}_{k=1}^{n} (a_k/b_k) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n}. \tag{1.4}$$

The symbol

$$\mathop{\mathrm{K}}_{n=1}^{\infty} (a_n/b_n) \tag{1.5}$$

is used to denote both the continued fraction (i.e., the sequence $\{S_n(0)\}$) and, when it converges, the value of its limit.

Let $V_0, V_1, V_2, \ldots$ denote a given sequence of closed circular or closed half-plane regions such that zero is an interior point of each V_n. In the present paper we shall prove some new convergence criteria for continued fractions $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$ which have the property

$$s_n(V_n) \subset V_{n-1}, \qquad n \geqq 1, \tag{1.6}$$

where $s_n(V_n)$ denotes the set of all images under (1.1) of points in V_n. Necessary and sufficient conditions for the elements a_n and b_n to satisfy the fundamental property (1.6) are established in §2. Also in §2 we show that the continued

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fractions with the property (1.6) subsume an important subclass of positive-definite continued fractions. It is easily seen that (1.6) is equivalent to

$$(1.7) \qquad S_n(V_n) \subset S_{n-1}(V_{n-1}) \subset V_0, \qquad n \geqq 2.$$

We remark that if zero is not contained in any of the sets $b_n + V_n$, $n \geqq 1$, then $a_n = 0$ for some n implies that $s_n(z) = 0$ for all $z \in V_n$. Hence, if $a_m \neq 0$, $m < n$, and $a_n = 0$, it is easily shown that $S_m(0) = S_{n-1}(0)$ for $m \geqq n$ so that the sequence $\{S_n(0)\}$ converges trivially. We shall extend the definition of a continued fraction to include this degenerate case. With one exception (Theorem 4.3), however, we have required (implicitly) the condition $0 \notin (b_n + V_n)$. Therefore, in the remainder of this paper (except in Theorem 4.3) it will suffice to consider only the case $a_n \neq 0$, $n \geqq 1$, so that the l.f.t.'s $s_n(z)$ and $S_n(z)$ will be non-singular.

In this study, extensive use will be made of a sequence of l.f.t.'s $\{T_n(z)\}$ related to the c.f.g. sequence $\{S_n(z)\}$ by a transformation

$$(1.8) \qquad T_n(z) = v_0 \circ S_n \circ v_n^{-1}(z),$$

where $v_n(z)$ is an l.f.t. which maps V_n onto the closed unit disk U: $|z| \leqq 1$; that is,

$$(1.9) \qquad v_n(V_n) = U, \qquad n \geqq 0.$$

Here "$\circ$" denotes functional composition (e.g., $f \circ g(z) = f[g(z)]$). It follows from (1.8) and (1.9) that

$$(1.10) \qquad T_n(U) \subset T_{n-1}(U) \subset U, \qquad n \geqq 2,$$

if and only if (1.6) holds. Also from (1.8)

$$(1.11) \qquad S_n(0) = v_0^{-1} \circ T_n(v_n(0)),$$

so that the continued fraction $\{S_n(0)\}$ converges if and only if the sequence $\{T_n(v_n(0))\}$ converges. By our initial assumption, zero is an interior point of each V_n, and so we may take $v_n(0) = 0$, $n \geqq 1$. This will be done in the remainder of the paper. Hence, to prove that the continued fraction $\{S_n(0)\}$ converges, it is sufficient to show that the sequence $\{T_n(z)\}$ converges in the interior of U.

In 1963, Thron (**6**) gave a characterization of the convergence behaviour of sequences of l.f.t.'s $\{T_n(z)\}$ having the property (1.10) and, in the same paper, he used results derived from that work to give new proofs of the Pringsheim criterion and the general parabola theorem. More recently, Hillam and Thron (**1**) applied the same type of analysis to obtain a new convergence criterion for continued fractions $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$.

In the present paper we have employed a modification of the method introduced by Thron. In §3 we have proved a generalization of the result of Hillam and Thron (**1**, Theorem 2), allowing for variable circular regions V_n. Section 4 contains a convergence criterion (Theorem 4.3) which extends an earlier result of Thron (**4**, Theorem A). Section 5 includes an extension

(Theorem 5.1)of the general parabola theorem (**6**, Theorem 8.1) for continued fractions $\mathrm{K}_{n=1}^{\infty}(a_n/1)$, which allows variable parabolic element regions. Theorems 4.3 and 5.1 have an overlapping relation with certain earlier results of Wall (e.g., **7**, Theorem 31.3). As an example of the usefulness of these general convergence criteria, we derive in §6 two new convergence theorems for a class of continued fraction expansions.

2. Basic lemmas. For use in the following sections we shall prove now two basic lemmas giving necessary and sufficient conditions for the property (1.6) to hold. In the first case, the regions V_n are closed circular disks; in the second, they are half-planes with the boundary included. In both cases the V_n contain zero as an interior point.

LEMMA 2.1. *Let*

$$s_n(z) = a_n/(b_n + z), \qquad a_n \neq 0,\ n \geqq 1, \tag{2.1}$$

and let V_n be the circular region defined by

$$V_n = \{z\colon |z - D_n| \leqq q_n,\ |D_n| < q_n\}, \qquad n \geqq 1, \tag{2.2}$$

where the D_n are complex numbers and the q_n are positive. Then

$$s_n(V_n) \subset V_{n-1}, \qquad n \geqq 2, \tag{2.3}$$

if and only if

$$|a_n(\bar{b}_n + \bar{D}_n) - D_{n-1}(|b_n + D_n|^2 - q_n^2)| + |a_n|q_n \leqq q_{n-1}(|b_n + D_n|^2 - q_n^2), \qquad n \geqq 2. \tag{2.4}$$

Before proving the lemma we remark that (2.3) implies that

$$|b_n + D_n| > q_n, \tag{2.5}$$

and for the special case $D_n = 0$, $q_n = 1$, $n \geqq 2$, condition (2.4) reduces to the Pringsheim criterion

$$|b_n| \geqq |a_n| + 1. \tag{2.6}$$

Proof. It is easily verified that (2.5) is a necessary condition for (2.3). Using (2.5) we obtain by direct computation that

$$s_n(V_n) = \{z\colon |z - D_n^*| \leqq q_n^*\}, \tag{2.7}$$

where

$$D_n^* = \frac{a_n(\bar{b}_n + \bar{D}_n)}{|b_n + D_n|^2 - q_n^2}, \tag{2.8}$$

and

$$q_n^* = \frac{|a_n|q_n}{|b_n + D_n|^2 - q_n^2}. \tag{2.9}$$

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Thus for (2.3) to hold it is necessary and sufficient that

$$|D_n^* - D_{n-1}| + q_n^* \leqq q_{n-1} \tag{2.10}$$

be satisfied. This is equivalent to (2.4).

LEMMA 2.2. *Let*

$$s_n(z) = \frac{a_n}{b_n + z}, \qquad a_n \neq 0,\ n \geqq 1,$$

and let V_n *be the half-plane region defined by*

$$V_n = \{z: \operatorname{Re}(z \exp(-i\psi_n)) \geqq -|P_n|,\ P_n \neq 0\}, \qquad n \geqq 1. \tag{2.11}$$

where $P_n = p_n \exp(i\psi_n)$, $p_n > 0$, *and* ψ_n *is real. Then*

$$s_n(V_n) \subset V_{n-1}, \qquad n \geqq 2, \tag{2.12}$$

if and only if

$$|a_n| - \operatorname{Re}[a_n \exp(-i(\psi_n + \psi_{n-1}))] \leqq 2p_{n-1}[\operatorname{Re}(b_n \exp(-i\psi_n)) - p_n], \qquad n \geqq 1. \tag{2.13}$$

Proof. It is readily shown that

$$\operatorname{Re}(b_n \exp(-i\psi_n)) \geqq p_n \tag{2.14}$$

is a necessary condition for (2.12). We shall consider the equality and inequality as separate cases. If equality holds in (2.14), then by a direct computation we obtain for $s_n(V_n)$ the half-plane

$$s_n(V_n) = \{z: \operatorname{Re}(z \exp(i(\psi_n - \arg a_n))) \geqq 0\}. \tag{2.15}$$

Thus it is easily seen that for (2.12) to hold it is necessary and sufficient to have

$$\arg a_n = \psi_n + \psi_{n-1}, \tag{2.16}$$

which, in this case, is equivalent to (2.13). If the inequality holds in (2.14), then one finds that

$$s_n(V_n) = \{z: |z - E_n| \leqq w_n\}, \tag{2.17}$$

where

$$E_n = w_n \exp(i(\arg a_n - \psi_n)), \tag{2.18}$$

and

$$w_n = \frac{|a_n|}{2[\operatorname{Re}(b_n \exp(-i\psi_n)) - p_n]}. \tag{2.19}$$

Now let $B(V_{n-1})$ denote the boundary of V_{n-1} and let l_{n-1} be the line passing through E_n and perpendicular to $B(V_{n-1})$. Then, clearly, (2.12) will hold if and only if we have both of the following conditions: (a) $E_n \in V_{n-1}$ and (b) $|E_n - d_{n-1}| \geqq w_n$, where d_{n-1} is the point at which l_{n-1} intersects $B(V_{n-1})$. One can easily show that (a) is equivalent to

(2.20) $$-\text{Re}[a_n \exp(-i(\psi_n + \psi_{n-1}))] \leqq 2p_{n-1}[\text{Re}(b_n \exp(-i\psi_n)) - p_n].$$

This condition is certainly implied by (2.13). The proof of the lemma is completed by showing that (b) is equivalent to (2.13), which follows from the fact that

(2.21) $$d_{n-1} = \exp(i\psi_{n-1})[-p_{n-1} + i\,\text{Im}(E_n \exp(-i\psi_{n-1}))],$$

and hence

(2.22) $$|E_n - d_{n-1}| = w_n \cos(\arg a_n - \psi_n - \psi_{n-1}) + p_{n-1}.$$

Before continuing with the development of convergence criteria in the following sections, we shall digress for a moment to show that the continued fractions (1.5), having the property (1.6) where the V_n are half-planes with the origin an interior point, subsume an important subclass of positive-definite continued fractions. We make use of the fact (**7**, Corollary 16.2) that positive-definite continued fractions are of the form

(2.23) $$\frac{1}{B_1 + z_1} - \frac{A_1^2}{B_2 + z_2} - \frac{A_2^2}{B_3 + z_3} - \ldots,$$

where the A_n and B_n are complex constants satisfying the conditions

(2.24) $$\text{Im}(B_n) \geqq 0, \qquad n \geqq 1,$$

and

(2.25) $$|A_n^2| - \text{Re}(A_n^2) \leqq 2\,\text{Im}(B_n)\text{Im}(B_{n+1})(1 - d_{n-1})d_n, \qquad n \geqq 1,$$

for some sequence of numbers $d_0, d_1, d_2, \ldots$ such that

(2.26) $$0 \leqq d_n \leqq 1, \qquad n \geqq 0.$$

The numbers $z_1, z_2, z_3, \ldots$ are complex variables. By use of Lemma 2.2 we may now prove the following theorem.

THEOREM 2.1. *Let a positive-definite continued fraction* (2.23) *be given such that*

(2.27) $$A_n \neq 0 \quad \textit{and} \quad \text{Im}(z_n) > 0, \qquad n \geqq 1.$$

Let

(2.28) $$a_1 = 1, \quad a_n = -A_{n-1}^2, \qquad n \geqq 2,$$
$$b_n = B_n + z_n, \qquad n \geqq 1.$$

Let V_n be the half-plane (2.11), *where*

(2.29) $$\psi_n = \pi/2, \qquad n \geqq 1,$$
$$p_n = \text{Im}(z_n) + \text{Im}(B_n)(1 - d_{n-1}), \qquad n \geqq 1,$$

where the d_n are the constants in the inequality (2.25) *and satisfy* (2.26). *If $s_n(z)$ is defined by* (1.1), *then* (1.6) *holds.*

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Proof. In view of (2.27), $p_n > 0$ and hence the half-plane V_n contains the origin in its interior. Now, using the notation (2.28) we may write (2.25) as

$$|a_{n+1}| + \operatorname{Re}(a_{n+1}) \leqq 2 \operatorname{Im}(B_n)\operatorname{Im}(B_{n+1})(1 - d_{n-1})d_n,$$

and it is easily shown, using (2.29), that

$$\begin{aligned}\operatorname{Im}(B_n)\operatorname{Im}(B_{n+1})(1 - d_{n-1})d_n &\leqq p_n[\operatorname{Im}(b_{n+1}) - p_{n+1}] \\ &= p_n[\operatorname{Re}(b_{n+1}\exp(-i\psi_{n+1})) - p_{n+1}].\end{aligned}$$

Thus (2.13) is satisfied by the a_n, b_n, ψ_n, and p_n and hence by Lemma 2.2 we have (2.12).

3. Variable circular regions. For use here and in the following two sections we shall develop some general properties of sequences of l.f.t.'s $\{T_n(z)\}$ having the property (1.10). First, it should be noted that $\{T_n(U)\}$ is a nested sequence of circular regions all contained in the unit disk U. Thus, if C_n and $r_n > 0$ denote the centre and radius of $T_n(U)$, respectively, then $T_n(z)$ can be written in the form

$$T_n(z) = C_n + R_n \frac{z + \bar{G}_n}{G_n z + 1}, \tag{3.1}$$

where

$$R_n = r_n \exp(i\omega_n), \quad r_n \to r \geqq 0,$$
$$G_n = g_n \exp(i\tau_n), \quad 0 \leqq g_n < 1,$$

and

$$|C_{n-1} - C_n| \leqq r_{n-1} - r_n.$$

It is easily shown that the sequence $\{C_n\}$ converges to a limit C, since $\{r_n\}$ converges monotonically to a limit $r \geqq 0$. The *limit point case* is said to occur when $r = 0$ and the *limit circle case* when $r > 0$.

Since our interest here lies in sequences $\{T_n(z)\}$ which are related to c.f.g. sequences $\{S_n(z)\}$ by a transformation (1.8), it is useful to write the characteristic property of c.f.g. sequences (1.3) in terms of the $T_n(z)$. By means of (1.8) this property becomes

$$T_n[v_n(\infty)] = T_{n-1}[v_{n-1}(0)]. \tag{3.2}$$

Now in view of (1.9), if V_n is a half-plane, ∞ lies on its boundary so that $|v_n(\infty)| = 1$. On the other hand, if V_n is a circular region, ∞ is in the exterior and hence $|v_n(\infty)| > 1$. As pointed out in the introduction, we shall choose the $v_n(z)$ so that $v_n(0) = 0$. With these conditions imposed, (3.2) implies that for each $n \geqq 2$, there exists a number K_n such that

$$\begin{aligned} T_n(K_n) &= T_{n-1}(0), \qquad n \geqq 2, \\ |K_n| &\geqq 1. \end{aligned} \tag{3.3}$$

We shall now prove the following useful result.

THEOREM 3.1. *Let* $\{T_n(z)\}$ *be a sequence of l.f.t.'s satisfying* (1.10) (*or equivalently* (3.1)) *and* (3.3). *If* $\lim r_n = r = 0$ (*i.e., limit point case*), *then* $\{T_n(z)\}$ *converges for all* z *in the closed unit disk* U: $|z| \leqq 1$. *If* $\lim r_n = r > 0$ (*i.e. limit circle case*), *then the following hold*:

(a) $\sum_{j=1}^{\infty} h_j$ *converges, where* $h_j = 1 - g_j$, *and hence* $g_j \to 1$;

(b) $\{T_n(z)\}$ *converges for all* z *such that* $|z| \neq 1$, *provided* $\{\exp(i(\omega_n - \tau_n))\}$ *converges*;

(c) $\{\exp(i(\omega_n - \tau_n))\}$ *converges if* $\sum_{j=1}^{\infty} R_j K_j (1 - g_j^2)/(G_j K_j + 1)$ *converges*;

(d) $\sum_{j=1}^{\infty} R_j K_j (1 - g_j^2)/(G_j K_j + 1)$ *converges if there exists an* $\epsilon > 0$ *such that*

$$|K_j| \geqq 1 + \epsilon, \qquad j \geqq 1. \tag{3.4}$$

Proof. In the limit point case it is clear that $\{T_n(z)\}$ converges for all z in U to a common limit. By use of (3.1), the equation in (3.3) becomes

$$C_n + R_n \frac{K_n + \bar{G}_n}{G_n K_n + 1} = C_{n-1} + R_{n-1} \bar{G}_{n-1}. \tag{3.5}$$

With the help of the inequality in (3.3), this implies that

$$\frac{r_n}{r_{n-1}} \leqq 1 - \frac{1 - g_{n-1}}{2},$$

from which it is easily deduced that

$$r_n \leqq l \prod_{j=1}^{n} \left(1 - \frac{h_j}{2}\right),$$

where l is a non-zero constant and $h_j = 1 - g_j$. Thus

$$r = \lim r_n \leqq \prod_{j=1}^{\infty} \left(1 - \frac{h_j}{2}\right),$$

so that if $\sum_{j=1}^{\infty} h_j$ diverges, the infinite product diverges to zero and hence $r = 0$ (limit point case). Therefore, in the limit circle case, $\sum_{j=1}^{\infty} h_j$ converges and we arrive at part (a) of the theorem.

To investigate the limit circle case further, it is useful to express $T_n(z)$ in the following form

$$T_n(z) = C_n + \frac{R_n}{G_n} \left[1 - \frac{1 - g_n^2}{G_n z + 1}\right]. \tag{3.6}$$

For then, since $C_n \to C$, $r_n \to r > 0$ and $g_n \to 1$, to prove the convergence of $\{T_n(z)\}$ for all z such that $|z| \neq 1$, it suffices to show that $\{\exp(i(\omega_n - \tau_n))\}$ converges (as asserted in part (b)), or, equivalently, that $\{R_n \bar{G}_n\}$ converges. Again making use of (3.5) we obtain

(3.7) $$R_n\bar{G}_n - R_{n-1}\bar{G}_{n-1} = (C_{n-1} - C_n) - R_nK_n\frac{1 - g_n^{\ 2}}{G_nK_n + 1}.$$

Successive application of this relation leads to

(3.8) $$R_n\bar{G}_n - R_m\bar{G}_m = (C_m - C_n) - \sum_{j=m+1}^{n} R_jK_j\frac{1 - g_j^{\ 2}}{G_jK_j + 1}, \qquad n > m,$$

from which part (c) follows. For m and n sufficiently large $(n > m)$,

(3.9) $$|R_n\bar{G}_n - R_m\bar{G}_m| \leqq |C_m - C_n| + 2(r + 1)\sum_{j=m+1}^{n} h_j\left|\frac{K_j}{G_jK_j + 1}\right|.$$

If the K_j are restricted by condition (3.4), then for j sufficiently large, the sequence

$$\left\{\frac{K_j}{G_jK_j + 1}\right\}$$

is bounded above. Therefore, the right side of (3.9) becomes arbitrarily small for all m and n sufficiently large since $\sum h_j$ converges. Hence, $\{R_n\bar{G}_n\}$ is a Cauchy sequence and the proof of the theorem is complete.

It is now a simple matter to prove the following theorem.

THEOREM 3.2. *If the elements a_n and b_n of the continued fraction* $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$ *satisfy condition* (2.4) *for some sequences of positive numbers $\{q_n\}$ and complex numbers $\{D_n\}$ such that*

(3.10) $$|D_n|/q_n \leqq 1 - \epsilon_1, \qquad \epsilon_1 > 0,\ n \geqq 1,$$

then the continued fraction converges to a value v such that

(3.11) $$\left|v - \frac{a_1(\bar{b}_1 + \bar{D}_1)}{|b_1 + D_1|^2 - q_1^{\ 2}}\right| \leqq \frac{|a_1|q_1}{|b_1 + D_1|^2 - q_1^{\ 2}}.$$

Proof. Let $\{V_n\}$ be the sequence of closed circular regions defined by (2.2). Then by Lemma 2.1, condition (2.4) implies (2.3). The transformation

(3.12) $$v_n(z) = \frac{-q_nz}{D_nz + (q_n^{\ 2} - |D_n|^2)}$$

satisfies (1.9) and

$$v_n(0) = 0, \qquad n \geqq 0.$$

Moreover, using (3.10) we obtain

$$|v_n(\infty)| = |q_n/D_n| \geqq 1 + \epsilon, \qquad n \geqq 0,$$

for some fixed positive number $\epsilon > 0$. Thus, $T_n(z)$, defined by (1.8) using (3.12), satisfies (1.10), (3.3), and (3.4). Hence, by Theorem 3.1 the sequence $\{T_n(z)\}$ converges at least for all z such that $|z| < 1$. In view of (1.11) this implies the convergence of the continued fraction $\{S_n(0)\}$. By (1.7), the limit v, to which the continued fraction converges, is contained in $S_1(V_1)$. This is the set described by (3.11), which completes the proof of the theorem.

We shall mention briefly a few interesting special cases of Theorem 3.2. By taking $D_n = 0$ we obtain

$$|a_n| \leqq q_{n-1}(|b_n| - q_n), \qquad 0 < \epsilon < q_n,\ n \geqq 1, \tag{3.13}$$

as a sufficient condition for convergence of the continued fraction $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$. The Pringsheim criterion (2.6) is obtained by letting $q_n = 1$, $n \geqq 1$, in (3.13). If we let $b_n = 1$, $0 < \epsilon < q_n < 1$, $n \geqq 1$, then (3.13) reduces to the well-known sufficient condition (7, p. 50)

$$|a_n| \leqq q_{n-1}(1 - q_n) \tag{3.14}$$

for convergence of the continued fraction $\mathrm{K}_{n=1}^{\infty}(a_n/1)$. Worpitzky's criterion $|a_n| \leqq \frac{1}{4}$ follows by taking $q_n = \frac{1}{2}$. Theorem 3.2 reduces to the result of Hillam and Thron (**1**, Theorem 2) by taking $D_n = D$ and $q_n = q$, $n \geqq 1$, where $|D|/q < 1$.

4. Variable half-plane regions. Theorem 3.1 sets forth some useful properties of sequences of l.f.t.'s $\{T_n(z)\}$ satisfying (1.10) and (3.3). These properties were applied in Theorem 3.2 to obtain a convergence criterion for continued fractions using variable circular regions V_n. In this section we shall develop some additional properties of the sequences $\{T_n(z)\}$ and shall apply these results together with Theorem 3.1 to obtain a continued fraction convergence criterion using variable half-plane regions V_n.

THEOREM 4.1. *Let* $\{T_n(z)\}$ *be a given sequence of l.f.t.'s satisfying* (1.10) *or, equivalently*, (3.1). *Let*

$$t_1(z) = T_1(z) \quad \textit{and} \quad t_n(z) = T_{n-1}^{-1}[T_n(z)], \qquad n \geqq 2. \tag{4.1}$$

Then

$$t_n(z) = \frac{\kappa_n z + \lambda_n}{\mu_n z + \nu_n}, \qquad n \geqq 1, \tag{4.2}$$

where

$$\begin{aligned} \kappa_n &= [G_n(C_{n-1} - C_n) + R_{n-1}\bar{G}_{n-1}G_n - R_n]M, \\ \lambda_n &= [(C_{n-1} - C_n) + R_{n-1}\bar{G}_{n-1} - R_n\bar{G}_n]M, \\ \mu_n &= -[G_nG_{n-1}(C_{n-1} - C_n) + R_{n-1}G_n - R_nG_{n-1}]M, \\ \nu_n &= -[G_{n-1}(C_{n-1} - C_n) + R_{n-1} - \bar{G}_nG_{n-1}R_n]M, \end{aligned} \tag{4.3}$$

where M is an arbitrary constant of proportionality. The transformation $t_n(z)$ will be normalized so that

$$\kappa_n\nu_n - \lambda_n\mu_n = 1 \tag{4.4}$$

if we choose

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(4.5) $$M = [R_{n-1}R_n(1 - g_n^2)(1 - g_{n-1}^2)]^{-1/2},$$

where $-\frac{1}{2}\pi < \arg M \leqq \frac{1}{2}\pi$.

The proof of this theorem consists of substituting the expression for $T_n(z)$ in (3.1) into (4.1) and carrying out the obvious computation.

THEOREM 4.2. *Let* $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$ *be a continued fraction with c.f.g. sequence* $\{S_n(z)\}$. *Let* $T_n(z)$ *be defined by* (1.8), *where* $v_n(z)$ *is an l.f.t. of the form*

(4.6) $$v_n(z) = \frac{\alpha_n z}{\gamma_n z + \delta_n}, \qquad n \geqq 0,$$

satisfying (1.9) *for some sequence of half-plane or circular regions* $\{V_n\}$ *each having zero as an interior point. If* $t_n(z)$ *is the l.f.t. of the form* (4.2) *defined by* (4.1), *then*

(4.7) $$\begin{aligned} \kappa_n &= [a_n K_{n-1}\gamma_{n-1}\gamma_n]M, \\ \lambda_n &= [-a_n K_{n-1}K_n\gamma_{n-1}\gamma_n]M, \\ \mu_n &= [a_n\gamma_{n-1}\gamma_n + b_n\delta_{n-1}\gamma_n - \delta_{n-1}\delta_n]M, \\ \nu_n &= [-a_n K_n\gamma_{n-1}\gamma_n - b_n K_n\gamma_n\delta_{n-1}]M, \end{aligned}$$

where

(4.8) $$K_n = \frac{\alpha_n}{\gamma_n} = v_n(\infty),$$

and M *is a constant of proportionality. The transformation* $t_n(z)$ *is normalized to satisfy* (4.4) *provided*

(4.9) $$M = [-a_n K_{n-1}K_n\gamma_{n-1}\gamma_n\delta_{n-1}\delta_n]^{-1/2}$$

with $-\frac{1}{2}\pi < \arg M \leqq \frac{1}{2}\pi$.

Proof. As pointed out in the introduction, it is possible to choose $v_n(z)$ so that $v_n(0) = 0$, since zero is an interior point of V_n. Thus, the form of $v_n(z)$ given by (4.6) is permissible. The remainder of the proof can be made by substitution of (1.1) and (4.6) into the right side of the equation

(4.10) $$t_n(z) = v_{n-1} \circ s_n \circ v_n^{-1}(z), \qquad n \geqq 1,$$

which follows from (4.1), (1.8), and (1.2).

THEOREM 4.3. *If the elements* a_n ($a_n \neq 0$) *and* b_n *of the continued fraction* $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$ *satisfy* (2.13) *for some sequences of positive numbers* $\{p_n\}$ *and real numbers* $\{\psi_n\}$, *and if the sequence*

(4.11) $$\left\{\frac{a_n}{p_n p_{n-1}}\right\}$$

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is bounded, then the continued fraction converges to a value v such that

$$\left| v - \frac{a_1 \exp(-i\psi_1)}{2[\mathrm{Re}(b_1 \exp(-i\psi_1)) - p_1]} \right| \leqq \frac{|a_1|}{2[\mathrm{Re}(b_1 \exp(-i\psi_1)) - p_1]}$$

(4.12) *if* $\mathrm{Re}(b_1 \exp(-i\psi_1)) > p_1$

and

$$\mathrm{Re}(v \exp(i(\psi_1 - \arg a_1))) \geqq 0 \quad \text{if} \quad \mathrm{Re}(b_1 \exp(-i\psi_1)) = p_1.$$

The restriction $a_n \neq 0$ *may be removed if we require the following additional condition*:

$$\mathrm{Re}(b_n \exp(-i\psi_n)) > p_n, \qquad n \geqq 1. \tag{4.13}$$

Proof. Let $\{V_n\}$ be the sequence of half-plane regions (2.11). Then (4.13) implies that zero is not contained in any of the sets $b_n + V_n$, $n \geqq 1$. Thus, in view of the remark in the introduction concerning the degenerate case ($a_n = 0$ for some n), it suffices to assume that $a_n \neq 0$ for all n. Then by Lemma 2.2, condition (2.13) implies (2.12). Let $T_n(z)$ be defined by (1.8), where $\{S_n(z)\}$ is the c.f.g. sequence associated with the continued fraction and where

$$v_n(z) = z/(z + 2P_n). \tag{4.14}$$

The transformation $v_n(z)$ is easily shown to satisfy (1.9) and hence $T_n(z)$ satisfies both (1.10) (or equivalently (3.1)) and (3.3) with $K_n = v_n(\infty) = 1$. In order to prove the convergence of the continued fraction it suffices to show that the sequence $\{T_n(z)\}$ converges for all z such that $|z| < 1$. But by Theorem 3.1, part (c), it will be sufficient to show that in the limit circle case the series

$$\sum_{n=1}^{\infty} R_n \frac{1 - g_n^2}{G_n + 1} \tag{4.15}$$

converges. By Theorems 4.1 and 4.2, taking $\alpha_n = \gamma_n = K_n = 1$, and $\delta_n = 2P_n$ and normalizing the parameters in (4.2) to satisfy (4.4), we have that

$$\lambda_n = \left[\frac{-a_n}{4P_{n-1}P_n}\right]^{1/2} = \frac{(C_{n-1} - C_n) + R_{n-1}\bar{G}_{n-1} - R_n\bar{G}_n}{[R_{n-1}R_n(1 - g_n^2)(1 - g_{n-1}^2)]^{1/2}}. \tag{4.16}$$

Formula (3.5) yields

$$(C_{n-1} - C_n) + R_{n-1}\bar{G}_{n-1} = R_n \frac{1 + \bar{G}_n}{1 + G_n}$$

and upon substitution into (4.16) we obtain

$$R_n \frac{1 - g_n^2}{G_n + 1} = \left[\frac{-a_n}{4P_{n-1}P_n}\right]^{1/2} [R_{n-1}R_n(1 - g_n^2)(1 - g_{n-1}^2)]^{1/2}. \tag{4.17}$$

From this it follows that, for all sufficiently large n,

$$
\left| R_n \frac{1 - g_n^{\,2}}{G_n + 1} \right| \leqq L \left| \frac{a_n}{P_{n-1}P_n} \right|^{1/2} (h_n h_{n-1})^{1/2} \tag{4.18}
$$
$$
\leqq \frac{L}{2} \left[\frac{|a_n|}{P_{n-1}P_n} \right]^{1/2} (h_n + h_{n-1}),
$$

where L is a constant independent of n. Thus, the series (4.15) converges since in the limit circle case $\sum h_n$ converges and by hypothesis the sequence (4.11) is bounded. This completes the proof of the theorem.

As noted in the introduction, an earlier theorem of Thron (**4**, Theorem A) may be obtained from Theorem 4.3 by taking $p_n = p > 0$ and $\psi_n = \psi$ real, $n \geqq 1$.

It was also mentioned in the introduction that Theorem 4.3 has an overlapping relation with an earlier theorem of Wall (**7**, Theorem 31.3). We shall now clarify that relationship. The theorem of Wall referred to states that: *A continued fraction* $\mathrm{K}(a_n/1)$ *converges if the elements* a_n *satisfy the conditions*

$$
|a_n| - \mathrm{Re}[a_n \exp(i(\phi_n + \phi_{n+1}))] \leqq \frac{2 \cos \phi_n \cos \phi_{n+1}(1 - g_{n-1})g_n}{(1 + \delta \sec \phi_n)(1 + \delta \sec \phi_{n+1})}, \tag{4.19}
$$

where

$$
\delta > 0, \quad -\tfrac{1}{2}\pi < \phi_n < \tfrac{1}{2}\pi, \quad 0 \leqq g_{n-1} \leqq 1, \quad n \geqq 1,
$$

and the series

$$
\sum \{ |a_n| (1 + \delta \sec \phi_n)(1 + \delta \sec \phi_{n+1}) \}^{-1/2} \tag{4.20}
$$

diverges. To connect this result with Theorem 4.3 we make the identification

$$
p_n = \cos \phi_{n+1}(1 - g_n), \qquad \psi_n = -\phi_{n+1}, \tag{4.21}
$$

so that inequality (4.19) becomes

$$
|a_n| - \mathrm{Re}[a_n \exp(-i(\psi_{n-1} + \psi_n))] \leqq \frac{2p_{n-1}(\cos \psi_n - p_n)}{(1 + \delta \sec \psi_{n-1})(1 + \delta \sec \psi_n)}. \tag{4.22}
$$

To ensure that $p_n > 0$ we must restrict g_n to $0 \leqq g_n < 1$. Then, by setting $b_n = 1$ in Theorem 4.3, it is clear that the parabolic region (4.19) is a subset of the interior of the parabolic region (2.13). However, the series (4.20) diverges whenever the sequence (4.11) is bounded. This can be seen by the fact that

$$
\left| \frac{a_n}{p_n p_{n-1}} \right| = \left| \frac{a_n \sec \phi_n \sec \phi_{n+1}}{(1 - g_{n-1})(1 - g_n)} \right| \leqq M
$$

implies that the nth term of (4.20) is not less than $1/\delta M^{1/2}$. Theorem 4.3 has the advantage of being more easily applied. Some examples of its use are given in §6.

5. Extension of the parabola theorem. The following theorem extends the general parabola theorem (**6**, Theorem 8.1) to allow for variable parabolic regions for the elements of the continued fraction.

THEOREM 5.1. *Let the elements a_n of the continued fraction $\mathrm{K}_{n=1}^{\infty}(a_n/1)$ with nth approximant $S_n(0)$ lie in parabolic regions defined by*

$$(5.1)\qquad |a_n| - \mathrm{Re}(a_n \exp(-i(\psi_n + \psi_{n-1}))) \leqq 2p_{n-1}(\cos\psi_n - p_n), \qquad n \geqq 1,$$

where $p_n > 0$ and ψ_n is real, and where

$$(5.2)\qquad |P_n - \tfrac{1}{2}| \leqq M < \tfrac{1}{2}, \qquad P_n = p_n \exp(i\psi_n), \qquad n \geqq 1,$$

for some fixed M. Then the sequences of even and odd approximants $\{S_{2n}(0)\}$ and $\{S_{2n+1}(0)\}$ both converge. The continued fraction $\{S_n(0)\}$ converges if and only if at least one of the two series

$$(5.3)\qquad \sum_{n=1}^{\infty}\left|\frac{a_2 \cdot a_4 \cdot \ldots \cdot a_{2n}}{a_3 \cdot a_5 \cdot \ldots \cdot a_{2n+1}}\right|, \qquad \sum_{n=1}^{\infty}\left|\frac{a_3 \cdot a_5 \cdot \ldots \cdot a_{2n+1}}{a_4 \cdot a_6 \cdot \ldots \cdot a_{2n+2}}\right|$$

diverges. If there exists a constant $K > 0$ such that $|a_n| < K$, $n \geqq 1$, then at least one of the series diverges so that the continued fraction converges.

Before proving the theorem we remark that the general parabola theorem (**6**, Theorem 8.1) is obtained by taking

$$(5.4)\qquad \psi_n = \psi \quad \text{and} \quad p_n = \tfrac{1}{2}\cos\psi,$$

where $-\frac{1}{2}\pi < \psi < \frac{1}{2}\pi$.

Theorem 5.1 also has an overlapping relation with the theorem of Wall stated in §4. Again, the connection is seen by means of the identification (4.21). The same relation holds between the parabolic regions as with Theorem 4.3 except for the further restriction (5.2) on the parameters $P_n = p_n \exp(i\psi_n)$. Since divergence of one of the series (5.3) is necessary for the convergence of $\mathrm{K}(a_n/1)$, that condition cannot be improved upon and is weaker than the condition that (4.20) diverges. We shall give two examples of convergent continued fractions $\mathrm{K}(a_n/1)$ to demonstrate the fact that either of the theorems may apply when the other does not.

Example 1. Let $a_{2n+1} = n^4$, $a_{2n} = 2n^4$, $\psi_n = -\phi_{n+1} = 0$, $n \geqq 1$. Then the first series in (5.3) diverges so that $\mathrm{K}(a_n/1)$ converges by Theorem 5.1. However, (4.20) converges so that Wall's theorem does not apply.

Example 2. $\psi_n = -\phi_{n+1}$, $\phi_{4n} = \phi_{4n+1} = \frac{1}{2}\pi - 1/n$, $\phi_{4n+2} = \phi_{4n+3} = 0$, $\arg a_n = -(\phi_n + \phi_{n+1})$, and $|a_{4n+2}| = n^2$. Then $\mathrm{K}(a_n/1)$ converges by Wall's theorem, since (4.20) diverges. However, Theorem 5.1 does not apply because of the restriction (5.2).

Proof. First we note that (5.1) is obtained from (2.13) by taking $b_n = 1$. Thus, if

$$(5.5)\qquad s_n(z) = a_n/(1 + z), \qquad n \geqq 1,$$

then by Lemma 2.2, condition (5.1) implies (2.12), where the V_n are the half-plane regions defined by (2.11). Let $T_n(z)$ be defined by (1.8), where $\{S_n(z)\}$

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is the c.f.g. sequence and the $v_n(z)$ are given by (4.14). Then, as in §4, the $T_n(z)$ satisfy (1.10) or, equivalently, (3.1) and

$$T_n(1) = T_{n-1}(0), \qquad n \geqq 2, \tag{5.6}$$

which is obtained from (1.3) and (1.8). Therefore by Theorem 3.1 it suffices for us to consider the convergence of $\{T_n(z)\}$ for all z such that $|z| < 1$ in the limit circle case (i.e., $r = \lim r_n > 0$).

In addition to (1.10) and (5.6), we shall also make use of the following two properties of the sequence $\{T_n(z)\}$:

$$T_n(J_n) = T_{n-1}(1) = T_{n-2}(0), \qquad n \geqq 3, \tag{5.7}$$

where

$$J_n = v_n(-1) = (1 - 2P_n)^{-1}, \qquad n \geqq 1, \tag{5.8}$$

and the property

$$T_n(0) = T_{n-1}[v_{n-1}(a_n)], \qquad n \geqq 2. \tag{5.9}$$

The first of these follows from (1.8) and

$$S_n(-1) = S_{n-1}(s_n(-1)) = S_{n-1}(\infty) = S_{n-2}(0).$$

Equation (5.9) follows from the corresponding property

$$S_n(0) = S_{n-1}(s_n(0)) = S_{n-1}(a_n).$$

Now by (5.2) and (5.8) there exists a constant $\epsilon > 0$ such that

$$|J_n| = \left|\frac{1}{1 - 2P_n}\right| \geqq 1 + \epsilon, \qquad n \geqq 1. \tag{5.10}$$

Therefore, using (5.7) and (5.10), we may conclude from Theorem 3.1 that the sequences $\{T_{2n}(z)\}$ and $\{T_{2n+1}(z)\}$ converge at least for all z such that $|z| < 1$. Hence, the sequences $\{S_{2n}(0)\}$ and $\{S_{2n+1}(0)\}$ both converge.

If we let $\sigma_n = \omega_n - \tau_n$, then we have shown in the preceding argument that

$$\sigma_{2n} \to \sigma \quad \text{and} \quad \sigma_{2n+1} \to \sigma'.$$

It follows from (3.6) that the sequence $\{T_n(z)\}$ will converge for $|z| < 1$ (and hence the continued fraction will converge) unless in the limit circle case (i.e., $r_n \to r > 0$)

$$\sigma \neq \sigma'. \tag{5.11}$$

We now show that

$$G_n \to -1 \tag{5.12}$$

is a necessary condition for divergence of the continued fraction. For, suppose there exists a subsequence $\{G_{k(n)}\}$ bounded away from -1, but with $g_{k(n)} \to 1$.

It follows that

$$\arg \frac{1 + \bar{G}_{k(n)}}{1 + G_{k(n)}} = -\tau_{k(n)} + o(k(n)).$$

From (3.1) and (5.6) we have that

$$C_n + R_n \frac{1 + \bar{G}_n}{1 + G_n} = C_{n-1} + R_{n-1}\bar{G}_{n-1} \tag{5.13}$$

and therefore

$$r_{k(n)-1}g_{k(n)-1} \exp(i\sigma_{k(n)-1}) = r_{k(n)} \left| \frac{1 + \bar{G}_{k(n)}}{1 + G_{k(n)}} \right| \exp(i\sigma_{k(n)} + o(k(n))) + (C_{k(n)} - C_{k(n)-1})$$

from which it is easily deduced that $\sigma = \sigma'$.

Making use of the preceding results, we shall now show that if the continued fraction diverges, then both series in (5.3) converge. To this end it is convenient to write

$$G_n = -(1 + \epsilon_n \exp(i\eta_n)), \qquad \epsilon_n > 0 \text{ and } \eta_n \text{ is real},$$

and we may assume (5.11), (5.12), and $r_n \to r > 0$ (limit circle case). Thus, $\epsilon_n \to 0$ and $\tau_n \to \pi$ so that $\omega_{2n} \to \omega$ and $\omega_{2n+1} \to \omega'$, where $\omega \neq \omega'$. Again using (5.13) we can show that

$$\lim 2\eta_{2n} = \omega - \omega' + \pi, \qquad \lim 2\eta_{2n+1} = \omega' - \omega + \pi.$$

Then, it follows from the same argument used by Thron (**6**, p. 125), that the convergence of $\sum h_k$ implies the convergence of $\sum \epsilon_k$. To interpret these results in terms of the elements a_n of the continued fraction we use property (5.9) with $v_n(z)$ given by (4.14) to write

$$\frac{a_n}{a_n + 2P_{n-1}} = T_{n-1}^{-1}[T_n(0)] = t_n(0),$$

or

$$\frac{a_n}{2P_{n-1}} = \frac{t_n(0)}{1 - t_n(0)}.$$

Taking $t_n(0) = \lambda_n/\nu_n$ as described by Theorem 4.1, we have that

$$\frac{a_n}{2P_{n-1}} = \frac{(R_{n-1}\bar{G}_{n-1} - R_n\bar{G}_n) - (C_n - C_{n-1})}{(1 + G_{n-1})(C_n - C_{n-1}) + (1 + G_{n-1})R_n\bar{G}_n - R_{n-1}(1 + \bar{G}_{n-1})}, \tag{5.14}$$

and from this we can eliminate $(C_n - C_{n-1})$, by use of (5.13) to obtain

$$\frac{a_n}{2P_{n-1}} = \frac{R_n(1 - g_n^2)}{R_{n-1}(g_{n-1}^2 - 1)(1 + G_n) - R_n(1 - g_n^2)(1 + G_{n-1})}. \tag{5.15}$$

Now from (5.7) we obtain

$$C_n + R_n \frac{J_n + \bar{G}_n}{G_n J_n + 1} = C_{n-2} + R_{n-2}\bar{G}_{n-2}. \tag{5.16}$$

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Using this and (5.13) with n replaced by $n-1$, we obtain

$$C_n + R_n \frac{J_n + \bar{G}_n}{G_n J_n + 1} = C_{n-1} + R_{n-1} \frac{1 + \bar{G}_{n-1}}{1 + G_{n-1}}, \tag{5.17}$$

which, when combined with (5.13), yields

$$\frac{R_{n-1}(1 - g_{n-1}^2)(1 + G_n)}{R_n(1 - g_n^2)} = \frac{(J_n - 1)(1 + G_{n-1})}{1 + G_n J_n}. \tag{5.18}$$

This equation may be used to eliminate R_n and R_{n-1} from (5.15) and we obtain

$$a_n = \frac{-2P_{n-1}(1 + G_n J_n)}{J_n(1 + G_{n-1})(1 + G_n)}. \tag{5.19}$$

Thus

$$\begin{aligned} \left| \frac{a_2 \cdot a_4 \cdot \ldots \cdot a_{2n}}{a_3 \cdot a_5 \cdot \ldots \cdot a_{2n+1}} \right| &= L_{2n+1} \prod_{k=1}^{n} \left| 1 + \frac{\epsilon_{2k} \exp(i\eta_{2k})}{2P_{2k}} \right| \\ &\quad \times \left\{ \prod_{k=1}^{n} \left| 1 + \frac{\epsilon_{2k+1} \exp(i\eta_{2k+1})}{2P_{2k+1}} \right| \right\}^{-1} \\ &\leqq L_{2n+1} \prod_{k=1}^{n} \left(1 + \frac{\epsilon_{2k}}{2p_{2k}} \right) \left\{ \prod_{k=1}^{n} \left(1 - \frac{\epsilon_{2k+1}}{2p_{2k+1}} \right) \right\}^{-1}, \end{aligned} \tag{5.20}$$

where

$$L_{2n+1} = \left| \frac{P_1}{P_{2n+1}(1 + G_1)} \right| \epsilon_{2n+1}, \tag{5.21}$$

and where we have used (5.8) to eliminate J_n. Since $\sum \epsilon_n$ converges and since by (5.2) $p_n = |P_n|$ is bounded away from zero, it follows that as $n \to \infty$, the products in the right side of the inequality in (5.20) converge to positive numbers and hence by (5.21) the first series in (5.3) converges. A similar argument holds for the second series and hence we have proved that divergence of the continued fraction implies convergence of both series (5.3). It is well known (**3**, p. 79) that the convergence of these two series is sufficient for the divergence of the continued fraction. Thus, the proof of the theorem is completed by showing that the continued fraction can diverge only if $a_n \to \infty$ and this is easily shown using (5.19) with (5.2), (5.8), (5.10), and (5.12).

It is useful to have Theorem 5.1 stated in terms of a continued fraction of the form $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$ as follows:

THEOREM 5.2. *Let the elements a_n and b_n of the continued fraction $\mathrm{K}_{n=1}^{\infty}(a_n/b_n)$ satisfy the conditions*

$$|a_n| - \mathrm{Re}[a_n \exp(-i(\psi_n + \psi_{n-1}))] \leqq 2p_{n-1}[\mathrm{Re}(b_n \exp(-i\psi_n)) - p_n], \quad n \geqq 1, \tag{5.22}$$

and

$$|b_n - H_M P_n| \leqq 2MH_M p_n, \tag{5.23}$$

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for some sequence $\{P_n\}$ *of non-zero complex numbers* $P_n = p_n \exp(i\psi_n)$, $p_n > 0$, ψ_n *is real, and*

$$H_M = 2(1 - 4M^2)^{-1}, \tag{5.24}$$

where M *is a fixed number such that* $0 < M < \frac{1}{2}$. *Let* $S_n(0)$ *denote the nth approximant of the continued fraction. Then the sequences of odd and even approximants,* $\{S_{2n+1}(0)\}$ *and* $\{S_{2n}(0)\}$, *converge. The continued fraction* $\{S_n(0)\}$ *converges if and only if at least one of the two series*

$$\sum_{n=1}^{\infty} \left| \frac{a_2 \cdot a_4 \cdot \ldots \cdot a_{2n}}{a_3 \cdot a_5 \cdot \ldots \cdot a_{2n+1}} \cdot b_{2n+1} \right|, \qquad \sum_{n=1}^{\infty} \left| \frac{a_3 \cdot a_5 \cdot \ldots \cdot a_{2n+1}}{a_4 \cdot a_6 \cdot \ldots \cdot a_{2n+2}} \cdot b_{2n+2} \right| \tag{5.25}$$

diverges. If there exists a fixed number K *such that*

$$\left| \frac{a_n}{b_n b_{n-1}} \right| < K, \qquad n \geqq 1, \tag{5.26}$$

then at least one of the series diverges so that the continued fraction converges.

Proof. Let $a_n^* = a_n/b_n b_{n-1}$, so that the continued fractions

$$\mathop{\mathrm{K}}_{n=1}^{\infty} (a_n/b_n) \quad \text{and} \quad \mathop{\mathrm{K}}_{n=1}^{\infty} (a_n^*/1)$$

are equivalent. Let

$$P_n^* = p_n^* \exp(i\psi_n^*) = P_n/b_n, \tag{5.27}$$

where $p_n^* > 0$ and ψ_n^* is real. Then it is sufficient to verify that (5.23) is equivalent to

$$|P_n^* - \tfrac{1}{2}| \leqq M < \tfrac{1}{2}, \qquad n \geqq 1,$$

and (5.22) is equivalent to

$$|a_n^*| - \mathrm{Re}[a_n^* \exp(-i(\psi_n^* + \psi_{n-1}^*))] \leqq 2p_{n-1}^*[\cos \psi_n^* - p_n^*].$$

6. Application to continued fraction expansions. The primary importance of the general convergence criteria studied in this paper is perhaps to derive convergence theorems for continued fraction expansions. As an illustration of this, we shall obtain from Theorem 4.3 two new results on the convergence of T-fractions. T-fractions (**2**; **5**) are the continued fractions of the form

$$1 + d_0 z + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{z}{1 + d_n z}, \tag{6.1}$$

where the d_n are complex constants and z is a complex variable. For our purpose we shall let

$$a_n = z = |z| \exp(i\theta), \quad b_n = 1 + d_n z, \qquad n \geqq 1. \tag{6.2}$$

Then, by Theorem 4.3, the continued fraction (6.1) will converge if for some

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sequence $\{P_n\}$ of non-zero complex numbers $P_n = p_n \exp(i\psi_n)$, $p_n > 0$, ψ_n is real, and some constant $K > 0$, the following properties hold:[1]

$$z \neq 0, \tag{6.3}$$

$$\operatorname{Re}(d_n \exp(i(\theta - \psi_n))) \geqq \frac{p_n - \cos \psi_n}{|z|} + \frac{1 - \cos(\theta - \psi_n - \psi_{n-1})}{2p_{n-1}}, \quad n \geqq 2, \tag{6.4}$$

$$\left| \frac{z}{p_n p_{n-1}} \right| \leqq K, \qquad n \geqq 2. \tag{6.5}$$

To obtain a theorem which holds for an angular opening in z, we shall make (6.4) independent of $|z|$ by setting

$$p_n = \cos \psi_n, \qquad -\tfrac{1}{2}\pi < \psi_n < \tfrac{1}{2}\pi, \tag{6.6}$$

and requiring the p_n to be bounded away from zero. For simplicity we shall choose $p_n = 1$ and $\psi_n = 0$, $n \geqq 1$, so that (6.5) is clearly satisfied for all fixed z and (6.4) becomes

$$\operatorname{Re}(d_n \exp(i\theta)) \geqq \tfrac{1}{2}(1 - \cos \theta). \tag{6.7}$$

For each fixed value of θ, (6.7) requires the d_n to lie within or on the boundary of the half-plane region

$$H_\theta = \{\xi\colon \operatorname{Re}(\xi \exp(i\theta)) \geqq \tfrac{1}{2}(1 - \cos \theta)\}. \tag{6.8}$$

Using this analysis, we arrive at the following theorem.

THEOREM 6.1. *The T-fraction* (6.1) *will converge for all* $z = |z| \exp(i\theta)$ *contained in the angular opening*

$$0 \leqq \theta_1 \leqq \theta \leqq \theta_2 < 2\pi, \tag{6.9}$$

where $0 < \theta_2 - \theta_1 < \pi$, *provided all* d_n *lie in the intersection* $H_{\theta_1} \cap H_{\theta_2}$ *of the half-planes* H_{θ_1} *and* H_{θ_2}.

We note that the restriction $\theta_2 - \theta_1 < \pi$ merely ensures that the region containing the d_n be non-null. To complete the proof of Theorem 6.1, it suffices to show that if θ_1, θ_2, and θ satisfy (6.9), then

$$H(\theta_1, \theta_2) \equiv \bigcap_{\theta_1 \leqq \theta \leqq \theta_2} H_\theta = H_{\theta_1} \cap H_{\theta_2}. \tag{6.10}$$

If we let ξ_0 denote the point of intersection of the boundaries of H_{θ_1} and H_{θ_2}, then it will be sufficient for our purpose to show that

$$\xi_0 \in H(\theta_1, \theta_2). \tag{6.11}$$

[1]The continued fraction (6.1) converges trivially for $z = 0$. Hence, we may exclude this case in the following discussion.

But it is easily shown that

$$\xi_0 = \left[\frac{\sin\theta_1 - \sin\theta_2}{2\sin(\theta_1-\theta_2)} - \frac{1}{2}\right] + i\left[\frac{\cos\theta_1 - \cos\theta_2}{2\sin(\theta_1-\theta_2)}\right], \tag{6.12}$$

and by an elementary analysis one can verify (6.11), thus completing the proof of our theorem.

Finally, we obtain by a similar argument the following theorem.

THEOREM 6.2. *If there exists a number* $A \geqq 0$ *such that*

$$\operatorname{Re}(d_n) \geqq A, \qquad n \geqq 1, \tag{6.13}$$

then the T-fraction (6.1) *will converge for all* z *such that*

$$0 < \cos(\arg z) \geqq (1 + 2A)^{-1} \tag{6.14}$$

or $z = 0$.

Proof. For any z satisfying (6.14), conditions (6.4) and (6.5) will hold if we take

$$p_n = \cos\psi_n, \qquad \psi_n = \theta = \arg z.$$

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