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ON THE EXTINCTION OF A CLASS OF POPULATION-SIZE-DEPENDENT BISEXUAL BRANCHING PROCESSES

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Abstract

In this paper, we study a class of bisexual Galton–Watson branching processes in which the law of offspring distribution is dependent on the population size. Under a suitable condition on the offspring distribution, we prove that the limit of mean growth-rate per mating unit exists. Based on this limit, we give a criterion to identify whether the process admits ultimate extinction with probability one.

Keywords: Bisexual Galton–Watson branching process; population-size-dependent branching process; extinction probability

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1. Introduction of bisexual Galton–Watson branching processes

The bisexual Galton–Watson process was first introduced by Daley (1968a) as a two-type branching model $\{(F_n, M_n), n = 1, 2, \ldots\}$, which can be described in the following way.

Let F_n and M_n respectively denote the number of females and males in the nth generation, and let $L(\cdot, \cdot)$ be a mating function that describes their mating rule. We denote by Z_n the number of mating units in the nth generation. Moreover, $\xi_{n,j}$ and $\eta_{n,j}$ are the numbers of females and males produced by the jth mating unit in the nth generation, respectively. To formulate the model, we begin with $Z_0 \in \mathbb{N}^+$ (the set of all nonnegative integers) and inductively define

$$(F_{n+1}, M_{n+1}) := \sum_{j=1}^{Z_n} (\xi_{n,j}, \eta_{n,j}), \qquad n = 0, 1, 2, \ldots,$$

$$Z_{n+1} := L(F_{n+1}, M_{n+1}), \qquad n = 0, 1, 2, \ldots.$$

(Here we use the convention that empty sums equal (0, 0)).

We make the usual assumptions on the model, as follows. The $(\xi_{n,j}, \eta_{n,j})$, where $n = 0, 1, \ldots$ and $j = 1, 2, \ldots$, are independent, identically distributed, bivariate random variables taking values in $\mathbb{N}^+ \times \mathbb{N}^+$, and the mating function $L : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing in each variable and satisfies the inequalities

$$L(x, y) \le xy,$$
$$L\Big(\sum_{i=1}^n x_i, \sum_{i=1}^n y_i\Big) \ge \sum_{i=1}^n L(x_i, y_i) \tag{1.1}$$

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for any $n \geq 2$ and any $(x, y), (x_i, y_i) \in \mathbb{R}^+ \times \mathbb{R}^+$. (The second inequality is known as the superadditive property.)

Bisexual branching processes have received much attention in the literature. The extinction problem has been studied by Daley (1968a), Hull (1982), (1984), Bruss (1984), Daley *et al.* (1986), and Alsmeyer and Rösler (1996). The main result, proved by Daley *et al.* (1986), is based on the concept of mean growth-rate per mating unit, i.e.

$$r_k := k^{-1}\,\mathrm{E}[Z_{n+1} \mid Z_n = k], \qquad k = 1, 2 \dots,$$

which was introduced by Bruss (1984). Daley *et al.* (1986) proved that, for a superadditive branching process, the asymptotic growth-rate $r := \lim_{k\to\infty} r_k$ exists, and that

$$\mathrm{P}(Z_n \to 0, n \to \infty \mid Z_0 = j) = 1, \qquad j = 1, 2, \dots,$$

if and only if, excluding trivial cases,

$$r \leq 1.$$

Molina *et al*. (2002) suggested a bisexual Galton–Watson model with population-size-dependent mating. They obtained a necessary and sufficient condition for the process to become extinct with probability 1.

In this paper, we are interested in the so-called population-size-dependent bisexual Galton–Watson processes (PSDBPs), i.e. the class of bisexual Galton–Watson processes whose offspring reproduction laws depend on the size of the population. The biological motivation for this model is that population size governs reproduction laws.

In section 2, the probabilistic model is described and basic concepts and necessary results are introduced. Then, in section 3, we give a criterion (see Theorem 3.1) to identify whether the process admits ultimate extinction with probability one.

2. The probabilistic model

We define a population-size-dependent bisexual Galton–Watson process via a two-type sequence $(F_n^\star, M_n^\star)_n$, as follows:

$$Z_0^\star := N,$$
$$(F_{n+1}^\star, M_{n+1}^\star) := \sum_{j=1}^{Z_n^\star} (\xi_{n,j}^{(Z_n^\star)}, \eta_{n,j}^{(Z_n^\star)}), \qquad n = 0, 1, \dots,$$
$$Z_{n+1}^\star := L(F_{n+1}^\star, M_{n+1}^\star), \qquad n = 0, 1, \dots.$$

Here, the law of offspring distribution depends only on the size of the parental generation. We assume that, for every $k = 1, 2, \dots$, the random variables $(\xi_{n,j}^{(k)}, \eta_{n,j}^{(k)})$, where $n = 0, 1 \dots$, $j = 1, 2, \dots$, are independent and have the same distribution as $(\xi_{0,1}^{(k)}, \eta_{0,1}^{(k)})$. As usual, the mating function $L(\cdot, \cdot)$ is assumed to be superadditive (see (1.1)).

Remark 2.1. It is not hard to check that $\{Z_n^\star, n \geq 0\}$ is a homogeneous Markov chain and that 0 is an absorbing state. However, $\{Z_n^\star, n \geq 0\}$ is not a stochastically monotone Markov chain in the sense of Daley (1968b).

Throughout this paper, we suppose that the sequence of offspring random variables $(\xi_{0,1}^{(k)}, \eta_{0,1}^{(k)})_k$ satisfies the following condition.

Condition 2.1. *The sequence* $(\xi_{0,1}^{(k)}, \eta_{0,1}^{(k)})_k$ *satisfies*

$$\mathrm{E}\, g(\xi_{0,1}^{(k+1)}, \eta_{0,1}^{(k+1)}) \le \mathrm{E}\, g(\xi_{0,1}^{(k)}, \eta_{0,1}^{(k)}) \tag{2.1}$$

for every bounded, componentwise-increasing function $g(\cdot, \cdot)$.

Remark 2.2. It follows from Kamae *et al.* (1977) that, simultaneously for all $k \ge 1$, there exist random variables $(\xi^{(k)}, \eta^{(k)})^\star$ and $(\xi^{(k+1)}, \eta^{(k+1)})^\star$, defined on the same probability space and having the same (respective) distributions as $(\xi_{0,1}^{(k)}, \eta_{0,1}^{(k)})$ and $(\xi_{0,1}^{(k+1)}, \eta_{0,1}^{(k+1)})$, such that

$$(\xi^{(k)}, \eta^{(k)})^\star = (\xi^{(k+1)}, \eta^{(k+1)})^\star + (\xi^{(k,k+1)}, \eta^{(k,k+1)}), \qquad k = 0, 1, \dots,$$

for nonnegative, integer-valued random variables $(\xi^{(k,k+1)}, \eta^{(k,k+1)})$.

In the following, we will consider the random variables $(\xi^{(k)}, \eta^{(k)})^\star$ given in this remark, instead of $(\xi_{0,1}^{(k)}, \eta_{0,1}^{(k)})$. However, in an abuse of notation, we will write $(\xi^{(k)}, \eta^{(k)})$ for $(\xi^{(k)}, \eta^{(k)})^\star$.

With this, we can immediately make the following proposition.

Proposition 2.1. *Under Condition 2.1,*

1. *the sequence* $(\xi^{(k)}, \eta^{(k)})_k$ *converges almost surely to a pair of nonnegative, integer-valued random variables* (ξ, η)*; and*
2. *the sequence* $(\mathrm{E}\, g(\xi^{(k)}, \eta^{(k)}))_k$ *is monotonic, nonincreasing, and converges to* $\mathrm{E}\, g(\xi, \eta)$*, where* $g(\cdot, \cdot)$ *is defined as in Condition 2.1.*

So, letting $g(x, y) = x$ and $g(x, y) = y$ in turn, we have, respectively,

$$\lim_{k\to\infty} \mathrm{E}\, \xi^{(k)} = \mathrm{E}\, \xi \quad \text{and} \quad \lim_{k\to\infty} \mathrm{E}\, \eta^{(k)} = \mathrm{E}\, \eta. \tag{2.2}$$

In order to obtain a sufficient condition for the process to become extinct, i.e. to have $Z_n^\star = 0$ with probability 1 for some positive integer n, we must define some characteristic quantities.

Definition 2.1. We define ultimate extinction to be the event $Q := \{Z_n^\star \to 0, n \to \infty\}$, and let

$$q(j) := \mathrm{P}(Q \mid Z_0^\star = j), \qquad j = 1, 2, \dots.$$

(Actually, $Q = \bigcup_{n=1}^{\infty} \{Z_n^\star = 0\}$.)

The following assertion originally comes from Molina *et al.* (2002), but the proof needs some modifications in our case.

Proposition 2.2. *Let* $\{Z_n^\star, n \ge 0\}$ *be a PSDBP satisfying Condition 2.1, and suppose that the mating function satisfies* $L(1, 1) = 1$*. If*

$$\mathrm{P}(Z_{n+1}^\star = j \mid Z_n^\star = j) < 1 \quad \textit{holds for all } j = 1, 2, \dots, \tag{2.3}$$

then

$$\mathrm{P}(Z_n^\star \to 0) + \mathrm{P}(Z_n^\star \to \infty) = 1.$$

Proof. It suffices to prove that, if $k \ne 0$, then k is transient in the Markov chain $\{Z_n^\star, n \ge 0\}$.

Step 1. Suppose that, for any $k = 1, 2, \dots$ and $j = 0, 1, \dots$, the offspring distribution satisfies

$$\mathrm{P}(\xi^{(k)} = 0, \eta^{(k)} = j) = \mathrm{P}(\xi^{(k)} = j, \eta^{(k)} = 0) = 0.$$

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This means that every mating unit has among its offspring at least one female and one male. Since $L(1, 1) = 1$, we have

$$Z_1^\star \le Z_2^\star \le Z_3^\star \le \cdots .$$

It follows that if

$$f_{kk}^\star := \mathrm{P}(Z_{n+m}^\star = k \text{ for some } m \ge 1 \mid Z_n^\star = k),$$

then

$$f_{kk}^\star = \mathrm{P}(Z_{n+1}^\star = k \mid Z_n^\star = k) < 1.$$

This implies that k is transient.

Step 2. Suppose that there exists a $k \ge 1$ and a $j \ge 0$ such that either

$$\mathrm{P}(\xi^{(k)} = 0, \eta^{(k)} = j) > 0 \quad \text{or} \quad \mathrm{P}(\xi^{(k)} = j, \eta^{(k)} = 0) > 0.$$

Then either

$$\mathrm{P}(\xi^{(k)} = 0) > 0 \quad \text{or} \quad \mathrm{P}(\eta^{(k)} = 0) > 0.$$

Let

$$N := \inf\{k \ge 1\colon \mathrm{P}(\xi^{(k)} = 0) > 0 \text{ or } \mathrm{P}(\eta^{(k)} = 0) > 0\}.$$

It follows, from Remark 2.1, that either

$$\mathrm{P}(\xi^{(N+m)} = 0) > 0 \quad \text{or} \quad \mathrm{P}(\eta^{(N+m)} = 0) > 0$$

for all $m = 0, 1, 2, \ldots$.

Since $L(x, y) \le xy$, we have $L(0, \cdot) = L(\cdot, 0) = 0$ and, therefore, for $k \ge N$,

$$\begin{aligned}
\mathrm{P}(Z_{n+1}^\star = 0 \mid Z_n^\star = k) &= \mathrm{P}\bigg(L\bigg(\sum_{j=1}^{k}(\xi_{n,j}^{(k)}, \eta_{n,j}^{(k)})\bigg) = 0\bigg) \\
&\ge \mathrm{P}\bigg(\sum_{j=1}^{k}\xi_{n,j}^{(k)} = 0 \text{ or } \sum_{j=1}^{k}\eta_{n,j}^{(k)} = 0\bigg) \\
&\ge \max\bigg\{\mathrm{P}\bigg(\sum_{j=1}^{k}\xi_{n,j}^{(k)} = 0\bigg), \mathrm{P}\bigg(\sum_{j=1}^{k}\eta_{n,j}^{(k)} = 0\bigg)\bigg\} \\
&= \max\{\mathrm{P}(\xi^{(k)} = 0)^k, \mathrm{P}(\eta^{(k)} = 0)^k\} \\
&> 0.
\end{aligned}$$

Since 0 is an absorbing state, we deduce that

$$f_{kk}^\star \le 1 - \mathrm{P}(Z_{n+1}^\star = 0 \mid Z_n^\star = k) < 1.$$

This implies that k is transient.

For $k < N$, by the definition of N, we have

$$\mathrm{P}(\xi^{(k)} = 0) = \mathrm{P}(\eta^{(k)} = 0) = 0.$$

So, $\mathrm{P}(\xi^{(k)} = 0, \eta^{(k)} = j) = \mathrm{P}(\xi^{(k)} = j, \eta^{(k)} = 0) = 0$ for all $j = 0, 1, \ldots$.

We claim that

$$P(Z_n^\star \ge N \text{ for some } n \ge 1 \mid Z_0^\star = k) > 0.$$

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Otherwise

$$P(Z_n^\star \geq N \mid Z_0^\star = k) = 0 \quad \text{for every } n \geq 1,$$

which implies that

$$Z_0^\star \leq Z_1^\star \leq Z_2^\star \leq \cdots < N,$$

contradicting (2.3). Hence, the states no smaller than N are accessible from k, and k is transient.

In summary, $k \neq 0$ is transient for the Markov chain $\{Z_n^\star, n \geq 0\}$, and the assertion of the proposition follows.

3. Extinction probability under Condition 2.1

In this section, we will study the extinction problem of a PSDBP under Condition 2.1. By developing some techniques similar to those of Daley *et al.* (1986) and Molina *et al.* (2002), we obtain a criterion analogous to theirs for superadditive branching processes.

For our purposes, we introduce the 'mean growth-rates per mating unit' for a PSDBP.

Definition 3.1. Let $\{Z_n^\star, n \geq 0\}$ be a PSDBP. For every positive integer k, we define the mean growth-rate per mating unit as

$$r_k^\star := \frac{1}{k} \operatorname{E}[Z_{n+1}^\star \mid Z_n^\star = k] = \frac{1}{k} \operatorname{E} L\biggl(\sum_{j=1}^{k} (\xi_{n,j}^{(k)}, \eta_{n,j}^{(k)})\biggr).$$

Proposition 3.1. *Assume that the PSDBP* $\{Z_n^\star, n \geq 0\}$ *satisfies Condition 2.1. Then the limit* $r^\star := \lim_{k\to\infty} r_k^\star$ *exists.*

Remark 3.1. Proposition 3.1 plays a key role in the derivation of the extinction probability. However, for the PSDBP, the function ψ defined by $\psi(j) := j r_j^\star$ is *not* superadditive, so the well-known method described in Daley *et al.* (1986) does not work in our case.

Proof of Proposition 3.1. The proof is done in two steps.

Step 1. For every $m \geq 1$, we define a bisexual Galton–Watson process $\{Z_n^{(m)}, n \geq 0\}$ as follows:

$$\begin{aligned} Z_0^{(m)} &:= N, \\ (F_{n+1}^{(m)}, M_{n+1}^{(m)}) &:= \sum_{j=1}^{Z_n^{(m)}} (\xi_{n,j}^{(m)}, \eta_{n,j}^{(m)}), \qquad n = 0, 1, 2, \ldots, \\ Z_{n+1}^{(m)} &:= L(F_{n+1}^{(m)}, M_{n+1}^{(m)}), \qquad n = 0, 1, 2, \ldots, \end{aligned}$$

where mating units of each generation produce offspring independently, but with the same offspring probability distribution as $(\xi^{(m)}, \eta^{(m)})$, i.e.

$$\operatorname{E} \xi^{(m)} = \operatorname{E} \xi_{n,j}^{(m)} \quad \text{and} \quad \operatorname{E} \eta^{(m)} = \operatorname{E} \eta_{n,j}^{(m)},$$

for all $n = 0, 1, \ldots,\ j = 1, 2, \ldots$. By (2.2), we have

$$\lim_{m\to\infty} \operatorname{E} \xi^{(m)} = \operatorname{E} \xi \quad \text{and} \quad \lim_{m\to\infty} \operatorname{E} \eta^{(m)} = \operatorname{E} \eta. \tag{3.1}$$

Let

$$r_k^{(m)} := \frac{1}{k}\,\mathrm{E}[Z_{n+1}^{(m)} \mid Z_n^{(m)} = k] = \frac{1}{k}\,\mathrm{E}\,L\bigg(\sum_{j=1}^{k}(\xi_{n,j}^{(m)}, \eta_{n,j}^{(m)})\bigg).$$

Then, by Theorem 1 of Daley *et al.* (1986), the limit $r^{(m)} := \lim_{k\to\infty} r_k^{(m)}$ exists, with $r^{(m)} = \sup_{k>0} r_k^{(m)}$. Furthermore, by Theorem 3.2 of Molina *et al.* (2002),

$$r^{(m)} = r(\mathrm{E}\,\xi^{(m)}, \mathrm{E}\,\eta^{(m)}),$$

where the function $r(x, y)$ is continuous for every nonnegative-valued (x, y) – see Proposition 3.2 of Molina *et al.* (2002). So, by (3.1), the limit $r := \lim_{m\to\infty} r^{(m)}$ exists, and

$$r = r(\mathrm{E}\,\xi, \mathrm{E}\,\eta). \tag{3.2}$$

Next, we consider a bisexual Galton–Watson process $\{\tilde{Z}_n, n \geq 0\}$ with the same offspring probability distribution as that of (ξ, η), i.e.

$$\tilde{Z}_0 := N,$$

$$(\tilde{F}_{n+1}, \tilde{M}_{n+1}) := \sum_{j=1}^{\tilde{Z}_n}(\xi_{n,j}, \eta_{n,j}),$$

$$\tilde{Z}_{n+1} := L(\tilde{F}_{n+1}, \tilde{M}_{n+1}).$$

(It is worth mentioning that $\{\tilde{Z}_n, n \geq 0\}$ is a stochastically monotone Markov chain.)

By Theorem 1 of Daley *et al.* (1986) and Theorem 3.2 of Molina *et al.* (2002), $\tilde{r} := \lim_{k\to\infty} k^{-1}\,\mathrm{E}[\tilde{Z}_{n+1} \mid \tilde{Z}_n = k]$ exists, with

$$\tilde{r} = \sup_{k\geq 1} \frac{1}{k}\,\mathrm{E}\,L\bigg(\sum_{j=1}^{k}(\xi_{n,j}, \eta_{n,j})\bigg) = r(\mathrm{E}\,\xi, \mathrm{E}\,\eta). \tag{3.3}$$

It follows, from (3.2) and (3.3), that

$$r = \tilde{r} = r(\mathrm{E}\,\xi, \mathrm{E}\,\eta). \tag{3.4}$$

Step 2. By Proposition 2.1(2), we have

$$\frac{1}{k}\,\mathrm{E}\,L\bigg(\sum_{j=1}^{k}(\xi_{n,j}^{(k)}, \eta_{n,j}^{(k)})\bigg) \geq \frac{1}{k}\,\mathrm{E}\,L\bigg(\sum_{j=1}^{k}(\xi_{n,j}, \eta_{n,j})\bigg), \qquad k \geq 1,$$

and so

$$\liminf_{k\to\infty} \frac{1}{k}\,\mathrm{E}\,L\bigg(\sum_{j=1}^{k}(\xi_{n,j}^{(k)}, \eta_{n,j}^{(k)})\bigg) \geq \tilde{r}. \tag{3.5}$$

For every $m \geq 1$, by applying Proposition 2.1(2), we find that

$$\limsup_{k\to\infty} \frac{1}{k}\,\mathrm{E}\,L\bigg(\sum_{j=1}^{k}(\xi_{n,j}^{(k)}, \eta_{n,j}^{(k)})\bigg) \leq \lim_{k\to\infty} \frac{1}{k}\,\mathrm{E}\,L\bigg(\sum_{j=1}^{k}(\xi_{n,j}^{(m)}, \eta_{n,j}^{(m)})\bigg).$$

This implies that

$$\limsup_{k\to\infty} \frac{1}{k} \operatorname{E} L\bigg(\sum_{j=1}^{k} (\xi_{n,j}^{(k)}, \eta_{n,j}^{(k)})\bigg) \le r^{(m)},$$

and so

$$\limsup_{k\to\infty} \frac{1}{k} \operatorname{E} L\bigg(\sum_{j=1}^{k} (\xi_{n,j}^{(k)}, \eta_{n,j}^{(k)})\bigg) \le r. \tag{3.6}$$

Combining (3.4), (3.5), and (3.6), we deduce that $r^\star = \lim_{k\to\infty} k^{-1} \operatorname{E} L(\sum_{j=1}^{k} (\xi_{n,j}^{(k)}, \eta_{n,j}^{(k)}))$ exists, with

$$r^\star = r = \tilde{r}. \tag{3.7}$$

Theorem 3.1. *For the PSDBP $\{Z_n^\star, n \ge 0\}$ satisfying Condition 2.1, the following assertions hold.*

1. *If $r^\star < 1$ then $q(j) = 1$ for $j = 1, 2, \ldots$.*
2. *If $r^\star > 1$ then $q(j) < 1$ for $j = 1, 2, \ldots$.*

Proof. 1. If $r^\star < 1$, we have $r = r^\star < 1$, by (3.7), recalling that $r = \lim_{m\to\infty} r^{(m)}$. Therefore, there exist at most finitely many ms such that $r^{(m)} \ge 1$, and we can thus suppose that there exists a positive integer k with $r^{(l+k)} < 1$ for all $l > 0$. Let $\alpha := \max\{1, r^{(1)}, r^{(2)}, \ldots, r^{(k)}\}$. Since $r^{(m)} = \sup_{k>0} r_k^{(m)}$, we have

$$\begin{aligned}
\operatorname{E} Z_{n+1}^\star &= \sum_{m=0}^{\infty} \operatorname{P}(Z_n^\star = m) \operatorname{E}[Z_{n+1}^\star \mid Z_n^\star = m] \\
&= \sum_{m=0}^{\infty} \operatorname{P}(Z_n^\star = m) \operatorname{E} L\bigg(\sum_{j=1}^{m} (\xi_{n,j}^{(m)}, \eta_{n,j}^{(m)})\bigg) \\
&= \sum_{m=1}^{\infty} \operatorname{P}(Z_n^\star = m) m r_m^{(m)} \quad \text{(by the definition of } r_k^{(m)}) \\
&= \sum_{m=1}^{k} \operatorname{P}(Z_n^\star = m) m r_m^{(m)} + \sum_{m=k+1}^{\infty} \operatorname{P}(Z_n^\star = m) m r_m^{(m)} \\
&\le \alpha \sum_{m=1}^{k} m \operatorname{P}(Z_n^\star = m) + \sum_{m=k+1}^{\infty} m \operatorname{P}(Z_n^\star = m) \\
&= \operatorname{E} Z_n^\star + (\alpha - 1) \sum_{m=1}^{k} m \operatorname{P}(Z_n^\star = m).
\end{aligned}$$

This implies that

$$\operatorname{E} Z_{n+1}^\star \le \operatorname{E} Z_n^\star + (\alpha - 1) \sum_{m=1}^{k} m \operatorname{P}(Z_n^\star = m), \qquad n \ge 0.$$

Iterating this inequality, we find that

$$\operatorname{E} Z_n^\star \le \operatorname{E} Z_0^\star + (\alpha - 1) \sum_{m=1}^{k} m \sum_{j=0}^{n-1} \operatorname{P}(Z_j^\star = m),$$

recalling that, if $m \neq 0$, m is transient for $\{Z_n^\star, n \geq 0\}$. Furthermore, if we define $\lambda_m := \sum_{j=0}^{\infty} \mathrm{P}(Z_j^\star = m) < \infty$, we have

$$\mathrm{E}\, Z_n^\star \leq \mathrm{E}\, Z_0^\star + (\alpha - 1) \sum_{m=1}^{k} m\lambda_m < \infty \quad \text{for all } n.$$

This implies that

$$\mathrm{P}(Z_n^\star \to \infty, n \to \infty) = 0,$$

i.e. $q(j) = 1$.

2. If $r^\star > 1$ then $\tilde{r} = r^\star > 1$, by (3.7). Recall that $\{Z_n^\star, n \geq 0\}$ and $\{\tilde{Z}_n, n \geq 0\}$ have, respectively, the offspring random variables $(\xi_{n,j}^{(m)}, \eta_{n,j}^{(m)})$ and $(\xi_{n,j}, \eta_{n,j})$, where m represents the present population size of $\{Z_n^\star, n \geq 0\}$.

By Proposition 2.1(2),

$$\mathrm{E}\, g(\xi_{n,j}^{(m)}, \eta_{n,j}^{(m)}) \geq \mathrm{E}\, g(\xi_{n,j}, \eta_{n,j}),$$

where $g(\cdot, \cdot)$ is as defined in (2.1). So, from our Remark 2.1 and Theorem 1 of Daley (1968b), we have

$$\mathrm{P}(Z_n^\star \to 0 \mid Z_0^\star = j) \leq \mathrm{P}(\tilde{Z}_n \to 0 \mid \tilde{Z}_0 = j).$$

Since $\tilde{r} > 1$, we see that $\mathrm{P}(\tilde{Z}_n \to 0 \mid \tilde{Z}_0 = j) < 1$. Therefore $\mathrm{P}(Z_n^\star \to 0 \mid Z_0^\star = j) < 1$, i.e. $q(j) < 1$.

In the following, we give an example to illustrate that the extinction property is not certain when $r^\star = 1$. This example demonstrates (see Proposition 3.2, below) that the extinction argument depends heavily on the convergence rate of $r^{(m)} \to 1$.

For the PSDBP $\{Z_n^\star, n \geq 0\}$ with promiscuous mating, i.e. where the mating function satisfies $L(x, y) = x \min\{1, y\}$, we have, by the definition of $\{Z_n^{(m)}, n \geq 0\}$,

$$\begin{aligned}
r^{(m)} &= \lim_{k\to\infty} \frac{1}{k} \mathrm{E}[Z_{n+1}^{(m)} \mid Z_n^{(m)} = k] \\
&= \lim_{k\to\infty} \frac{1}{k} \mathrm{E}\, L\Bigg(\sum_{j=1}^{k} (\xi_{n,j}^{(m)}, \eta_{n,j}^{(m)})\Bigg) \\
&= \lim_{k\to\infty} \frac{1}{k} \mathrm{E}\Bigg[\Bigg(\sum_{j=1}^{k} \xi_{n,j}^{(m)}\Bigg) \min\Bigg\{1, \sum_{j=1}^{k} \eta_{n,j}^{(m)}\Bigg\}\Bigg] \\
&= \lim_{k\to\infty} \mathrm{E}\, \xi_{n,j}^{(m)}\, \mathrm{P}\Bigg(\sum_{j=1}^{k} \eta_{n,j}^{(m)} > 0\Bigg) \\
&= \lim_{k\to\infty} \mathrm{E}\, \xi_{n,j}^{(m)} [1 - \mathrm{P}(\eta_{n,j}^{(m)} = 0)^k] \\
&= \mathrm{E}\, \xi_{n,j}^{(m)} \\
&= \mathrm{E}\, \xi^{(m)}.
\end{aligned}$$

Let $\{F_n^\star, n \geq 0\}$ be an asexual Galton–Watson process defined by $F_{n+1}^\star := \sum_{j=1}^{F_n^\star} \xi_{n,j}^{(F_n^\star)}$. This is actually a population-size-dependent Galton–Watson process (see Klebaner (1984)). Define

$q' := \mathrm{P}(F_n^\star \to 0, n \to \infty)$ and let $\mu_1^{(m)}$ and $\sigma_1^{(m)}$ be, respectively, the mean and the variance of the offspring distribution, where m is the population size. It is easy to see that $\mu_1^{(m)} = \mathrm{E}\,\xi^{(m)} = r^{(m)}$. By letting $g(x, y) = x^2$ in Proposition 2.1(2), we see that $\sigma_1 := \lim_{m\to\infty} \sigma_1^{(m)}$ exists.

Let us assume that one further condition holds.

Condition 3.1. $\lim_{m\to\infty} m(r^{(m)} - 1) = c$ *for* $0 \le c < \infty$.

Then the conditions of Theorem 1.4 of Klebaner (1984) are satisfied, and we have the following proposition.

Proposition 3.2. *Suppose that* $c \neq \sigma_1$. *Then,*

(a) *if* $c > \sigma_1$ *then* $q' < 1$*; and*

(b) *if* $0 < \sigma_1$ *and* $\limsup_{m\to\infty} m^\alpha (r^{(m)} - 1) < \infty$, $\alpha \ge 2$, *then* $q' = 1$.

The PSDBP $\{Z_n^\star, n \ge 0\}$ with promiscuous mating is a 'killed' Markov chain, which kills the process $\{F_n^\star, n \ge 0\}$ at state j with probability $k(j)$, where $k(j) = r(j)^j$ and $r(j) = \mathrm{P}(\eta^{(j)} = 0)$ for $j \in N$. Indeed, if $F_0^\star = j$, this process is killed if no males are produced, which happens with probability $k(j)$. It is thus easy to see that

$$\mathrm{P}(F_n^\star \to 0,\ n \to \infty \mid F_0^\star = j) \le \mathrm{P}(Z_n^\star \to 0,\ n \to \infty \mid Z_0^\star = j). \tag{3.8}$$

Theorem 3.2. *For the PSDBP* $\{Z_n^\star, n \ge 0\}$ *with promiscuous mating satisfying Conditions 2.1 and 3.1, we have the following:*

(a) $c > \sigma_1$ *implies that* $q(j) < 1$*; and*

(b) $0 < \sigma_1$ *and* $\limsup_{m\to\infty} m^\alpha (r^{(m)} - 1) < \infty$, $\alpha \ge 2$, *imply that* $q(j) = 1$.

Proof. (a) By the definition of $q(j)$,

$$\begin{aligned} q(j) &= 1 - \mathrm{P}(Z_n^\star \ge 1 \text{ for all } n \ge 0 \mid Z_0^\star = j) \\ &= 1 - \sum_{i=1}^{\infty} \mathrm{P}(Z_1^\star = i \mid Z_0^\star = j)\, \mathrm{P}(Z_n^\star \ge 1 \text{ for all } n \ge 1 \mid Z_1^\star = i) \\ &= 1 - \mathrm{E}_j[(1 - k(j))\, \mathrm{P}(Z_n^\star \ge 1 \mid Z_1^\star = F_1^\star)] \quad (\text{where } \mathrm{E}_j[\cdot] := \mathrm{E}[\cdot \mid Z_0^\star = j]) \\ &= 1 - \mathrm{E}_j \prod_{n\ge 0} (1 - k(F_n^\star)). \end{aligned}$$

Let $Q' := \{F_n^\star \to \infty, n \to \infty\}$. By applying Proposition 3.2, we find that $\mathrm{P}(Q') > 0$. Therefore, we need only show that $\prod_{n\ge 0}(1 - k(F_n^\star)) > 0$ on Q'. Note that

$$\begin{aligned} \mathrm{E}_j \sum_{n\ge 0} k(F_n^\star) &= \mathrm{E}_j \sum_{n\ge 0} r(F_n^\star)^{F_n^\star} \\ &\le \mathrm{E}_j \sum_{n\ge 0} r(1)^{F_n^\star} \\ &= \sum_{n\ge 0} \sum_{k=1}^{\infty} r(1)^k\, \mathrm{P}_j(F_n^\star = k) \quad (\text{where } \mathrm{P}_j(\cdot) := \mathrm{P}(\cdot \mid Z_0^\star = j)) \\ &= \sum_{k=1}^{\infty} r(1)^k \sum_{n\ge 0} \mathrm{P}_j(F_n^\star = k). \end{aligned}$$

Let $G_{jk} := \sum_{n\geq 0} \mathrm{P}_j(F_n^\star = k)$ be the Green function of the chain $\{F_n^\star, n \geq 0\}$. Then $G_{jk} < \infty$ for all $k \in N$, since k is transient (see Klebaner (1984), p. 32). Thus, we obtain

$$\mathrm{E}_j \sum_{n\geq 0} k(F_n^\star) \leq \sum_{k=1}^{\infty} r(1)^k G_{jk}.$$

On the other hand, $\sum_{k=1}^{\infty} k^{-2} G_{jk} < \infty$ (see Klebaner (1984), p. 35), so we have

$$\mathrm{E}_j \sum_{n\geq 0} k(F_n^\star) < \infty \quad \text{on } Q'.$$

This implies that $\prod_{n\geq 0}(1 - k(F_n^\star)) > 0$ on Q'.

(b) By Proposition 3.2, we have

$$\mathrm{P}(F_n^\star \to 0,\ n \to \infty \mid F_0^\star = j) = 1.$$

So, by (3.8), we see that $q(j) = \mathrm{P}(Z_n^\star \to 0,\ n \to \infty \mid Z_0^\star = j) = 1$. This completes the proof of Theorem 3.2.

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