J. Austral. Math. Soc. (*Series A*) **61** (1996), 119–128

DECOMPOSING CUBES

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(Received 8 December 1994)

Communicated by L. Caccetta

Abstract

A graph H decomposes into a graph G if one can write H as an edge-disjoint union of graphs isomorphic to G. H decomposes into D, where D is a family of graphs, when H can be written as a union of graphs each isomorphic to some member of D, and every member of D is represented at least once. In this paper it is shown that the d-dimensional cube Q_d decomposes into any graph G of size d each of whose blocks is either an even cycle or an edge. Furthermore, Q_d decomposes into D, where D is any set of six trees of size d.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 05C70; secondary 05C99.

1. Introduction

We use the standard ideas of graph theory. All graphs are finite, simple and undirected.

A graph H decomposes into a graph G if H can be written as an edge-disjoint union of copies of G.

It has been thought for a long time that the general graph decomposition problem is hard. This was confirmed when Dolinski and Tarsi [1] proved that unless G is of the form $tK_2 \cup nP_3$, the G-decomposition problem is NP-complete. In view of their result it is not surprising that there is an interest in restricted decomposition problems. One of the most famous conjectures is that of Ringel [4]:

CONJECTURE. The complete graph on $2n + 1$ vertices decomposes into any tree of size n.

Many partial results have been obtained and recently an analogue of the conjecture has been proved independently by Fink [2] and Jacobson, Truszczynski and Tuza [3].

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THEOREM. ([2, 3]) *The d-dimensional cube Q_d decomposes into any tree of size d.*

The Theorem can be generalized in three ways: one can replace 'cube' by a more general graph, replace 'tree' by a more general graph, or consider decompositions into families rather than decompositions into a single graph. In [3] the following generalization of the first kind is proposed.

CONJECTURE. Every d-regular bipartite graph decomposes into any tree of size d.

In the present paper we focus on a generalization of the second kind, namely we show that if the graph G of size d has the property that any block is either an even cycle or an edge then Q_d decomposes into G. In view of this result and other supporting evidence we believe that the following conjecture could be true.

CONJECTURE. If G is a graph of size d embeddable into Q_d, then Q_d decomposes into G.

This conjecture is of course an analog of Wilson's Theorem [5] that for fixed λ and G, the λ-fold complete multigraph $K_n^{(\lambda)}$ decomposes into G provided n is sufficiently large and the obvious divisibility conditions hold.

In [2] Fink discusses a generalization of the third kind. Let F be a set of graphs. Then it is said that there is an *F-decomposition* of a graph G if G can be partitioned into subgraphs each of which is isomorphic to a member of F such that every graph from F is represented at least once in the decomposition of G. He asks what is the largest number n such that, for any set F of n trees of size d, there is an F-decomposition of Q_d, and shows that $n \geq 2$. In the second part of this paper we shall prove that $n \geq 6$.

2. Balanced decompositions of cubes

A *decomposition* of a graph G into a graph H is a system of mutually edge disjoint subgraphs $G_1, \dots, G_n$ of G such that $E(G_1) \cup \dots \cup E(G_n) = E(G)$ and G_i is isomorphic to H for $i = 1, \dots, n$. In this paper we deal with decompositions of the n-dimensional cube which we denote by Q_n. There are many ways to represent an n-cube. The following one is the most suitable for our purposes. The *vertex set* is the set A^n, the set of all ordered n-tuples of 0's and 1's, and two vertices are *connected* if they differ in precisely one coordinate. By O and I we denote the n-tuples $(0, 0, \dots, 0)$ and $(1, 1, \dots, 1)$ respectively. For $\alpha \in A^n$ we denote by α_i the ith coordinate of α, and ℓ_i is the n-tuple with $(\ell_i)_i = 1$ and $(\ell_i)_j = 0$ for $j \neq i$. If α and β belong to A^n, the sum $\alpha + \beta$ is also in A^n and is the componentwise sum (mod 2). Further $\alpha \in A^n$ is called *even* or *odd* according to whether the number of non-zero coordinates of α is even or odd. Finally, let G be a graph of size n. Then a decomposition

$D = \{G_1, \ldots, G_{2^{n-1}}\}$ of Q_n into G is said to be *balanced* if there exist isomorphisms $\varphi_i : G \to G_i$ such that, for any $v \in V(G)$, $\{\varphi_i(v), i = 1, \ldots, 2^{n-1}\}$ coincides either with the set of all even vertices of Q_n or with the set of all odd vertices of Q_n.

Clearly, if there is a balanced decomposition of Q_n into G then there is a balanced decomposition of Q_n into G such that for a given vertex w of G the images of w under the φ_i's occupy all even vertices of Q_n.

LEMMA 1. *There is a balanced decomposition of Q_{2n} into cycles of length $2n$.*

PROOF. **Suppose first that n is even, say $2n = 4k$.** Let T and T' denote the vertices $T = \ell_1 + \ell_2 + \cdots + \ell_{2k}$ and $T' = \ell_{2k+1} + \cdots + \ell_{4k}$. Consider two $4k$-cycles of C_1 and C_2 Q_{4k}, where

$$C_1 = O, \ell_1, \ell_1 + \ell_2, \ldots, T, T + \ell_1, T + \ell_1 + \ell_2, \ldots, T + \ell_1 + \ell_2 + \cdots + \ell_{2k} \ (= O)$$

and C_2 is obtained from C_1 by exchanging any ℓ_i in the definition of C_1 with ℓ_{i+2k} (that is, C_2 is obtained from C_1 by cyclicly shifting the coordinates of any vertex of C_1 by $2k$ to the right). For example, if $d = 8$, where $d = 2n$ is the dimension of the cube,

C_1		C_2	
	00000000		00000000
	10000000		00001000
	11000000		00001100
	11100000		00001110
	11110000		00001111
	01110000		00000111
	00110000		00000011
	00010000		00000001
	00000000		00000000

It is obvious that the mapping $\varphi_\alpha : A^d \to A^d$, defined by $\varphi_\alpha(\beta) = \beta + \alpha$ for $\beta \in A^d$, is an automorphism of Q_d. This implies that $C_i + \alpha$ is a d-cycle of Q_d for any $\alpha \in A^d$, $i = 1, 2$. To finish the proof we show that $\mathscr{C} = \{C_1 + \alpha; \alpha \in A^d, \alpha \text{ is even}, \alpha_{2k} = 0\} \cup \{C_2 + \beta; \beta \in A^d, \beta \text{ is even}, \beta_{4k} = 0\}$ is a decomposition of Q_d in C_d's.

As $\mathscr{C}$ contains 2^{d-1} cycles it suffices to prove that they are edge disjoint. Suppose, to the contrary, that there is an edge f of Q_d which belongs to two different cycles of $\mathscr{C}$.

We consider two cases.

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CASE 1. There exist $\alpha, \beta \in A^d, \alpha \neq \beta$, such that f belongs to both $C_1 + \alpha$ and $C_1 + \beta$. Let $f = st$, where $s = t + \ell_j$, that is, s and t differ precisely in the jth coordinate. In C_1, there are two edges such that their end vertices differ in the jth coordinate; denote them by $g_1 = v_1 w_1$, $g_2 = v_2 w_2$. Then f must be the image of g_1 or g_2. By the definition of C_1 we can assume

(1) $$v_1 + v_2 = w_1 + w_2 = T,$$

(2) $$v_1 + w_2 = v_2 + w_1 = \ell_j + T.$$

Hence, either

$$f = g_1 + \alpha = g_1 + \beta,$$

or

$$f = g_1 + \alpha = g_2 + \beta.$$

In the former case: either $s = v_1 + \alpha = v_1 + \beta$, implying $\alpha = \beta$, which is impossible; or $s = v_1 + \alpha = w_1 + \beta$ (whence $v_1 + w_1 = \alpha + \beta$), so $\ell_j = \alpha + \beta$, contradicting the fact that $\alpha + \beta$ is even. In the latter case: either $s = v_1 + \alpha = v_2 + \beta$, and by (1), $\alpha + \beta = T$, contradicting $\alpha_k = \beta_k = 0$; or $s = v_1 + \alpha = w_2 + \beta$, which, by (2), yields $T + \ell_j = \alpha + \beta$ contradicting $\alpha + \beta$ is even. So case 1 is impossible.

CASE 2. There exist $\alpha, \beta \in A^d$ such that $f = st$ belongs to $C_1 + \alpha$, implying s and t differ in the jth coordinate, $j \leq 2k$, and also f belongs to $C_2 + \beta$ implying s and t differ in the jth coordinate for $j > 2k$. So case 2 is impossible.

To finish the proof it is necessary to show that $\mathscr{C}$ is a balanced decomposition. Because of the symmetry of a cycle and the way we have defined C it is sufficient to pick, for any cycle $C \in \mathscr{C}$, a vertex $v_C \in C$ such that the set $\{v_C; C \in \mathscr{C}\}$ is the set of all even vertices of Q_d. It is a matter of routine to verify that the following choice has that property:

For $C_1 + \alpha$ pick $O + \alpha$ if $\alpha_{4k} = 0$; otherwise pick $T + \alpha$.
For $C_2 + \alpha$ pick $O + \alpha$ if $\alpha_{2k} = 1$; otherwise pick $T' + \alpha$.

Suppose now $2n = 4k + 2$. We represent Q_d as in Figure 1, where the four squares stand for copies of Q_{d-2} induced by d-tuples with the same last two coordinates. These last two coordinates are written down above each cube. Consider the decomposition of Q_{d-2} into cycles of length $d - 2$ given in the first part of the proof. We take the same decomposition for all four Q_{d-2}'s. By means of these decompositions we generate a decomposition of Q_d into cycles of length d. We write v_{ij} for the vertex v from the Q_{d-2} whose last two coordinates are ij.

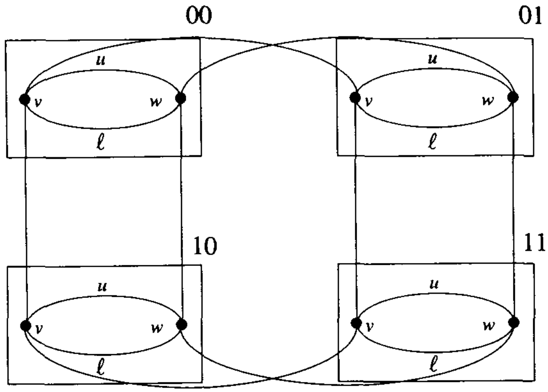

FIGURE 1.

Consider a cycle of $\mathscr{C}$ which is of the form $C_1 + \alpha$. One of these is depicted in Figure 1. By vuw ($v\ell w$) we denote the 'upper' ('lower') part of the cycle. Then the cycle generates four cycles of length d, namely

$$K_1 = v_{00}uw_{00}w_{01}uv_{01}v_{00}; \qquad K_2 = v_{00}\ell w_{00}w_{10}\ell v_{10}v_{00};$$
$$K_3 = v_{10}uw_{10}w_{11}uv_{11}v_{10}; \qquad K_4 = v_{01}\ell w_{01}w_{11}\ell v_{11}v_{01}.$$

We choose v and w be the vertices $O + \alpha$ and $T + \alpha$, respectively. Clearly, the set of vertices $\{O + \alpha, T + \alpha; \alpha \text{ is even}, \alpha_{2k} = 0\}$ is the set of all even vertices of Q_{d-2}. On the other hand, consider a cycle of $\mathscr{C}$ which is of the form $C_2 + \alpha$. In this case we choose as v and w the vertices $O + \alpha + \ell_1$ and $T' + \alpha + \ell_1$, respectively. Then the set $\{O + \alpha + \ell_1, T' + \alpha + \ell_1; \alpha \text{ is even}, \alpha_{4k} = 0\}$ is the set of all odd vertices of Q_{d-2}. Hence, the cycles generated in the above stated manner form a decomposition D of Q_d.

We now show that the decomposition is balanced. From the way we constructed the decomposition, it is sufficient to pick from any $C \in D$ a vertex v_C and that $\{v_C; C \in D\}$ is the set of all even vertices of Q_d. In the case that the underlying cycle from $\mathscr{C}$ is of the form $C_1 + \alpha$ we choose as v_C the vertex v_{00} for K_1, w_{00} for K_2, v_{11} for K_3, and w_{11} for K_4. For a cycle of $\mathscr{C}$ of the form $C_2 + \alpha$ we choose v_{01} for K_1, w_{10} for K_2, v_{01} for K_3, and w_{01} for K_4.

3. Decomposing a cube into a graph

Our first main result is the following generalization of the theorem of Fink, Jacobson, Truzczyński and Tuza.

https://doi.org/10.1017/S1446788700000112 Published online by Cambridge University Press

THEOREM 2. *Let G be a graph of size n, each block of which is either a cycle or an edge. If G is embeddable into Q_n then Q_n can be decomposed into G.*

REMARK. Since Q_n is bipartite G is embeddable into Q_n if and only if each cycle of G has even length.

PROOF. We proceed by induction on the number of blocks of G. To be able to carry out the second step of the induction we prove a stronger statement, namely that there is a balanced decomposition of Q_n into G.

Firstly, suppose the number m of blocks equals 1. If G is a single edge the statement is obvious. If G is a cycle then it must be of even length and the claim follows from Lemma 1.

Now we assume $m > 1$. Suppose first that G is connected and let w be a cutpoint of G. We split G at w into two connected subgraphs F and H of sizes k and $n-k$ respectively. Any block of G belongs entirely to F or entirely to H and w is the only vertex which belongs to both F and H. Now we consider two decompositions A and B of the set of vertices of Q_n. A is a decomposition into 2^k classes where two vertices of Q_n belong to the same class when their last k coordinates coincide. B has 2^{n-k} classes; two vertices are in the same class of B when their first $n-k$ coordinates coincide. Clearly, the subgraph of Q_n induced by any class in A is an $(n-k)$-dimensional cube. Analogously, a k-dimensional cube is induced by any class in B.

It is straightforward that the $2^k + 2^{n-k}$ cubes induced by A and B form an edge-decomposition of Q_n. Take a balanced decomposition into H of any $(n-k)$-dimensional cube Q^* given by a class of A and a balanced decomposition into F of any k-dimensional cube Q^* given by a class of B such that the images of the vertex w occupy even vertices. (Note that Q^* is a k- (or $(n-k)$-) dimensional cube but any of its vertices has n coordinates and the phrase 'even vertex' refers to the number of non-zero coordinates in this description of vertices of Q^*.) Thus we get 2^{n-1} subgraphs of G isomorphic to F and 2^{n-1} subgraphs of G isomorphic to H with vertex w occupying any even vertex twice.

For each even vertex, take the subgraphs isomorphic to F and H which have that vertex as w and paste them together (at w) to form a graph isomorphic to G. Then these graphs form a decomposition of Q_n into G, and clearly it is balanced.

If G is disconnected we proceed as above, where F is a component of G and $H = G - F$, and we skip over the last step of pasting copies of F and H.

It is obvious from the proof that Theorem 2 could be strengthened in the following way: let $\mathscr{H}$ be a family of graphs such that for any graph $H \in \mathscr{H}$ there is a balanced decomposition of $Q_{|H|}$ into H. Then for any graph G all of whose blocks are from $\mathscr{H}$ there is a balanced decomposition of $Q_{|G|}$ into G.

4. Decomposing a cube into a family

In order to prove our theorem on decomposing cubes into families of trees, we need a result on decomposing the union of two disjoint copies of Q_3 into rooted trees of size 3. We consider the set $S = \{P_3^m, P_3^t, C^c, C^p\}$, where:

P_3^m is a path of length 3 rooted at a midpoint;
P_3^t is a path of length 3 rooted at a terminal point;
C^c is a claw of size 3 rooted at the center;
C^p is a claw of size 3 rooted at a pendant vertex.

We write Q_3^1 and Q_3^2 for two disjoint copies of Q_3, with vertices $\{000^1, \dots, 111^1\}$ and $\{000^2, \dots, 111^2\}$ respectively. $H = Q_3^1 \cup Q_3^2$.

LEMMA 3. *Suppose $\mathscr{F}^1$ is the collection $\{T_1, T_2, T_3, T_4, T_5, T_6\}$, where each T_i is a member of S. Then one can choose T_7 and T_8 in $\mathscr{F}^1$ so that there is a decomposition $H = \bigcup_{i=1}^{8} T_i$ with the property that the roots of the T_i which lie in Q_3^1 form a set V^1 and the roots of the T_i which lie in Q_3^2 form a set V^2 where $V^2 = \{v^2 : v^1 \notin V^1\}$.*

PROOF. First we state four propositions which can be easily verified by the reader. Denote by E and O the sets of even or odd vertices of Q_3, respectively.

PROPOSITION 1. *For any tree F in $\mathscr{S}$ there is a decomposition of Q_3 into F so that the roots of the F's occupy the set E.*

PROPOSITION 2. *Let F_1, F_2 be a pair of trees of $\mathscr{S}$ such that $\{F_1, F_2\} \neq \{C^c, C^p\}$. Then there is a decomposition of Q_3 into two copies of F_1 and two copies of F_2 so that the roots occupy the set E.*

PROPOSITION 3. *Let $\{F_1, F_2\} = \{C^c, C^p\}$. Then there is a decomposition of Q_3 into three copies of F_1 and a copy of F_2 so that the roots occupy the vertices of the set $S = \{000, 100, 001, 111\}$.*

PROPOSITION 4. *There is an $\mathscr{S} = \{P_3^m, P_3^t, C^c, C^p\}$ decomposition of Q_3 so that the roots occupy the set E.*

In each case, the symmetry of Q_3 means that the proposition remains true if the set of root positions is replaced by its complement (S by its complement in Proposition 3, E by O in the others).

To exhibit the required decompositions of H we use the notation

$$(a, b, c, d) \to (a', b', c', d') : (a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2).$$

Here (a, b, c, d) means that $\mathscr{F}^1$ contains a copies of A, b copies of B, c copies of C and d of D (where A, B, C, D are P_3^m, P_3^l, C^c, C^p in some order); $a+b+c+d=6$. (a', b', c', d') exhibits the same information when trees T_7 and T_8 are included, so $a'+b'+c'+d'=8$. The quadruple (a_1, b_1, c_1, d_1) gives the same information for the trees in Q_3^1, and (a_2, b_2, c_2, d_2) for Q_3^2, so $a_1+a_2=a'$, and so on. For example, $(4,2,0,0) \to (4,4,0,0) : (4,0,0,0)+(0,4,0,0)$ means that in the family $\mathscr{F}^1 = \{T_i^1; i=1,\dots,6\}$ one tree from $\mathscr{S}$ occurs 4 times, one other tree occurs 2 times and the other two are not represented. As T_7^1 and T_8^1 we choose two copies of the tree of $\mathscr{S}$ which is $\mathscr{F}^1$ twice. In the decomposition of H we have 4 copies of the first tree in Q_3^1 and four copies of the second tree in Q_3^2. By Proposition 1 we can carry out the first decomposition so that the four roots are the members of E, and the second so that the roots are the members of O, so the roots have the required property.

Below we list all possibilities and corresponding choices of T_7^1 and T_8^1 and decompositions of H.

$$\begin{aligned}
&(6,0,0,0) \to (8,0,0,0) : (4,0,0,0)+(4,0,0,0) \\
&(5,1,0,0) \to (6,2,0,0) : (4,0,0,0)+(2,2,0,0) \quad \text{otherwise} \\
&\qquad\qquad\qquad\qquad\quad : (3,1,0.0)+(3,1,0,0) \quad \text{if } \{A,B\}=\{C^c,C^p\} \\
&(4,2,0,0) \to (4,4,0,0) : (4,0,0,0)+(0,4,0,0) \\
&(4,1,1,0) \to (4,2,2,0) : (4,0,0,0)+(0,2,2,0) \quad \text{otherwise} \\
&\qquad\qquad\qquad\qquad\quad : (2,2,0,0)+(2,0,2,0) \quad \text{if } \{B,C\}=\{C^c,C^p\} \\
&(3,3,0,0) \to (4,4,0,0) : (4,0,0,0)+(0,4,0,0) \\
&(3,2,1,0) \to (4,2,2,0) : \text{ as above} \\
&(3,1,1,1) \to (5,1,1,1) : (4,0,0,0)+(1,1,1,1) \\
&(2,2,1,1) \to (2,2,2,2) : (1,1,1,1)+(1,1,1,1).
\end{aligned}$$

In each case it follows from Propositions 1–4 that the roots can be placed appropriately.

THEOREM 4. *Suppose $\mathscr{D} = \{T_1, T_2, T_3, T_4, T_5, T_6\}$, where the T_i are trees of size d. Then Q_d decomposes into $\mathscr{D}$.*

PROOF. We represent the cube Q_d as shown in Figure 2. The eight squares stand for subcubes Q_{d-3}; Q_{ijk} is formed from all vertices with last three binary digits ijk. Any eight vertices with the same first $d-3$ coordinates induce a subcube of dimension 3. One of these 2^{d-3} subcubes is represented in the Figure 2; v is a vertex of Q_{d-3} (the same vertex in each case) and the lines represent the Q_3. We write $Q_3(v)$ for this 3-cube.

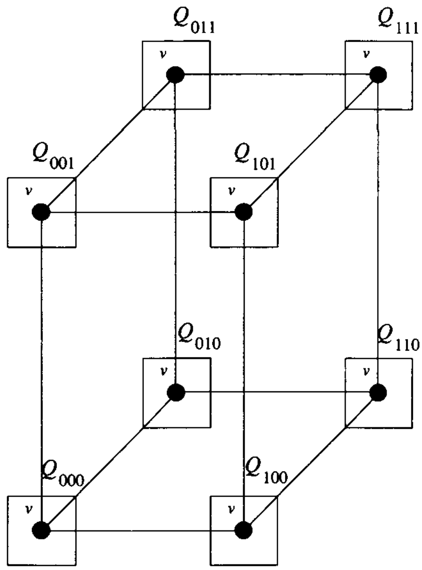

FIGURE 2.

For each i select a vertex x_i of T_i such that we can split T_i at x_i into two subtrees T_i^1, of size 3, and T_i^2. We view T_i^1 and T_i^2 as rooted at x_i; T_i^1 must be isomorphic to one of P_3^m, P_3^t, C^c or C^p. Write $\mathscr{F}^1 = \{T_1^1, T_2^1, T_3^1, T_4^1, T_5^1, T_6^1\}$. Select two trees T_7^1 and T_8^1 from $\mathscr{F}^1$, and find a decomposition of a graph $H = Q_3^1 + Q_3^2$, as in Lemma 3.

The subcube $Q_3(v)$ is decomposed as Q_3^1 if v is an even vertex, and as Q_3^2 if v is odd. For each ijk we choose a balanced decomposition of Q_{ijk} into T_s^2, where s is the index such that the tree whose root was placed at either ijk^1 or ijk^2 in the decomposition of H was a T_s^1. (If $s = 7$ or 8 we take T_s^2 the tree T_r^2 where T_s^1 is isomorphic to T_r^1.) If it was at ijk^1 then the roots of the copies of T_s^2 will occupy all the even vertices of Q_{ijk}, otherwise they occupy all the odd vertices of Q_{ijk}. In either case, at each root we glue together a copy of T_s^1 and T_s^2 to form a member of $\mathscr{D}$. These trees form the desired decomposition of Q_d.

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