A GENERALIZATION OF CERTAIN RINGS OF A. L. FOSTER

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1. Introduction. The concept of a Boolean ring, as a ring A in which every element is idempotent (i.e., $a^2 = a$ for all a in A), was first introduced by Stone [4]. Boolean algebras and Boolean rings, though historically and conceptually different, were shown by Stone to be equationally interdefinable. Indeed, let $(A, +, x)$ be a Boolean ring with unit 1, and let $(A, \cup, \cap, ')$ be a Boolean algebra, where $\cup, \cap, '$, denote "union", "intersection", and "complement". The equations which convert the Boolean ring into a Boolean algebra are:

(I) $a \cup b = a + b - ab$; $a \cap b = ab$; $a' = 1 - a$.
Conversely, the equations which convert the Boolean algebra into a Boolean ring are:

(II) $a + b = (a \cap b') \cup (a' \cap b)$; $ab = a \cap b$.

With this equational interdefinability as motivation, Foster [1] introduced the concept of a Boolean-like ring as a commutative ring A with unit 1 such that, for all a, b, in A,

(III) $a + b = (a \cap b') \cup (a' \cap b)$.

In view of (I) above, it is readily verified that (III) reduces to

(IV) $(ab)^2 - ab^2 - a^2b + 3ab = 0$.

Moreover, by setting $b = 1$ in (IV), it is easily seen that (IV), in turn, is equivalent to

(V) $(ab)^2 - ab^2 - a^2b + ab = 0$, and (VI) $2a = 0$, for all a, b, in R.

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It is noteworthy that (V) and (VI) combined are equivalent to (IV) (as well as to (III)) without any assumption of commutativity. In fact, it turns out that any ring with unit which satisfies (V) and (VI) is necessarily commutative (see Theorem 2). Hence, we have the following equivalent definition of a Boolean-like ring.

Definition. A ring A with unit 1 is called a Boolean-like ring if, and only if, for every a and b in A,

$$(V)\ (ab)^2 - ab^2 - a^2b + ab = 0\ , \quad \text{and} \quad (VI)\ 2a = 0\ .$$

The object of the present note is to study a certain class of rings which include the Boolean-like rings of Foster as special cases. Such rings we call n-like rings. In fact, Boolean-like rings are now easily seen to reduce to our 2-like rings ($n = 2$). In section 3, we prove that an n-like ring is necessarily commutative, and, in addition, we give a simple characterization of an n-like ring. The results given here generalize some previous theorems which the author proved for the case where $n = p =$ prime (see [5]). In this note, however, n is assumed to be any integer (> 1), not necessarily prime.

2. Preliminary definitions and lemmas. Suppose $n > 1$ is a fixed integer, not necessarily prime, throughout.

Definition. A ring A with unit 1 is called an n-like ring if, and only if, for every a and b in A,

$$(1)\ (ab)^n - ab^n - a^nb + ab = 0\ , \quad \text{and} \quad (2)\ na = 0\ .$$

We now have the following

LEMMA 1. In an n-like ring, we have (i) $(a-a^n)^2 = 0$, $a^{n^2-n+2} = a^2$, $(a^n)^n = a^n$; (ii) a is nilpotent if, and only if, $a^2 = 0$; (iii) The product of any two nilpotent elements is zero; (iv) $(ab)^n = a^nb^n$; (v) Every nilpotent element is in the center.

Proof. Setting $b = a$ in (1), we get $(a-a^n)^2 = 0$. Hence, $a^{2n} = 2a^{n+1} - a^2$. Therefore, $a^{2n}a^{n-1} = 2a^{n+1}a^{n-1} - a^2a^{n-1}$,

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$a^{3n-1} = 2a^{2n} - a^{n+1} = 2(2a^{n+1} - a^2) - a^{n+1} = 3a^{n+1} - 2a^2$.
Hence, $a^{3n-1} = 3a^{n+1} - 2a^2$. Multiplying the last equation by a^{n-1} and simplifying as in above, we obtain, $a^{4n-2} = 4a^{n+1} - 3a^2$.
Repeating this process a suitable number of times, we get,
$a^{kn-(k-2)} = ka^{n+1} - (k-1)a^2$, where k is any positive integer.
Hence, in particular, $a^{nn-(n-2)} = na^{n+1} - (n-1)a^2$. Therefore, $a^{n^2-n+2} = a^2$, since by (2), $na = 0$. Hence, also, $(a^n)^n = a^{n^2-n+2}a^{n-2} = a^2a^{n-2} = a^n$. This proves (i).

To prove (ii), let $a^r = 0$, and choose k so large that $k(n^2-n) + 2 \geq r$. Then, by (i), $a^2 = a^2a^{n^2-n} = a^2a^{n^2-n}a^{n^2-n} = \ldots = a^2a^{k(n^2-n)} = a^{k(n^2-n)+2} = 0$. This proves (ii).

To prove (iii), let a and b be nilpotent. Then, by (ii), $a^2 = 0 = b^2$. Hence by (1), $(ab)^n + ab = 0$. But, by (1) and (2), we also have, since $a^2 = 0 = b^2$,

$$0 = \{a(b+1)\}^n - a(b+1)^n + a(b+1) = (ab+a)^n + ab =$$
$$(ab)^n + (ab)^{n-1}a + ab. \quad \text{Hence,} \quad (ab)^{n-1}a = 0, \; (ab)^n = 0, \; ab = 0,$$

and (iii) is proved.

To prove (iv), we have, by (i), $(a-a^n)^2 = 0 = (b-b^n)^2$.
Hence, by (iii), $0 = (a-a^n)(b-b^n) = ab - ab^n - a^nb + a^nb^n$.
Part (iv) now follows at once from (1).

To prove (v), let a be nilpotent. By (i), $b^n - b$ is also nilpotent, for any b. Therefore, by (iii), $a(b^n-b) = 0 = (b^n-b)a$. Hence,

(3) $ab^n = ab$; $b^na = ba$ (a nilpotent, b arbitrary).

Since a is nilpotent, therefore, by (ii) and (iv), we have, $0 = a^nb^n = (ab)^n$. Hence, by (ii) again, $(ab)^2 = 0$. Similarly, $(ba)^2 = 0$. We have thus shown

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(4) $(ab)^2 = 0 = (ba)^2$, (a nilpotent, b arbitrary).

Now, using (3), (4), (iii) we obtain,

$(ab+b)^n = (ab)b^{n-1} + b(ab)b^{n-2} + b^2(ab)b^{n-3} + \ldots + b^{n-1}(ab) + b^n$, $(ba+b)^n = (ba)b^{n-1} + b(ba)b^{n-2} + \ldots + b^{n-2}(ba)b + b^{n-1}(ba) + b^n$.

Hence, $(ab+b)^n - (ba+b)^n = (ab)b^{n-1} - b^{n-1}(ba) = ab^n - b^n a = ab - ba$.

Moreover, by (iv), (2), and the hypothesis that a is nilpotent (and hence, by (ii), $a^2 = 0$), we have, $(ab+b)^n = \{(a+1)b\}^n = (a+1)^n b^n = b^n$. Similarly, $(ba+b)^n = b^n$. Hence, $ab - ba = (ab+b)^n - (ba+b)^n = 0$. This proves (v), and the proof of Lemma 1 is complete.

3. The main theorems. We are now in a position to prove the following main

THEOREM 2. *An n-like ring A is commutative.*

Proof. Let $x \in A$. Then, by Lemma 1 (i), $x^n - x$ is nilpotent. Hence, by Lemma 1 (v), $x^n - x$ is in the center of A. Therefore, by a well-known theorem of Herstein [2], the ring A is commutative, and the theorem is proved.

We shall now give a simple characterization of n-like rings.

THEOREM 3. *Let A be a ring with unit 1, and let n be any positive integer (n > 1). A is an n-like ring if, and only if, the following conditions are satisfied:*

(1^o) *A is commutative,*

(2^o) $na = 0$ *for all* a *in* A,

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(3°) *For every* x *in* A, $\exists\, y, \eta$ *in* A *such that* $x = y + \eta$, *where* $y^n = y$ *and* η *is nilpotent*,

(4°) *The product of any two nilpotent elements in* A *is zero*.

Proof. The necessity of conditions (1°) -- (4°) follows from Theorem 2, Lemma 1, the definition of an n-like ring, and the observation that $x = x^n + (x - x^n)$.

Sufficiency: Let A be a ring with unit 1 satisfying conditions (1°) - (4°), and let $x \in A$. By (3°), $\exists\, y, \eta \in A$ such that $x = y + \eta$, $y^n = y$, η nilpotent. Then, by (1°), (2°), (4°), $x - \eta = y = y^n = (x-\eta)^n = x^n$. Hence, $\eta = x - x^n$ is nilpotent. Therefore, by (4°), for any x, y in A, we have, $(x-x^n)(y-y^n) = 0$, which is equivalent to (1) in the definition of an n-like ring, since, by (1°), A is commutative. The proof is now completed by observing (2°).

4. *Examples*. We shall conclude this note by giving some examples of n-like rings.

Example 1. Let A be any Boolean ring with unit. More generally, let A be any p-ring with unit (p prime) (see [3]). It is easily seen that A is an n-like ring also, for suitably chosen n.

Example 2. Let A be a p-ring with unit (p prime), and let x be such that $x^2 = 0$, $x \neq 0$. Then, $A[x]$, the ring obtained by adjoining x to A, is easily seen to be a p-like ring ($n = p$ = prime) but *not* a p-ring. Indeed, $A[x]$ contains some non-zero nilpotent elements (namely, x, for example). This example also shows that, for every prime p, there exists a p-like ring (with proper nilpotent elements) which is *not* a p-ring.

Example 3. Let F_p be the ring (field) of residue classes (mod p), where p is a prime integer. Let $A = F_3 \oplus F_5$ be

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the direct sum of F_3 and F_5. Then A is an n-like ring. Indeed, one may take $n = 45$ in this case.

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