Proceedings of the Edinburgh Mathematical Society (1997) 40, 325–330 ©

BIORTHOGONALITY IN THE REAL SEQUENCE SPACES ℓ^p

by ANTHONY J. FELTON and H. P. ROGOSINSKI*

(Received 14th August 1995)

In this paper we generalise some of the results obtained in [1] for the n-dimensional real spaces $\ell^p(n)$ to the *infinite dimensional* real spaces ℓ^p. Let $p > 1$ with $p \neq 2$, and let $\mathbf{x}$ be a non-zero real sequence in ℓ^p. Let $\mathcal{E}(\mathbf{x})$ denote the closed linear subspace spanned by the set $\{\mathbf{x}\}^{\pm}$ of all those sequences in ℓ^p which are biorthogonal to $\mathbf{x}$ with respect to the *unique* semi-inner-product on ℓ^p consistent with the norm on ℓ^p. In this paper we show that codim $\mathcal{E}(\mathbf{x}) = 1$ unless *either* $\mathbf{x}$ has exactly two non-zero coordinates which are equal in modulus, *or* $\mathbf{x}$ has exactly three non-zero coordinates α, β, γ with $|\alpha| \geq |\beta| \geq |\gamma|$ and $|\alpha|^p > |\beta|^p + |\gamma|^p$. In these exceptional cases codim $\mathcal{E}(\mathbf{x}) = 2$. We show that $\{\mathbf{x}\}^{\pm}$ is a linear subspace if, and only if, $\mathbf{x}$ has *either* at most two non-zero coordinates *or* $\mathbf{x}$ has exactly three non-zero coordinates which satisfy the inequalities stated above.

1991 *Mathematics subject classification:* 46C50.

0. Introduction

Throughout this paper, p denotes a real number with $p > 1$ and $p \neq 2$. Consider the *real* normed linear space ℓ^p, and note that there exists a unique semi-inner-product on ℓ^p consistent with the norm. In fact for $\mathbf{x}, \mathbf{y} \in \ell^p$

$$[\mathbf{x}, \mathbf{y}] = \frac{1}{\|\mathbf{y}\|^{p-2}} \sum_{i=1}^{\infty} x_i |y_i|^{p-1} \operatorname{sgn} y_i.$$

For a discussion of semi-inner-products and semi-inner-product spaces we refer the reader to [2] and [3]. The following definitions are given in [1]. If $\mathbf{x}, \mathbf{y} \in \ell^p$ then $\mathbf{x}$ and $\mathbf{y}$ are said to be *biorthogonal* if $[\mathbf{x}, \mathbf{y}] = [\mathbf{y}, \mathbf{x}] = 0$. Further for fixed $\mathbf{x} \in \ell^p(n)$, $\tau(\mathbf{x})$ is defined to be the number of elements in a maximal linearly independent set of vectors biorthogonal to $\mathbf{x}$. The following theorem is the main result (Theorem 4.5) of [1].

Theorem 0.1. *Let* $n \geq 2$, *and let* $\mathbf{x} \in \ell^p(n)$. *Let* r *be the number of non-zero coordinates of* $\mathbf{x}$.

(i) *If* $r = 0$ *then* $\tau(\mathbf{x}) = n$.

(ii) *If* $r = 1$ *or* $r \geq 4$ *then* $\tau(\mathbf{x}) = n - 1$.

* This research was funded by the University of Wales.

(iii) *If* $r = 2$ *then* $\tau(\mathbf{x}) = n - 1$ *if the two non-zero coordinates have equal modulus, and* $\tau(\mathbf{x}) = n - 2$ *otherwise.*

(iv) *If* $r = 3$, *let* $\{\alpha, \beta, \gamma\}$ *be a permutation of the three non-zero coordinates such that* $|\alpha| \geq |\beta| \geq |\gamma|$. *Then* $\tau(\mathbf{x}) = n - 1$ *if* $|\alpha|^p \leq |\beta|^p + |\gamma|^p$ *and* $\tau(\mathbf{x}) = n - 2$ *otherwise.*

Definition 0.2. For $\mathbf{x} \in \ell^p$, define $\mathcal{E}(\mathbf{x})$ to be the smallest closed linear subspace in ℓ^p which contains every vector biorthogonal to $\mathbf{x}$.

Remark 0.3. Let $\mathbf{x} \in \ell^p$. Then $\mathcal{E}(\mathbf{x}) \subseteq \{\mathbf{y} : [\mathbf{y}, \mathbf{x}] = 0\}$. (This follows immediately from the left-linearity and left-continuity of the semi-inner-product.)

In the next section we shall show that $\mathcal{E}(\mathbf{x})$ has finite codimension, and we shall determine codim $\mathcal{E}(\mathbf{x})$ for all non-zero $\mathbf{x}$ in ℓ^p.

1. The space $\mathcal{E}(\mathbf{x})$

We introduce the following notation.

Notation. For $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\ell^p(n)$, denote by $\hat{\mathbf{x}}$ the sequence $(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ in ℓ^p. For $\mathbf{x} = (x_1, x_2, \ldots)$ in ℓ^p, denote by $\mathbf{x}^{[n]}$ the sequence $(x_1, x_2, \ldots, x_n)$ in $\ell^p(n)$.

Theorem 1.1. *Let* $\mathbf{x}$ *be a non-zero vector in* ℓ^p. *Then* codim $\mathcal{E}(\mathbf{x}) = 1$ *unless either*

(i) $\mathbf{x}$ *has exactly two non-zero coordinates* α *and* β *with* $|\alpha| \neq |\beta|$

or

(ii) $\mathbf{x}$ *has exactly three non-zero coordinates* α, β *and* γ *with* $|\alpha| \geq |\beta| \geq |\gamma|$ *and* $|\alpha|^p > |\beta|^p + |\gamma|^p$.

If either of the conditions (i) *or* (ii) *holds then* codim $\mathcal{E}(\mathbf{x}) = 2$.

Proof. Let $\mathbf{x} \in \ell^p$. Suppose first that $\mathbf{x}$ has *infinitely* many non-zero coordinates. Choose N so that $\mathbf{x}^{[n]}$ has at least four non-zero coordinates when $n \geq N$. Then by Theorem 0.1(ii) for $n \geq N$, $\tau(\mathbf{x}^{[n]}) = n - 1$, and we can find $(n-1)$ linearly independent vectors $\mathbf{f}_{i,n}(1 \leq i \leq n-1)$ in $\ell^p(n)$ which are biorthogonal to $\mathbf{x}^{[n]}$. The vector $\mathbf{x}^{[n]}$ is *not* a linear combination of these vectors since every such linear combination is left-orthogonal to $\mathbf{x}^{[n]}$. Hence $\{\mathbf{x}^{[n]}, \mathbf{f}_{1,n}, \ldots, \mathbf{f}_{n-1,n}\}$ is a basis for $\ell^p(n)$.

Let $\mathbf{y} \in \ell^p$, and let $n \geq N$. Then there exists scalars $\lambda_{i,n}$ $(0 \leq i \leq n-1)$ so that

$$\mathbf{y}^{[n]} = \lambda_{0,n}\mathbf{x}^{[n]} + \sum_{i=1}^{n-1} \lambda_{i,n}\mathbf{f}_{i,n}. \tag{1}$$

https://doi.org/10.1017/S0013091500023762 Published online by Cambridge University Press

By the left-linearity of the semi-inner-product,

$$[\mathbf{y}^{[n]}, \mathbf{x}^{[n]}] = \lambda_{0,n}\|\mathbf{x}^{[n]}\|^2, \tag{2}$$

and so

$$\lim_{n\to\infty} \lambda_{0,n} = \frac{[\mathbf{y}, \mathbf{x}]}{\|\mathbf{x}\|^2}. \tag{3}$$

Let

$$\mathbf{z}_n = \sum_{i=1}^{n-1} \lambda_{i,n}\hat{\mathbf{f}}_{i,n,} \qquad (n \geq N). \tag{4}$$

Then $\mathbf{z}_n \in \mathcal{E}(\mathbf{x})$ since each of the vectors $\hat{\mathbf{f}}_{i,n,}$ is biorthogonal to $\mathbf{x}$. By (1),

$$\widehat{\mathbf{y}^{[n]}} = \lambda_{0,n}\widehat{\mathbf{x}^{[n]}} + \mathbf{z}_n,$$

and so, using (3) and the observation that $\widehat{\mathbf{x}^{[n]}} \to \mathbf{x}$, and $\widehat{\mathbf{y}^{[n]}} \to \mathbf{y}$,

$$\lim_{n\to\infty} \mathbf{z}_n = \mathbf{y} - \frac{[\mathbf{y}, \mathbf{x}]}{\|\mathbf{x}\|^2}\mathbf{x}.$$

Hence, since $\mathcal{E}(\mathbf{x})$ is closed,

$$\mathbf{y} - \frac{[\mathbf{y}, \mathbf{x}]}{\|\mathbf{x}\|^2}\mathbf{x} \in \mathcal{E}(\mathbf{x}). \tag{5}$$

Noting that $\mathbf{x} \neq 0$, it follows from Remark 0.3 that $\mathbf{x} \notin \mathcal{E}(\mathbf{x})$. Since (5) holds for all $\mathbf{y}$ in ℓ^p, we deduce that

$$\ell^p = \mathcal{E}(\mathbf{x}) \oplus \mathcal{F}(\mathbf{x}),$$

where $\mathcal{F}(\mathbf{x})$ is the one-dimensional subspace generated by $\mathbf{x}$. It follows that codim $\mathcal{E}(\mathbf{x}) = 1$.

Suppose now that $\mathbf{x} = (x_1, x_2, \ldots)$ has *finitely* many non-zero coordinates. Choose n_0 so that $x_i = 0$ when $i > n_0$. Let $\tau_0 = \tau(\mathbf{x}^{[n_0]})$, and let $\tau_1 = n_0 - \tau_0$. Then we can find a basis $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{n_0}\}$ in $\ell^p(n_0)$ with $\mathbf{f}_i$ biorthogonal to $\mathbf{x}^{[n_0]}$ when $\tau_1 + 1 \leq i \leq n_0$. Let $\mathbf{y} = (y_1, y_2, \ldots) \in \ell^p$. Then there exist scalars λ_i $(1 \leq i \leq n_0)$ so that

$$\mathbf{y} = \sum_{i=1}^{n_0} \lambda_i\hat{\mathbf{f}}_i + \sum_{i=n_0+1}^{\infty} y_i\mathbf{e}_i, \tag{6}$$

where $\mathbf{e}_i$ is the i^{th} standard basis vector in ℓ^p. We can write

$$\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2, \tag{7}$$

where

$$\mathbf{y}_1 = \sum_{i=\tau_1+1}^{n_0} \lambda_i \hat{\mathbf{f}}_i + \sum_{i=n_0+1}^{\infty} y_i \mathbf{e}_i, \quad \text{and} \quad \mathbf{y}_2 = \sum_{i=1}^{\tau_1} \lambda_i \hat{\mathbf{f}}_i.$$

Since each of the vectors $\hat{\mathbf{f}}_i$ $(\tau_1 + 1 \leq i \leq n_0)$ and each of the vectors $\mathbf{e}_i$ $(i \geq n_0 + 1)$ is biorthogonal to $\mathbf{x}$,

$$\mathbf{y}_1 \in \mathcal{E}(\mathbf{x}). \tag{8}$$

If $\mathcal{F}(\mathbf{x})$ is the τ_1-dimensional subspace of ℓ^p spanned by the vectors $\hat{\mathbf{f}}_i$ $(1 \leq i \leq \tau_1)$ then

$$\mathbf{y}_2 \in \mathcal{F}(\mathbf{x}). \tag{9}$$

Noting that the map $\mathbf{z} \to \mathbf{z}^{[n_0]}$ is a continuous linear map from ℓ^p onto $\ell^p(n_0)$, and also that $\mathbf{z}$ is biorthogonal to $\mathbf{x}$ if, and only if, $\mathbf{z}^{[n_0]}$, is biorthogonal to $\mathbf{x}^{[n_0]}$, it is easy to see that if $\mathbf{z} \in \mathcal{E}(\mathbf{x})$ then $\mathbf{z}^{[n_0]} \in \mathcal{E}(\mathbf{x}^{[n_0]})$. Now let $\mathbf{z} \in \mathcal{E}(\mathbf{x}) \cap \mathcal{F}(\mathbf{x})$. Then $\mathbf{z}^{[n_0]}$ belongs to $\mathcal{E}(\mathbf{x}^{[n_0]})$, and so $\mathbf{z}^{[n_0]}$ is a linear combination of the vectors $\mathbf{f}_i$ $(\tau_1 + 1 \leq i \leq n_0)$. On the other hand, since $\mathbf{z}$ is a linear combination of the vectors $\hat{\mathbf{f}}_i$ $(1 \leq i \leq \tau_1)$, $\mathbf{z}^{[n_0]}$ must also be a linear combination of the vectors $\mathbf{f}_i$ $(1 \leq i \leq \tau_1)$. It follows that $\mathbf{z}^{[n_0]}$ is the zero-vector in $\ell^p(n_0)$, and so $\mathbf{z}(= \widehat{\mathbf{z}^{[n_0]}})$ is the zero-vector in ℓ^p. Hence

$$\mathcal{E}(\mathbf{x}) \cap \mathcal{F}(\mathbf{x}) = \{\mathbf{0}\}. \tag{10}$$

By (7), (8), (9) and (10) we see that

$$\ell^p = \mathcal{E}(\mathbf{x}) \oplus \mathcal{F}(\mathbf{x}),$$

and hence codim $\mathcal{E}(\mathbf{x}) = \dim \mathcal{F}(\mathbf{x}) = \tau_1$. An application of Theorem 0.1 shows that $\tau_1 = 1$ unless $\mathbf{x}$ satisfies either of the conditions (i) and (ii) in which case $\tau_1 = 2$. □

Remark 1.2. If E and F are proper linear subspaces of X with codim $E = 1$ and $E \subseteq F$ then $E = F$.

Let $[.\,,\,.]$ be a semi-inner-product on the normed space X which is consistent with the norm on X.

Theorem 1.3. (i) *Let* $\mathbf{x}$ *be a non-zero vector in* X. *Then* $\mathcal{E}(\mathbf{x})$ *has codimension* 1 *in* X *if, and only if,*

https://doi.org/10.1017/S0013091500023762 Published online by Cambridge University Press

$$\mathcal{E}(\mathbf{x}) = \{\mathbf{y} \in X : [\mathbf{y}, \mathbf{x}] = 0\}.$$

(ii) *If the set of all vectors biorthogonal to* $\mathbf{x}$ *is a linear subspace of* X *with codimension* 1 *then every vector which is left-orthogonal to* $\mathbf{x}$ *is also right-orthogonal to* $\mathbf{x}$.

Proof. Write $\{\mathbf{x}\}^L = \{\mathbf{y} \in X : [\mathbf{y}, \mathbf{x}] = 0\}$.

(i) If $\mathcal{E}(\mathbf{x}) = \{\mathbf{x}\}^L$ then $\mathcal{E}(\mathbf{x})$ is the kernel of a non-zero continuous linear functional on X, and so has codimension 1. Suppose conversely that $\mathcal{E}(\mathbf{x})$ has codimension 1. Let $E = \mathcal{E}(\mathbf{x})$ and $F = \{\mathbf{x}\}^L$. Applying Remarks 0.3 and 1.2 we see that $\mathcal{E}(\mathbf{x}) = \{\mathbf{x}\}^L$.

(ii) Apply Remark 1.2 with $E = \{\mathbf{y} : \mathbf{x} \pm \mathbf{y}\}$ and $F = \{\mathbf{x}\}^L$. □

2. The subspace problem

For $\mathbf{x} \in \ell^p$, let $\{\mathbf{x}\}^\pm$ denote the set of all those sequences in ℓ^p which are biorthogonal to $\mathbf{x}$. In this section we consider the problem of characterising those $\mathbf{x}$ for which $\{\mathbf{x}\}^\pm$ is a linear subspace. We begin with the following lemma.

Lemma 2.1. (i) *Let* $\mathbf{x} \in \ell^p(3)$. *If all of the coordinates of* $\mathbf{x}$ *are non-zero then there exists a vector in* $\ell^p(3)$ *which is left-orthogonal but not right-orthogonal to* $\mathbf{x}$.

(ii) *Let* $\mathbf{x} \in \ell^p$. *If* $\mathbf{x}$ *has at least three non-zero coordinates then there exists a vector in* ℓ^p *which is left-orthogonal but not right-orthogonal to* $\mathbf{x}$.

Proof. We shall only prove (i) since (ii) then follows as an obvious consequence.

Noting the fact that the semi-inner-product is homogeneous, we can assume without loss of generality that $\mathbf{x} = (a, b, 1)$, with a and b *non-zero*. Suppose for a contradiction that every vector which is left-orthogonal to $\mathbf{x}$ is right-orthogonal to $\mathbf{x}$. The vector $(1, 0, -|a|^{p-1}\text{sgn } a)$ is left-orthogonal to $\mathbf{x}$, and so by our supposition right-orthogonal to $\mathbf{x}$. This implies that

$$a + |a|^{(p-1)^2}\text{sgn}(-\text{sgn } a) = 0,$$

and so $|a| = |a|^{(p-1)^2}$. Hence $|a| = 1$ since $p \neq 2$. Similarly the left-orthogonality and consequent right-orthogonality of $(0, 1, -|b|^{p-1}\text{sgn } b)$ to $\mathbf{x}$ implies that $|b| = 1$. Since $|a| = |b| = 1$, the vector $(2\text{sgn } a, -\text{sgn } b, -1)$ is left-orthogonal to $\mathbf{x}$, and a simple calculation shows that the right-orthogonality of this vector to $\mathbf{x}$ implies that $2^{p-1} = 2$. Since $p \neq 2$ we obtain the desired contradiction. □

Theorem 2.2. *For given* $\mathbf{x} \in \ell^p$, $\{\mathbf{x}\}^\pm$ *is a linear subspace if, and only if, either* $\mathbf{x}$ *has at most two non-zero coordinates or* $\mathbf{x}$ *has exactly three non-zero coordinates* α, β, γ *with* $|\alpha| \geq |\beta| \geq |\gamma|$ *and* $|\alpha|^p > |\beta|^p + |\gamma|^p$.

Proof. If $\mathbf{x} = \mathbf{0}$ then $\{\mathbf{x}\}^\pm = \ell^p$. If $\mathbf{x}$ has exactly one non-zero coordinate x_{n_1} then

$$\{\mathbf{x}\}^{\pm} = \{(y_1, y_2, \ldots) \in \ell^p : y_{n_1} = 0\}.$$

If $\mathbf{x}$ has exactly two non-zero coordinates x_{n_1} and x_{n_2} then it is easily verified that

$$\{\mathbf{x}\}^{\pm} = \{(y_1, y_2, \ldots) \in \ell^p : y_{n_1} = y_{n_2} = 0\} \quad \text{if} \quad |x_{n_1}| \neq |x_{n_2}|,$$

and

$$\{\mathbf{x}\}^{\pm} = \left\{(y_1, y_2, \ldots) \in \ell^p : y_{n_1} = -\operatorname{sgn}\left(\frac{x_{n_1}}{x_{n_2}}\right) y_{n_2}\right\} \quad \text{if} \quad |x_{n_1}| = |x_{n_2}|.$$

Hence $\{\mathbf{x}\}^{\pm}$ is a linear subspace if $\mathbf{x}$ has at most two non-zero coordinates. If $\mathbf{x}$ has exactly three non-zero coordinates x_{n_1}, x_{n_2} and x_{n_3} with $|x_{n_1}| \geq |x_{n_2}| \geq |x_{n_3}|$ and $|x_{n_1}|^p > |x_{n_2}|^p + |x_{n_3}|^p$ then $\tau(x_{n_1}, x_{n_2}, x_{n_3}) = 1$ and

$$\{\mathbf{x}\}^{\pm} = \{(y_1, y_2, \ldots) \in \ell^p : (y_{n_1}, y_{n_2}, y_{n_3}) \in \mathcal{V}\},$$

where $\mathcal{V}$ is the *one-dimensional* linear subspace of $\ell^p(3)$ consisting of all those vectors which are biorthogonal to $(x_{n_1}, x_{n_2}, x_{n_3})$. Hence also in this case $\{\mathbf{x}\}^{\pm}$ is a linear subspace.

In all of the remaining cases, Lemma 2.1 shows that there exists a vector which is left-orthogonal but not right-orthogonal to $\mathbf{x}$. Moreover in all of these cases Theorem 1.1 shows that codim $\mathcal{E}(\mathbf{x}) = 1$. Hence $\{\mathbf{x}\}^{\pm}$ is *not* a linear subspace, since otherwise $\mathcal{E}(\mathbf{x}) = \{\mathbf{x}\}^{\pm}$ and Theorem 1.3(ii) leads to a contradiction. □

Acknowledgement. We would like to thank the referee for his helpful comments concerning the presentation of this paper.

REFERENCES

1. ANTHONY J. FELTON and H. P. ROGOSINSKI, Biorthogonality in the Banach Spaces $\ell^p(n)$, *Proc. Edinburgh Math. Soc.* **39** (1996), 325–336.

2. J. R. GILES, Classes of semi-inner-product spaces, *Trans. Amer. Math. Soc.* **129** (1967), 436–446.

3. G. LUMER, Semi-inner-product spaces, *Trans. Amer. Math. Soc.* **100** (1961), 29–43.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WALES, SWANSEA
SINGLETON PARK, SA2 8PP
WALES