ON NON-AVERAGING SETS OF INTEGERS

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1. Introduction. Let S be a set of positive integers no three of which are in arithmetical progression, i.e., if A, B, C are distinct elements of S, $A + B \neq 2C$. We call such a set a non-averaging set. Let $\nu(n)$ denote the maximum number of elements not exceeding n in any non-averaging set. The problem of finding bounds for $\nu(n)$ has been treated by several authors [**1, 3, 5, 6, 7**]. The question first arose in connection with a theorem of van der Waerden [**8**]. Van der Waerden's theorem states that if one separates the integers 1, 2, 3, . . . , N into k disjoint classes, then for every l, there exists at least one class which contains an arithmetical progression of l terms, if $N = (k, l)$ is sufficiently large. This theorem was used by Brauer [**2**] to prove the existence of sequences of l consecutive quadratic residues and l consecutive non-residues for every sufficiently large prime. In van der Waerden's theorem the $N(k, l)$ is extremely large, and it was thought that a study of $\nu(n)$ would yield better bounds for N. Unfortunately this hope has not as yet been fulfilled.

G. Szekeres conjectured that $\nu\{(3^k + 1)/2\} = 2^k$ and this was proved [**3**] for $k < 5$. This would make

$$\nu(n) < cn^{\log 2/\log 3}$$

for some fixed c. The conjecture was proved false by Salem and Spencer [**6**] who showed that for every $\epsilon > 0$ and sufficiently large n,

$$\nu(n) > n^{1-(\log 2 + \epsilon)/(\log \log n)}. \tag{1.1}$$

This result was refined by Behrend [**1**] who proved that for $\epsilon > 0$ and sufficiently large n,

$$\nu(n) > n^{1-(2\sqrt{(2 \log 2)} + \epsilon)/\sqrt{(\log n)}}. \tag{1.2}$$

In Behrend's method the set S depends upon n, i.e., the set used for $n = 1000$ might be quite different from that for $n = 1001$. Furthermore, the argument makes use of the Dirichlet's principle of drawers and hence is not constructive. In §2 we give a constructive definition of an infinite sequence R which has no three terms in arithmetic progression, and which yields, for n sufficiently large,

$$\nu(n) > n^{1-c/\sqrt{(\log n)}}, \tag{1.3}$$

where c is a fixed constant.

P. Erdös and P. Turán gave some upper bounds for $\nu(n)$ in [**3**]. They proved that for $n \geqslant 8$,

$$\nu(2n) \leqslant n. \tag{1.4}$$

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Also for every $\epsilon > 0$ and n sufficiently large they proved

$$\nu(n) < (\tfrac{4}{9} + \epsilon)n. \tag{1.5}$$

Finally they stated without proof that

$$\nu(n) < (\tfrac{3}{8} + \epsilon)n. \tag{1.6}$$

In §3 we shall use a different method to prove

$$\nu(n) < \tfrac{2}{5}n + 3. \tag{1.7}$$

This has the advantage of being free of ϵ. We shall then use the same method but with a much longer argument to prove that

$$\nu(n) < \tfrac{4}{11}n + 5, \tag{1.8}$$

which is stronger than (1.6).

It has long been conjectured that $\nu(n) = o(n)$, but this has only recently been proved [**5**]. In this connection it is interesting to note the following theorem proved by Redheffer [**4**]. A necessary and sufficient condition that every non-averaging set $\{\lambda_n\}$ have zero density is that

$$\{e^{i\lambda_n x}\}$$

be always incomplete on every interval.

2. Lower bounds for $\nu(n)$. We shall now define an infinite sequence R, show that no three of its elements are in arithmetical progression, and show that if $\nu^*(n)$ denotes the number of elements in R and not exceeding n, then

$$\nu^*(n) > n^{1-c/\sqrt{(\log n)}},$$

where c is a fixed constant.

Given a number x, written in the denary scale, we decide whether x is in R on the basis of the following rules:

First we enclose x in a set of brackets, putting the first digit (counting from right to left) in the first bracket, the next two in the second bracket, the next three in the third bracket, and so on. If the last non-empty bracket (the bracket furthest to the left which does not consist entirely of zeros) does not have a maximal number of digits, we fill it with zeros. For instance, the numbers

$$A = 32653200200, \quad B = 100026100150600, \quad C = 100086600290500$$

would be bracketed as follows:

$$A = (00003)\ (2653)\ (200)\ (20)\ (0), \qquad B = (10002)\ (6100)\ (150)\ (60)\ (0),$$
$$C = (10008)\ (6600)\ (290)\ (50)\ (0).$$

Now suppose the rth bracket in x contains non-zero digits, but all further brackets to the left are zero. Call the number represented by the digits in the ith bracket x_i, $i = 1, 2, \ldots, r-2$. Further, denote by $\bar{x}$ the number represented by the digits in the last two brackets taken together, but excluding the last digit. For x to belong to R we require:

(2.1) The last digit of x must be 1;

(2.2) x_i must begin with 0 for $i = 1, 2, \ldots, r-2$;

(2.3) $$\sum_{i=1}^{r-2} x_i^2 = \bar{x}.$$

In particular, we note that A satisfies (2.2) but violates (2.1) and (2.3) and so is not in R, but B and C satisfy all three conditions and so are in R. To check (2.3) for B we note that $60^2 + 150^2 = 26100$.

We next prove that no three integers of R are in arithmetic progression. First note that if two elements of R have a different number of non-empty brackets then their arithmetic mean cannot satisfy (2.1). Thus we need only consider averages of elements of R having the same number of non-empty brackets. From conditions (2.1) and (2.3) it follows that two elements of R can be averaged bracket by bracket for the first $r-2$ brackets and also for the last two brackets taken together. Thus in our example

$$\tfrac{1}{2}(60 + 50) = 55, \quad \tfrac{1}{2}(150 + 290) = 220,$$

$$\tfrac{1}{2}(100026100 + 100086600) = 100056350,$$

$$\tfrac{1}{2}(B + C) = (10005)\ (6350)(220)(55)(0).$$

This violates (2.3) and so cannot be in R. In general we will prove that if x and y are in R, then $z = \frac{1}{2}(x + y)$ will violate (2.3).

Since x and y are in R,

$$\bar{z} = \frac{\bar{x} + \bar{y}}{2} = \sum_{i=1}^{r-2} \frac{x_i^2 + y_i^2}{2}.$$

On the other hand z in R implies

$$\bar{z} = \sum_{i=1}^{r-2} z_i^2 = \sum_{i=1}^{r-2} \left(\frac{x_i + y_i}{2}\right)^2.$$

Hence if z is in R then

$$\sum_{i=1}^{r-2} \frac{x_i^2 + y_i^2}{2} = \sum_{i=1}^{r-2} \left(\frac{x_i + y_i}{2}\right)^2.$$

Thus

$$\sum_{i=1}^{r-2} \left(\frac{x_1 - y_i}{2}\right)^2 = 0,$$

which implies $x_i = y_i$ for $i = 1, 2, \ldots, r-2$. This together with (2.1) and (2.2) implies that x and y are not distinct.

Szekeres' sequence starts with 1, 2, 4, 5, 10, 11, 14, 28, 29, ... Our sequence starts with

100000, 1000100100, 1000400200, 100250500, ...

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Nevertheless, it will be proved that the terms of this sequence eventually become smaller than the corresponding terms of the first sequence.

The first sequence is maximal in the sense that no number can be added to it without introducing an arithmetical progression of three terms. Our sequence is not maximal in this sense and indeed it is easy to find numbers, among them 1, which may be adjoined. However, there seems little point in doing this since it improves (1.3) only by an ϵ in the constant.

We now estimate how many integers in R contain exactly r brackets. Given r brackets we can make the first digit in each of the first $r - 2$ brackets 0. We then fill up the first $r - 2$ brackets in an arbitary manner. This can be done in

$$10^{0+1+2+\ldots+(r-2)} = 10^{\frac{1}{2}(r-2)(r-3)}$$

ways. The last two brackets can then be filled in such a way as to satisfy (2.1) and (2.3). To see this we need only check that the last two brackets will not be overfilled, and that the last digit, which we shall set equal to 1, will not be interfered with. This follows from the inequality

$$(10^1)^2 + (10^2)^2 + \ldots + (10^{r-2})^2 < 10^{2(r-1)}.$$

For a given n let r be the integer determined by

$$10^{\frac{1}{2}r(r+2)} \leqslant n < 10^{\frac{1}{2}(r+1)(r+2)}. \tag{2.4}$$

Since all the integers with at most r brackets will not exceed n, and since r brackets can be filled to specification in $10^{\frac{1}{2}(r-3)(r-2)}$ ways, we have

$$\nu^*(n) \geqslant 10^{\frac{1}{2}(r-2)(r-3)}. \tag{2.5}$$

From the right-hand side of (2.4) we obtain, using logarithms to base 10,

$$r + 2 > \sqrt{(2 \log n)}$$

so that (2.5) implies, for sufficiently large n,

$$\nu^*(n) \geqslant 10^{\frac{1}{2}(r-2)(r-3)} > 10^{\log n - 9\sqrt{(2 \log n)}} > 10^{\log n(1-c/\sqrt{(\log n)})} = n^{1-c/\sqrt{(\log n)}}.$$

This proves (1.3) since $\nu(n) \geqslant \nu^*(n)$.

Use of base 2 instead of base 10, together with a more refined treatment of the inequalities, would yield a better value for the constant.

3. Upper bounds for $\nu(n)$. Let $a_1 < a_2 < \ldots < a_r$ denote the even elements, and by $b_1 < b_2 < \ldots < b_s$ the odd elements of S, not exceeding n. Denote the integers $\frac{1}{2}(a_i + a_j)$ by (i, j). It follows from the definition of S in §1 that (i, j) is not in S.

Now denote the following set of $2r - 3$ integers by $\bar{S}$:

$$(1, 2) < (1, 3) < (2, 3) < (2, 4) < \ldots < (i, i + 1) < (i, i + 2) < \ldots$$
$$< (r - 2, r - 1) < (r - 2, r) < (r - 1, r).$$

The integers of $\bar{S}$ are clearly under n and not in S. Furthermore, we shall see that at least one of the integers (1, 4) and (1, 5) which are not in S, is also not in $\bar{S}$.

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To see this note that if (1, 4) is in $\bar{S}$, then (1, 4) = (2, 3). If also (1, 5) is in $\bar{S}$, then either (1, 5) = (2, 3) or (1, 5) = (2, 4), or (1, 5) = (3, 4). Now (1, 4) = (2, 3) together with (1, 5) = (2, 3) implies $a_4 = a_5$. Also (1, 4) = (2, 3) together with (1, 5) = (2, 4) implies $a_3 + a_5 = 2a_4$. Finally, (1, 4) = (2, 3) together with (1, 5) = (2, 4) implies $a_3 + a_5 = 2a_4$. Similarly, at least one of each pair

$$(i, i + 3), \quad (i, i + 4), \qquad\qquad i = 1, 2, \ldots, r - 4$$

must be absent from S and differ from any in $\bar{S}$. Thus we have at least $(2r - 3) + (r - 4) = 3r - 7$ numbers under n and not in S. Present in S are exactly $r + s$ numbers. Hence we have

$$3r - 7 + r + s \leqslant n. \tag{3.1}$$

Let us assume that $r \geqslant s$. Then (3.1) yields

$$\tfrac{3}{2}(r + s) - 7 + (r + s) \leqslant n; \tag{3.2}$$

and since $\nu(n) = r + s$, this yields

$$\nu(n) < \tfrac{2}{5}n + 3.$$

If $r < s$ then we can deal with the b's instead of the a's and the same conclusion will follow. Thus in any case (1.5) is proved.

Our final object is to prove

$$\nu(n) < \tfrac{4}{11}n + 5.$$

We use the same notation as before. Again we have the $2r - 3$ integers of $\bar{S}$ not in S. This time however we will prove that of the integers

$$(1, 4), (1, 5), (1, 6), (1, 7), (2, 5), (2, 6), (2, 7),$$

which we will denote by T (and which are not in S) at least three distinct ones are not in $\bar{S}$.

The argument will then closely follow the lines used in the previous theorem. We will first prove the following

LEMMA. *Two different equations of the type* $(i_1, i_2) = (i_3, i_4)$ *involving only five distinct integers under* r, *imply a contradiction.*

Proof. Let the five different numbers involved be $i_1 < i_2 < i_3 < i_4 < i_5$. Clearly the only possible equations of the required type involving these are:

A: $(i_1, i_4) = (i_2, i_3)$ B: $(i_1, i_5) = (i_2, i_3)$ C: $(i_1, i_5) = (i_2, i_4)$
D: $(i_1, i_5) = (i_3, i_4)$ E: $(i_2, i_5) = (i_3, i_4)$

But now A together with B implies $a_{i_4} = a_{i_5}$; A together with C implies

$$a_{i_3} + a_{i_5} = 2a_{i_4}.$$

Similarly each of the other eight combinations easily leads to a contradiction.

We shall now prove that at least three distinct numbers in T are not in $\bar{S}$. We consider three cases.

Case 1. Suppose that (2, 5) is in $\bar{S}$. Now (2, 5) in $\bar{S}$ implies (2, 5) = (3, 4).

Note that we need not concern ourselves here with possibilities like $(2, 5) = (1, i)$ since then $i > 5$ and $(1, i)$ will not be in $\bar{S}$. If $(1, 4)$ is in $\bar{S}$, then $(1, 4) = (2, 3)$ which by the lemma is incompatible with $(2, 5) = (3, 4)$. Hence $(1, 4)$ is not in $\bar{S}$.

If $(1, 5)$ is in $\bar{S}$ then we have one of

$$(1, 5) = (2, 5), \quad (1, 5) = (2, 4), \quad (1, 5) = (3, 4)$$

but each of these possibilities is incompatible with $(2, 5) = (3, 4)$ by the lemma. Hence $(1, 5)$ is not in $\bar{S}$. If $(2, 6)$ is in $\bar{S}$, then we have one of

$$(2, 6) = (3, 4), \quad (2, 6) = (3, 5), \quad (2, 6) = (4, 5),$$

but each of these is incompatible with $(2, 5) = (3, 4)$ by the lemma. Hence $(2, 6)$ is not in $\bar{S}$.

Thus, in case (1) the three numbers $(1, 4)$ $(1, 5)$ $(2, 6)$ are not in $\bar{S}$ and we are finished. We may therefore assume $(2, 5)$ is not in $\bar{S}$, and need only show that two other numbers of T are not in $\bar{S}$.

Case 2. Suppose $(1, 4)$ is in $\bar{S}$. Now $(1, 4)$ in $\bar{S}$ implies $(1, 4) = (2, 3)$ and $(1, 5)$ in $\bar{S}$ implies one of

$$(1, 5) = (2, 3), \quad (1, 5) = (2, 4), \quad (1, 5) = (3, 4)$$

each of which is incompatible with $(1, 4) = (2, 3)$ by the lemma. Hence $(1, 5)$ is not in $\bar{S}$, and it will suffice to show that at least one other number in T is not in $\bar{S}$. If this be false then we must have at least one inequality in each of the following columns:

A:	$(1, 6) = (2, 3)$	A:	$(2, 6) = (3, 4)$	A:	$(2, 7) = (3, 4)$
A:	$(1, 6) = (2, 4)$	C:	$(2, 6) = (3, 5)$		$(2, 7) = (3, 5)$
A:	$(1, 6) = (3, 4)$		$(2, 6) = (4, 5)$	D:	$(2, 7) = (4, 5)$
	$(1, 6) = (3, 5)$			D:	$(2, 7) = (4, 6)$
B:	$(1, 6) = (4, 5)$			D:	$(2, 7) = (5, 6)$

The possibilities marked A are out by $(1, 4) = (2, 3)$ and the lemma. B: $(1, 6) = (4, 5)$ is out for then no possibility for $(2, 6)$ makes $(2, 6) > (1, 6)$. Now we have $(1, 6) = (3, 5)$ and this with the lemma eliminates the possibility marked C. Now $(2, 6) = (4, 5)$ so that by the lemma the possibilities marked D are out. But now the remaining possibilities for $(2,7)$, namely $(2, 7) = (3, 5)$ makes $(2, 7) < (2, 6)$ which is impossible. Hence in this case we are finished, and so we may assume $(1, 4)$ is not in $\bar{S}$.

Case 3. We have already $(1, 4)$ and $(2, 5)$ not in $\bar{S}$. Hence it will suffice to show that at least one other number is in T and not in $\bar{S}$. If this be false we must have at least one equality in each of the following columns:

G:	$(1, 5) = (2, 3)$	A:	$(1, 6) = (2, 3)$	D:	$(2, 6) = (3, 4)$
	$(1, 5) = (2, 4)$	B:	$(1, 6) = (2, 4)$		$(2, 6) = (3, 5)$
E:	$(1, 5) = (3, 4)$		$(1, 6) = (3, 4)$	F:	$(2, 6) = (4, 5)$
		C:	$(1. 6) = (3, 5)$		
		A:	$(1, 6) = (4, 5)$		

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D: (1, 7) = (2, 3)	
D: (1, 7) = (2, 4)	A: (2, 7) = (3, 4)
D: (1, 7) = (3, 4)	G: (2, 7) = (3, 5)
(1, 7) = (3, 5)	H: (2, 7) = (4, 5)
H: (1, 7) = (4, 5)	(2, 7) = (4, 6)
E: (1, 7) = (4, 6)	G: (2, 7) = (5, 6)
A: (1, 7) = (5, 6)	

Now (1, 6) = (2, 3) would not leave any consistent possibility for (1, 5) while (1, 6) = (4, 5) would not leave any consistent possibility for (2, 6). Similarly (1, 7) = (5, 6) and (2, 7) = (3, 4) are out. These four possibilities are marked A.

If B: (1, 6) = (2, 4) then by the lemma, the only possibility for (2, 6) is (2, 6) = (3, 5).

Also (1, 6) = (2, 4) and (1, 6) > (1, 5) gives as only possibility for (1, 5), (1, 5) = (2, 3).

Now (2, 6) = (3, 5) and (1, 5) = (2, 3) are incompatible by the lemma Hence B is out.

If C: (1, 6) = (3, 5) then by the lemma, the only possibility for (1, 5) is (1, 5) = (2, 4).

Also (1, 6) = (3, 5) and (2, 6) > (1, 6) gives as only possibility for (2, 6), (2, 6) = (4, 5).

Now (1, 5) = (2, 4) and (2, 6) = (4, 5) are incompatible by the lemma; hence C is out.

We now have (1, 6) = (3, 4), and since (2, 7) > (1, 7) > (1, 6) and (2, 6) > (1, 6) the possibilities marked D are out. Furthermore, by the lemma (1, 6) = (3, 4) eliminates the possibilities marked E.

Now F: (2, 6) = (4, 5) leaves every choice for (2, 7) either not larger than (2, 6) or incompatible with (2, 6) = (4, 5) by the lemma; hence F is out.

We now have (2, 6) = (3, 5) which by the lemma eliminates the possibilities marked G.

This leaves (1, 5) = (2, 4) which by the lemma eliminates the possibilities marked H.

We thus have (2, 6) = (3, 5) = (1, 7) but (2, 6) = (1, 7) eliminates the remaining possibilities for (2, 7). Hence case 3 is complete.

The theorem which we have thus proved for the set T:

$$(1, 4),\ (1, 5),\ (1, 6),\ (1, 7),\ (2, 5),\ (2, 6),\ (2, 7)$$

will go through with only an obvious change in notation for the set T_i:

$$(i,\ i+3),\ (i,\ i+4),\ (i,\ i+5),\ (i,\ i+6)\ \ (i+1,\ i+4),\ (i+1,\ i+5),\ (i+1, i+6), \qquad i = 1, 2, \dots, r-6.$$

Furthermore, the sets T_i are disjoint for $i = 1, 3, 5, \dots, k$ where k is the largest odd integer in $r - 6$. Certainly then we may take $k \geqslant \frac{1}{2}(r-6)$.

Thus we have at least $\frac{3}{2}\,(r-6)$ integers not in S and not in $\bar{S}$. Altogether then, we have

$$(2r - 3) + \tfrac{3}{2}(r - 6) = \tfrac{7}{2}r - 12$$

integers not in S.

In S we have exactly $r + s$ integers, hence

$$\tfrac{7}{2}r - 12 + r + s \leqslant n.$$

Assuming $r \geqslant s$ gives

$$\tfrac{7}{4}(r + s) + (r + s) - 12 = \tfrac{11}{4}(r + s) - 12 \leqslant n;$$

and since $\nu(n) = r + s$ we have

$$\nu(n) \leqslant \tfrac{4}{11}(n + 12) < \tfrac{4}{11}n + 5.$$

On the other hand, if $r < s$, the same result can be obtained by working with the b's instead of the a's.

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