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DERIVATIVES AND LENGTH-PRESERVING MAPS

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ABSTRACT. Let a be a constant, $|a| = 1$. We shall prove meromorphic (M) and bounded-holomorphic (BH) versions of the following prototype: (P) Let f and g be holomorphic in a domain D. Then, $|f'| = |g'|$ in D if and only if there exist constant a, b with $f = ag + b$ in D. (M) Let f and g be meromorphic in D. Then, $|f'|/(1 + |f|^2) = |g'|/(1 + |g|^2)$ in D if and only if there exist a, b with $|b| \leqq \infty$ such that $f = a(g - b)/(1 + \bar{b}g)$. (BH) Let f and g be holomorphic and bounded, $|f| < 1$, $|g| < 1$, in D. Then, $|f'|/(1 - |f|^2) = |g'|/(1 - |g|^2)$ in D if and only if there exist a, b with $|b| < 1$, such that $f = a(g - b)/(1 - \bar{b}g)$.

1. **Results**. Let D be a domain in the complex plane $\mathbb{C}$ and let Φ_1 be the family of functions $az + b$, where $|a| = 1$ and $b \in \mathbb{C}$. A prototype for the present observation is the following which can be easily proved.

(I) *Let f and g be holomorphic in D. Then, $|f'| = |g'|$ in D if and only if there exists $T \in \Phi_1$ such that $f = T \circ g$ in D.*

Each $T \in \Phi_1$ preserves the Euclidean metric. Let Φ_2 be the family of functions

$$a(z - b)/(1 + \bar{b}z),$$

where $|a| = 1$ and $b \in \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$; if $b = \infty$ then a/z should be considered instead. For f meromorphic in D we set

$$\begin{aligned} f^{\#}(z) &= |f'(z)|/(1 + |f(z)|^2) \quad \text{if} \quad f(z) \neq \infty; \\ &= |(1/f)'(z)| \quad \text{if} \quad f(z) = \infty. \end{aligned}$$

Our first result is:

(II) *Let f and g be meromorphic in D. Then, $f^{\#} = g^{\#}$ in D if and only if there exists $T \in \Phi_2$ such that $f = T \circ g$ in D.*

The "if" part is obvious. Let

$$0 \leqq \tan^{-1} x \leqq \pi/2, \qquad 0 \leqq x \leqq \infty,$$

and set

$$\sigma_S(z, w) = \tan^{-1}(|z - w|/|1 + \bar{z}w|);$$

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this is the spherical metric on $\bar{\mathbb{C}}$. To explain this, let Σ be the Riemann sphere of diameter one touching $\mathbb{C}$ at the origin from above. On identifying Σ with $\bar{\mathbb{C}}$ via the stereographic projection, we observe that, the great circle passing through z and w is divided into two arcs by z and w. The smaller of the lengths of these arcs is $\sigma_S(z, w)$. Each $T \in \Phi_2$ preserves σ_S. Furthermore,

$$\sigma_S(f(w), f(z))/|w - z| \to f^{\#}(z) \qquad \text{as } w \to z.$$

Let Φ_3 be the family of functions

$$a(z - b)/(1 - \bar{b}z),$$

where $|a| = 1$ and $b \in \Delta = \{|z| < 1\}$. For f holomorphic and bounded, $|f| < 1$, in D, we set $f^* = |f'|/(1 - |f|^2)$. Our next result is

(III) *Let f and g be holomorphic and bounded, $|f| < 1$, $|g| < 1$, in D. Then, $f^* = g^*$ in D if and only if there exists $T \in \Phi_3$ such that $f = T \circ g$ in D.*

The "if" part is obvious. The Poincaré metric in Δ is

$$\sigma_P(z, w) = \tanh^{-1}(|z - w|/(1 - \bar{z}w|).$$

Each $T \in \Phi_3$ preserves σ_P. Furthermore,

$$\sigma_P(f(w), f(z))/|w - z| \to f^*(z) \qquad \text{as } w \to z.$$

2. **Proofs**. To prove the "only if" parts of (II) and (III) we shall make use of the lemma due to E. Landau and J. Dieudonné; see [5, Theorem VI.10, p. 259].

LEMMA. *Let f be holomorphic and bounded, $|f| < M$, in Δ with $f(0) = f'(0) - 1 = 0$. Then, f is univalent and starlike in $\Delta(M) = \{|z| < \lambda(M)\}$ with $\lambda(M) = M - (M^2 - 1)^{1/2}$.*

"Starlike" here means that for each $z \in \Delta(M)$,

$$\{tf(z);\ 0 \leqq t \leqq 1\} \subset f(\Delta(M));$$

we note that $1 = |f'(0)| \leqq M$.

To prove the "only if" part of (II) we may assume that f is nonconstant. Then, there exists $w \in D$ such that

$$f(w) \neq \infty \neq g(w) \quad \text{and} \quad f^{\#}(w) \neq 0.$$

Set

$$F = (f - f(w))/(1 + \overline{f(w)}f),$$

$$G = (g - g(w))/(1 + \overline{g(w)}g),$$

in D. It suffices to show that there exists a_1, $|a_1| = 1$, such that

$$F(z) \equiv a_1 G(z) \quad \text{in} \quad D.$$

Obvious computations then complete the proof.

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We fix a constant $R > 0$ such that $\{|z - w| \leqq R\} \subset D$, and we note that

$$K = \max\{f^{\#}(z); |z - w| \leqq R\}$$

is positive and finite because $f^{\#}$ is continuous. Let $0 < r < R$ and $rK < \pi/4$. To verify that $|F| < 1$ and $|G| < 1$ in $D_1(w) = \{|z - w| < r\}$ we let

$$\alpha(w, z) = \{(1 - t)w + tz; 0 \leqq t \leqq 1\}, \qquad z \in D_1(w).$$

Now, $F^{\#} = f^{\#} = g^{\#} = G^{\#}$ in D and $F(w) = G(w) = 0$. We then have

$$\tan^{-1}|F(z)| = \sigma_S(F(z), F(w)) \leqq \int_{\alpha(w,z)} F^{\#}(\zeta)|d\zeta| \leqq |w - z|K \leqq rK < \pi/4,$$

whence $|F(z)| < 1$ for $z \in D_1(w)$. Similarly we have $|G(z)| < 1$ in $D_1(w)$.

Since $|F'(w)| = |G'(w)| = f^{\#}(w)$, it follows that the holomorphic functions

$$\phi(z) = F(rz + w)/(rF'(w)),$$

$$\psi(z) = G(rz + w)/(rG'(w)), \qquad z \in \Delta$$

are bounded,

$$|\phi| \leqq M = 1/(rf^{\#}(w)), \quad |\psi| \leqq M \quad \text{in } \Delta.$$

By the lemma, both ϕ and ψ are univalent and starlike in $\Delta(M)$.

Restricting F to $D_2(w) = \{|z - w| < r\lambda(M)\}$, we let $\beta(w, z)$ be the inverse image of

$$\{tF(z); 0 \leqq t \leqq 1\} \subset F(D_2(w)), \qquad z \in D_2(w).$$

Then,

$$\tan^{-1}|F(z)| = \sigma_S(F(z), F(w)) = \int_{\beta(w,z)} F^{\#}(\zeta)|d\zeta|$$

$$= \int_{\beta(w,z)} G^{\#}(\zeta)|d\zeta| \geqq \sigma_S(G(z), G(w)) = \tan^{-1}|G(z)|,$$

whence $|F(z)| \geqq |G(z)|$ for $z \in D_2(w)$. We can replace F by G in the above argument, so that we obtain

$$|F(z)| \equiv |G(z)| \quad \text{in} \quad D_2(w),$$

and hence $F(z) \equiv a_1 G(z)$ in $D_2(w)$. The unicity theorem yields that $F = a_1 G$ in the whole D.

For the case of (III) we may assume that there exists $w \in D$ with $f^*(w) \neq 0$. This time, we consider

$$F = (f - f(w))/(1 - \overline{f(w)}f),$$

$$G = (g - g(w))/(1 - \overline{g(w)}g),$$

in D to prove that $F = a_2 G$ ($|a_2| = 1$) in D. First, $F^* = f^* = g^* = G^*$ in D. For

$R > 0$ with $\{|z - w| \leqq R\} \subset D$, we may consider

$$\phi(z) = F(Rz + w)/(RF'(w)),$$
$$\psi(z) = G(Rz + w)/(RG'(w)), \qquad z \in \Delta.$$

We can then apply the lemma to ϕ and ψ with $M = 1/(Rf^*(w))$. Then, ϕ and ψ are univalent and starlike in $\Delta(M)$. In this case, for $z \in D_3(w) = \{|z - w| < R\lambda(M)\}$, we let $\gamma(w, z)$ be the inverse image of

$$\{tF(z); 0 \leqq t \leqq 1\} \subset F(D_3(w))$$

by F restricted to $D_3(w)$. Then,

$$\tanh^{-1}|F(z)| = \sigma_P(F(z), F(w)) = \int_{\gamma(w,z)} F^*(\zeta)|d\zeta|$$
$$= \int_{\gamma(w,z)} G^*(\zeta)|d\zeta| \geqq \sigma_P(G(z), G(w)) = \tanh^{-1}|G(z)|,$$

so that $|F(z)| \geqq |G(z)|$ for $z \in D_3(w)$. Similarly, $|F(z)| \leqq |G(z)|$ in $D_3(w)$. The unicity theorem now proves the requested.

3. **Real-part surfaces**. The real-part surface of f holomorphic in D is the set of vectors, $V(x,y) = (x, y, Re\, f(z))$, $z = x + iy \in D$, in the space $\mathbb{R}^3$. The Gauss curvature at $V(x, y)$ is then

$$K_f(z) = -(f')^{\#}(z)^2;$$

see [2], [1, Satz 3].

If $g' = af'$, $|a| = 1$, then $K_g = K_f$; in particular, $K_{(-if)} = K_f$; see [2, Satz 3.1]. E. Kreyszig and A. Pendl [3, Satz 3] proved much more; see also [4, Lemma 2].

(KP) *Let f and g be holomorphic in D such that*

(L) $$g' = (\alpha f' + \beta)/(\gamma f' + \delta) \quad in \quad D,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $\alpha\delta - \beta\gamma \neq 0$. Then, $K_f = K_g$ in D if and only if

$$g' = (Af' + B)/(-\bar{B}f' + \bar{A}) \quad in \quad D,$$

where $A, B \in \mathbb{C}$, $|A|^2 + |B|^2 > 0$.

We can drop the condition (L) in the proposition (KP). This is obvious for the "if" part. With the aid of (II) applied to f' and g' with $K_f = K_g$ or $f'^{\#} = g'^{\#}$, we can show that (L) is superfluous for the "only if" part also. The details are left as exercises.

4. **Further applications**. We begin with an entire f.

(IV) *If $f \in \Phi_1$, then for each $T \in \Phi_1$, $|f|^2 - |T|^2$ is harmonic in $\mathbb{C}$. Conversely if there exists $T \in \Phi_1$ such that $|f|^2 - |T|^2$ is harmonic in a domain D, then $f \in \Phi_1$.*

The first half is obvious by direct computations. If f and g are holomorphic in D, then

$$\Delta(|f|^2 - |g|^2) = 4(|f'|^2 - |g'|^2) \quad \text{in} \quad D.$$

Therefore, $|f'| = |T'|$ in D in the second half. By (I), together with the unicity theorem, we have $f \in \Phi_1$. Also a direct proof is possible.

By the similar observations we propose applications (V) and (VI) of (II) and (III), respectively. Perhaps (VI) is more interesting than (V).

Let f and g be meromorphic in D. Then, there exist holomorphic functions f_1 and f_2 (g_1 and g_2) with no common zero in D such that $f = f_1/f_2$ ($g = g_1/g_2$) in D. Then,

$$\Delta\{\log(|f_1|^2 + |f_2|^2) - \log(|g_1|^2 + |g_2|^2)\} = 4(f^{\#2} - g^{\#2}) \quad \text{in} \quad D.$$

(V) *Let $f = f_1/f_1$ be meromorphic in $\mathbb{C}$, where f_1 and f_2 are entire with no common zero. If $f \in \Phi_2$, then for each $T(z) = a(z - \beta)/(1 + \bar{b}z) \in \Phi_2$, there exists a harmonic function h such that*

(i) $$|f_1(z)|^2 + |f_2(z)|^2 = (|z - b|^2 + |1 + \bar{b}z|^2)e^{h(z)}$$

in $\mathbb{C}$ with $1 + |z|^2$ for the parentheses on the right in case $b = \infty$. Conversely if there exist $T \in \Phi_2$ and a harmonic function h in D such that (i) *holds in D, then $f \in \Phi_2$.*

PROOF. The first half. Because $f^{\#}(z) = (1 + |z|^2)^{-1} = T^{\#}(z)$ in $\mathbb{C}$. The second half. Since $f^{\#} = T^{\#}$ in D, it follows that there exists $T_1 \in \Phi_2$ with $f = T_1 \circ T$ ($\in \Phi_2$) in D by (II). By the unicity theorem, $f \in \Phi_2$.

(VI) *Let f be holomorphic and bounded, $|f| < 1$, in Δ. If $f \in \Phi_3$, then for each $T \in \Phi_3$, there exists a harmonic function h such that*

(ii) $$1 - |f|^2 = (1 - |T|^2)e^h$$

in Δ. Conversely if there exist $T \in \Phi_3$ and a harmonic function h in a subdomain Δ_1 of Δ such that (ii) *holds in Δ_1, then $f \in \Phi_3$.*

In general, if f and g are holomorphic and bounded, $|f| < 1$, $|g| < 1$, in D, then

$$\Delta\{\log(1 - |f|^2) - \log(1 - |g|^2)\} = 4(g^{*2} - f^{*2})$$

in D. Therefore, $f^* = g^*$ in D if and only if there exists a harmonic function h in D such that

$$1 - |f|^2 = (1 - |g|^2)e^h \quad \text{in } D.$$

PROOF OF (VI). The first half. Because $f^*(z) = (1 - |z|^2)^{-1} = T^*(z)$ in Δ. The second half. By (III), there exists $T_1 \in \Phi_3$ with $f = T_1 \circ T$ in Δ_1, and hence in Δ.

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