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NON-AMPHICHEIRAL CODIMENSION 2 KNOTS

F. GONZÁLEZ-ACUÑA AND JOSÉ M. MONTESINOS

1. Introduction. An n-knot (S^{n+2}, S^n) is *amphicheiral* if there is an orientation reversing autohomeomorphism of S^{n+2} leaving S^n invariant as a set. It is *invertible* if there is an orientation preserving autohomeomorphism of S^{n+2} whose restriction to S^n is an orientation reversing autohomeomorphism of S^n onto itself.

In 1961 Fox [**8**, Problem 35] asked if there exist non-amphicheiral locally flat 2-knots. We will prove the following

THEOREM 1. *For any integer n there are smooth n-knots which are neither amphicheiral nor invertible.*

2. Preliminaries. A knot (S^{n+2}, S^n) is $+$*amphicheiral* (resp. $-$*amphicheiral*) if there is an orientation reversing autohomeomorphism f of S^{n+2} leaving S^n invariant such that $f|S^n$ preserves (resp. reverses) orientation.

The following can be proved using Alexander duality:

LEMMA 1. *Let $f:(S^{n+2}, S^n) \to (S^{n+2}, S^n)$ be a homeomorphism. Then f reverses the orientation of S^{n+2} if and only if precisely one of the automorphisms*

$$f_*: H_n(S^n) \to H_n(S^n),\ f_*: H_1(S^{n+2} - S^n) \to H_1(S^{n+2} - S^n)$$

is the identity.

Thus a knot is $-$amphicheiral if and only if there is an orientation reversing homeomorphism of S^{n+2} leaving S^n invariant such that $f_*: H_1(S^{n+2} - S^n) \to H_1(S^{n+2} - S^n)$ is the identity.

Let (S^{n+2}, S^n) be a knot and let $h:(S^{n+2}, S^n, S^{n+2} - S^n) \to (S^{n+2}, S^n, S^{n+2} - S^n)$ be a homotopy equivalence of triples. Denote by $\tilde{h}$ a lifting of h to the universal abelian covering $\tilde{X}$ of $X = S^{n+2} - S^n$, and call $p:\tilde{X} \to X$ the projection. The proof of the following lemma is easy and we omit it.

LEMMA 2. *$\tilde{h}t = t^\delta \tilde{h}$ with $\delta = \pm 1$, and $\delta = 1$ if and only if $h_*: H_1(S^{n+2} - S^n) \to H_1(S^{n+2} - S^n)$ is the identity, where t is a generator of the group of covering transformations of $p:\tilde{X} \to X$.*

Remark. Suppose (S^{n+2}, S^n) is a knot such that S^n has a neighborhood homeomorphic to $S^n \times D^2$, where S^n corresponds to $S^n \times \{0\}$ (for $n \neq 2$ all locally flat knots satisfy this condition [**14**]). Let $h:(S^{n+2}, S^n, S^{n+2} - S^n) \to (S^{n+2}, S^n, S^{n+2} - S^n)$ be a homotopy equivalence of triples. Then there is a

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map homotopic to the identity $k(S^{n+2}, S^n, S^{n+2} - S^n) \to (S^{n+2}, S^n, S^{n+2} - S^n)$ which is the identity on S^n and such that, for some tubular neighborhood T of S^n, $kh(T) = T$ and $kh(E) = E$, where E is the closure of $S^{n+2} - T$. Notice that h and kh have the same orientation features.

Let P be an $(n + 1)$-manifold whose boundary is homeomorphic to S^n. Let $\alpha : P \to P$ be a homeomorphism which is the identity on a neighborhood of ∂P. In $P \times [0, 1]$ we identify $(x, 1)$ with $(\alpha(x), 0)$, for $x \in P$, and identify (x, t) with $(x, 0)$, for $x \in \partial P$, $t \in [0, 1]$. Denote the resulting space, which is an $(n + 2)$-manifold, by $M(\alpha)$ and let $\eta : P \times [0, 1] \to M(\alpha)$ be the identification map. This map sends $P \times \{t\}$ homeomorphically onto its image. If $M(\alpha)$ is homeomorphic to S^{n+2} we call the pair $(M(\alpha), \eta(\partial P \times \{0\}))$ a *fibered knot* with Int P as *fiber* and *monodromy* α. If U is a collar of ∂P in P such that $\alpha | U$ is the identity, then $(\eta(U \times [0, 1]), \eta(\partial P \times \{0\}))$ is homeomorphic to $(S^n \times D^2, S^n \times \{0\})$. Thus, by the remark above, if (S^{n+2}, S^n) is a fibered knot and $h : (S^{n+2}, S^n, S^{n+2} - S^n) \to (S^{n+2}, S^n, S^{n+2} - S^n)$ is a homotopy equivalence of triples, we may assume $hT = T$ and $hE = E$, where $T = \eta(U \times [0, 1])$, and E is the closure of $S^{n+2} - T$.

Denote by $\tilde{h}$ a lifting of h to the universal abelian covering $\tilde{E}$ of E, and call $p : \tilde{E} \to E$ the projection; $\tilde{E}$ is homeomorphic to $F \times \mathbf{R}$, where $F = \eta(P \times \{0\}) \cap E$. A lifting $q : F \to \tilde{E}$ of the inclusion $i : F \to E$ is a homotopy equivalence. Let $r : \tilde{E} \to F$ be a homotopy inverse of q. We have the commutative diagram of Figure 1 in which all arrows are isomorphisms, j_*, k_* are induced by inclusion and $h' = r\tilde{h}q$.

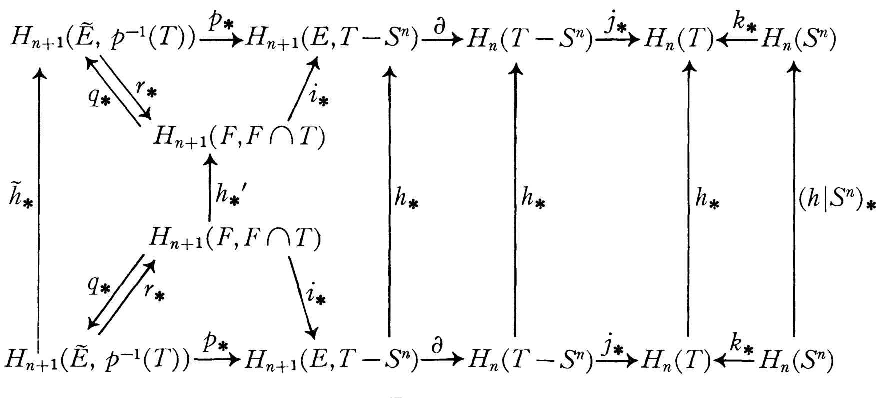

FIGURE 1

LEMMA 3. *If the closure of the fiber does not admit an orientation reversing homotopy equivalence leaving its boundary fixed as a set, then there is no homotopy equivalence of the triple $(S^{n+2}, S^n, S^{n+2} - S^n)$ reversing the orientation of S^n.*

LEMMA 4. *If $n = 2q$, then $[h_*'x, h_*'y] = \epsilon[x, y]$, where $x, y \in T_q =$ torsion $H_q(F)$, ϵ is the degree of $h|S^n$ and $[\ ,\ ] : T_q \times T_q \to \mathbf{Q}/\mathbf{Z}$ is the linking pairing.*

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Proof of Lemma 3. Since h' is a self homotopy equivalence of $(F, F \cap T)$, from the hypothesis it follows that

$$h_*' : H_{n+1}(F, F \cap T) \rightarrow H_{n+1}(F, F \cap T)$$

is the identity. Hence $(h|S^n)_* = (k_*^{-1} j_* \partial i_*) h_*' (k_*^{-1} j_* \partial i_*)^{-1}$ is the identity. This proves Lemma 3.

Proof of Lemma 4. The linking pairing may be described as follows (see for example [**22**]). We denote by φ the composition of isomorphisms

$$T_q \xrightarrow[\approx]{} \text{Torsion } H_q(F, \partial F) \xleftarrow[\approx]{\mu \cap} \text{Torsion } H^{q+1}(F)$$

$$\xleftarrow[\approx]{\beta} \text{coker } \theta \xrightarrow[\approx]{\omega} \text{Hom } (T_q; \mathbf{Q}/\mathbf{Z}),$$

where $\mu \in H_{n+1}(F, \partial F)$ is the fundamental class, so that $\mu_\cap$ is the Poincare duality isomorphism, θ is the homomorphism from $H^q(F; \mathbf{Q})$ to $H^q(F; \mathbf{Q}/\mathbf{Z})$, β is induced by the Bockstein associated to the sequence $0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Q} \rightarrow \mathbf{Q}/\mathbf{Z} \rightarrow 0$ and ω is induced by the universal coefficient theorem. Then $[\eta, \xi] = \varphi(\xi)(\eta)$. The lemma follows from the diagram of Figure 2

$$\begin{array}{ccccccccc}
T_q & \xrightarrow{\approx} & \text{Torsion } H_q(F, \partial F) & \xleftarrow{\mu\cap} & \text{Torsion } H^{q+1}(F) & \xleftarrow{\beta} & \text{coker } \theta & \xrightarrow{\omega} & \text{Hom } (T_q, \mathbf{Q}/\mathbf{Z}) \\
\downarrow{\scriptstyle h_*'} & & \downarrow{\scriptstyle h_*'} & & \uparrow{\scriptstyle h'^*} & & \uparrow{\scriptstyle h'^*} & & \uparrow{\scriptstyle h_*'^*} \\
T_q & \xrightarrow{\approx} & \text{Torsion } H_q(F, \partial F) & \xleftarrow{\mu\cap} & \text{Torsion } H^{q+1}(F) & \xleftarrow{\beta} & \text{coker } \theta & \xrightarrow{\omega} & \text{Hom } (T_q, \mathbf{Q}/\mathbf{Z})
\end{array}$$

FIGURE 2

in which the second square is commutative (resp. anticommutative) if $\epsilon = 1$ (resp. $\epsilon = -1$), and the remaining squares are commutative.

We now recall the definition of the lens space $L(p; q_1, q_2, \ldots, q_m)$, where $p, q_1, q_2, \ldots, q_m$ are integers such that $(p, q_i) = 1$ for $1 \leqq i \leqq m$. Let

$$S^{2m-1} = \{(z_1, \ldots, z_m) \in \mathbf{C}^m | \textstyle\sum_{i=1}^m z_i \bar{z}_i = 1\}$$

and let $g : S^{2m-1} \rightarrow S^{2m-1}$ be the diffeomorphism defined by

$$g(z_1, \ldots, z_m) = (e^{2\pi i (q_1/p)} z_1, \ldots, e^{2\pi i (q_m/p)} z_m).$$

Then $L(p; q_1, \ldots, q_m)$ is the smooth $(2m - 1)$-manifold S^{2m-1}/G, where G is the cyclic group of diffeomorphisms generated by g. Let $L_0(p; q_1, \ldots, q_m)$ be the lens space $L(p; q_1, \ldots, q_m)$ minus the interior of a smooth $(2m - 1)$-ball. The proof of the following lemma can be found in [**5**; 29.5].

LEMMA 5. *$L_0(p; q_1, \ldots, q_m)$ admits an orientation reversing homotopy equivalence leaving its boundary fixed as a set only if, for some integer b, $b^m \equiv -1 \bmod p$.*

Remark. The converse of Lemma 5 is valid.

LEMMA 6 [**2**; 5.3]. *Let f be a homotopy equivalence from $L = L(p; q_1, \ldots, q_m)$ into itself such that $f_*: H_1(L) \to H_1(L)$ is multiplication by an integer a satisfying $(a^r - 1, p) = 1$, for $1 \leqq r < m$, and $a^m \equiv 1 \bmod p$. Then the degree of f is 1 and, for $1 \leqq i < 2m - 1$, $f^* - I^*: H^i(L) \to H^i(L)$ is an isomorphism, where I is the identity map of L.*

Sketch of the proof. Using the fact that L is the $(2m - 1)$-skeleton of an Eilenberg-MacLane space of type $(\mathbf{Z}_p, 1)$ it is seen that $f^*: H^{2j}(L) \to H^{2j}(L)$ is multiplication by a^j and that $\deg f \equiv 1 \bmod p$.

LEMMA 7. *Let m be an even natural number. Then there exists a positive prime p such that:*

i) $p \equiv 1 \bmod m$.

ii) *there is no integer b such that $b^m \equiv -1 \bmod p$.*

Proof. Since $(2m, m + 1) = 1$ because m is even, by Dirichlet's theorem [**15**, p. 79] there is a prime $p = 2mk + m + 1$ for some positive integer k. The multiplicative group F_p^* of nonzero residue classes modulo p is cyclic of order $p - 1$ [**21**, p. 128]. The subgroup of F_p^* consisting of m-th powers has odd order $(p - 1)/m$ and, therefore, -1 is not such a power.

3. Proof of Theorem 1.

Case I. *n odd.* The result for $n = 1$ was established by Trotter [**20**]. We therefore assume $n \geqq 3$.

Let (S^{n+2}, S^n) be a smooth knot such that

i) Every automorphism of $G = \pi_1(S^{n+2}, S^n)$ induces the identity on G/G';

ii) (S^{n+2}, S^n) represents an element of order >2 in the cobordism group C_n^{TOP} of n-knots [**4**], [**16**].

Then i) implies that the knot is neither +amphicheiral nor invertible (compare [**8**, problem 35]), and ii) implies that the knot is not −amphicheiral [**16**, p. 231].

As Kinoshita observed [**8**, problem 35], knots having an Alexander polynomial $\Delta(t)$ which is not symmetric satisfy i). Also the examples of [**11**] satisfy i) even though their Alexander polynomials are symmetric.

One can construct knots satisfying i) and ii) as follows. Take a slice smooth n-knot K_1 with a group G satisfying i); for instance, take one of the examples exhibited in [**18**]. Take the connected sum of K_1 with a smooth knot K_2 with group $\mathbf{Z}$ and order >2 in the smooth cobordism group C_n. Such a knot K_2 can be constructed by [**12**]. Since the natural homomorphism $C_n \to C_n^{\mathrm{TOP}}$ is a monomorphism [**4**], then $K_1 \# K_2$ satisfies i) and ii).

Case II. $n \equiv 0 \bmod 4$. Consider the finite module $T = \Lambda/\langle p, (t+1)^2\rangle$, where $\Lambda = \mathbf{Z}[t, t^{-1}]$, $p \in \mathbf{Z}$ is a prime $\equiv 3 \bmod 4$ and $\langle \ldots \rangle$ denotes the ideal generated by . . . .

Define the skew-symmetric form $[\ ,\ ]: T \times T \to \mathbf{Q}/\mathbf{Z}$ by $[1, t] = 1/p$. By [**17**, § 0.13] a fibered n-knot $K = (S^{n+2}, S^n)$ can be constructed such that $H_q(F) = T$, where $q = n/2$, F is the closure of a fiber of K and $[\ ,\ ]$ is the linking pairing. Here the structure of $H_q(F)$ as a Λ-module is defined by $t\xi = \alpha_*\xi$ where $\alpha: F \to F$ is the monodromy.

Let $h: (S^{n+2}, S^n) \to (S^{n+2}, S^n)$ be a homeomorphism which reverses the orientation of S^{n+2}. Let e be a generator of $H_q(F)$ (as a Λ-module) and let $\beta = \beta(t) \in \Lambda$ be such that $h_*'(e) = \beta e$ where h' is as in Lemma 4. Then, by this lemma,

(1) $[\lambda e, \epsilon e] = \epsilon[\lambda e, e] = [h_*'(\lambda e), h_*'(e)]$, for any $\lambda \in \Lambda$,

where ϵ is the degree of $h|S^n$. From Lemma 2 and the diagram

$$\begin{array}{ccccccc} F & \xrightarrow{q} & \tilde{E} & \xrightarrow{\tilde{h}} & \tilde{E} & \xrightarrow{r} & F \\ \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle t} & & \uparrow{\scriptstyle t} & & \uparrow{\scriptstyle\alpha} \\ F & \xrightarrow[q]{} & \tilde{E} & \xrightarrow[\tilde{h}]{} & \tilde{E} & \xrightarrow[r]{} & F, \end{array}$$

in which the first and last squares are commutative up to homotopy, it follows that

(2) $h_*'(te) = t^\delta h_*'(e),\ \delta = \pm 1,$

where $\delta = 1$ if and only if $h_*: H_1(S^{n+2} - S^n) \to H_1(S^{n+2} - S^n)$ is the identity.

Hence, if $(\epsilon, \delta) = (-1, +1)$, from (1) and (2) we obtain

$$[\lambda e, \epsilon e] = [\lambda\beta e, \beta e] = [\lambda e, \bar{\beta}\beta e],$$

where $\bar{\beta} = \beta(t^{-1})$. Then $\epsilon e = \bar{\beta}\beta e$, hence $\epsilon \equiv \beta(t^{-1})\beta(t) \bmod \langle p, (t+1)^2\rangle$. For $t = -1$ this yields $-1 \equiv \beta(-1)^2 \bmod (p)$ which is impossible because -1 is not a quadratic residue mod p.

If $(\epsilon, \delta) = (+1, -1)$ then $[\lambda e, \epsilon e] = [\bar{\lambda}\beta e, \beta e] = [\lambda e, -\bar{\beta}\beta e]$ and we obtain the same contradiction. Thus K is not amphicheiral.

To obtain a fibered knot which in addition is not invertible, it suffices to take the connected sum of K with a fibered knot K' such that $H_q(F') = \Lambda/\langle\lambda\rangle$, where F' is the closure of a fiber of K' and $\lambda = \lambda(t)$ is a non-symmetric monic polynomial such that $\lambda(0) = \pm 1$. Such a knot K' can be constructed by [**19**, Corollary 3.4]. Notice that $H_q(F')$ has no $\mathbf{Z}$-torsion (by [**6**]) so that the previous argument shows that $K \# K'$ is still not amphicheiral (and non-invertible).

Case III. $n \equiv 2 \bmod 4$. We first assume $n > 2$. Let p be a positive integer which is the product of positive prime numbers which are congruent to 1 modulo $m = (n + 2)/2$ and such that $b^m \not\equiv -1 \bmod p$, for every integer b (such a p exists by Lemma 7). Then there is a positive integer a such that $a^m \equiv 1 \bmod p$ and $(a^r - 1, p) = 1$, for $1 \leqq r < m$ (see, for instance [**2**]). Consider the lens space $L = L(p; 1, a, \ldots, a^{m-1})$ and the diffeomorphism $g: S^{2m-1} \to S^{2m-1}$, defined by

$$g(z_1, \ldots, z_m) = (e^{2\pi i/p}z_1, e^{2\pi ia/p}z_2, \ldots, e^{2\pi ia^{m-1}/p}z_m).$$

If we define $T: S^{2m-1} \to S^{2m-1}$ by $T(z_1, z_2, \ldots, z_m) = (z_2, z_3, \ldots, z_m, z_1)$, then $Tg = g^aT$, so that T induces a diffeomorphism from L onto itself, which is isotopic to a diffeomorphism $f: L \to L$ which is the identity on a neighborhood of a smooth $(2m - 1)$-ball B. Since the induced automorphism $f_*: H_1(L) \to H_1(L)$ is multiplication by a, Lemma 6 can be applied; in particular f is orientation preserving. Let α be the restriction of f to $L_0 = L - \operatorname{Int} B$. By using the Mayer-Vietoris and Van-Kampen theorems it is readily seen that the smooth manifold $M(\alpha)$, defined above, is homeomorphic to S^{n+2}. We therefore have a fibered knot (S^{n+2}, S^n) with $\operatorname{Int} L_0$ as fiber. We may take S^{n+2} to be diffeomorphic to the standard smooth $(n + 2)$-sphere by changing the differentiable structure in an $(n + 2)$-ball in $S^{n+2} - S^n$. By Lemmas 5 and 3 (S^{n+2}, S^n) is not $-$amphicheiral.

Now, using the fact that p does not divide $a^2 - 1$, it is readily seen that every automorphism of

$$\pi_1(S^{n+2} - S^n) = ||t, x: x^p = 1, txt^{-1} = x^a||$$

induces the identity on its abelianization. Thus Lemma 1 implies that (S^{n+2}, S^n) is neither $+$amphicheiral nor invertible.

It remains to establish the case $n = 2$.

Let p be an odd positive integer such that $a^2 \not\equiv -1 \bmod p$, for every integer a. If q is relatively prime to p then there is a smooth fibered 2-knot K_1 with fiber $\operatorname{Int} L_0(p; 1, q)$, obtained by 2-twist spinning a suitable two bridge knot [**24**].

Consider a second smooth 2-knot K_2 such that

i) The Alexander polynomial $\Delta(t)$ of K_2 is not symmetric.

ii) Hom $(T_1, \mathbf{Z}_{p^2}) \to$ Hom $(T_1, \mathbf{Z}_p)$ is onto, where $T_1 = \operatorname{Torsion} H_1(\tilde{E}_2)$.

Of course ii) is satisfied if $H_1(\tilde{E}_2)$ has no $\mathbf{Z}$-torsion.

At the end of the proof we will give examples of knots satisfying i) and ii).

Condition ii) is equivalent to the condition $\beta = 0$, where

$$\beta: H^1(\tilde{E}_2, \partial\tilde{E}_2; \mathbf{Z}_p) \to H^2(\tilde{E}_2, \partial\tilde{E}_2; \mathbf{Z}_p)$$

is the Bockstein associated to the sequence $0 \to \mathbf{Z}_p \to \mathbf{Z}_{p^2} \to \mathbf{Z}_p \to 0$. This is a consequence of the commutative diagram of Figure 3 in which the Ext terms in the third row are 0 by [**9**, § 63].

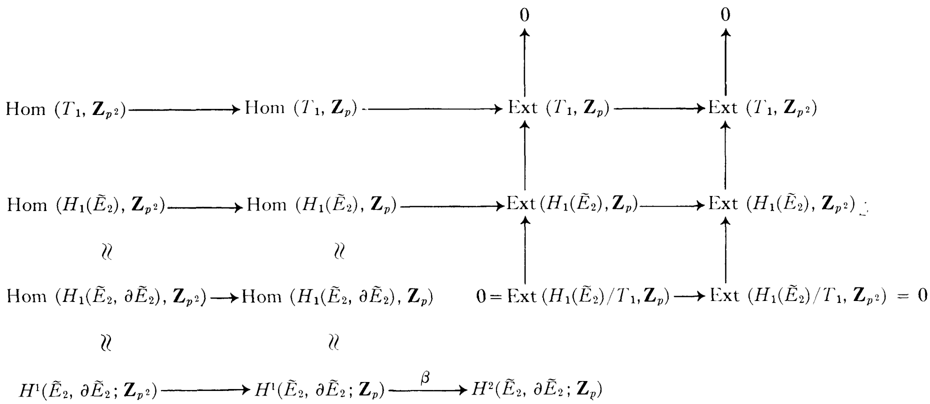

FIGURE 3

The sum $K_1 \# K_2 = (S^4, S^2)$ has Alexander polynomial $\Delta(t)$ so that (S^4, S^2) is neither $+$amphicheiral nor invertible.

Let $h:(S^4, S^2, S^4 - S^2) \to (S^4, S^2, S^4 - S^2)$ be a homotopy equivalence of triples. We may assume the knot has a tubular neighborhood T such that $h(T) = T$ and $h(E) = E$, where E is the closure of $S^4 - T$. Let S^3 be a 3-sphere in S^4 splitting S^4 into two balls B_1, B_2 such that, for $i = 1, 2$ $(B_i \cap S^2) \cup D^2$ is the knot K_i where D^2 is a 2-disk contained in S^3. We choose S^3 so that $S^3 \cap T$ is a tubular neighborhood of $S^3 \cap S^2$ in S^3. Write $E_i = B_i - \text{Int}\, T$ and $\tilde{E}_i = \eta^{-1}(E_i)$, where $\eta:\tilde{E} \to E$ is the infinite cyclic covering of E. Let $\tilde{h}:\tilde{E} \to \tilde{E}$ be a lifting of $h:E \to E$.

Since K_1 is a fibered knot we can identify $(\tilde{E}_1, \partial\tilde{E}_1)$ with $(L_0, \partial L_0) \times \mathbf{R}$, which is homotopy equivalent to $(L_0, \partial L_0)$, L_0 being $L_0(p; 1, q)$. We have the relation $x \cup \beta(x) = q\mu$, where x is a generator of $H^1(\tilde{E}_1, \partial\tilde{E}_1; \mathbf{Z}_p)$, $\beta:H^1(\tilde{E}_1, \partial\tilde{E}_1; \mathbf{Z}_p) \to H^2(\tilde{E}_1, \partial\tilde{E}_1; \mathbf{Z}_p)$ is the Bockstein homomorphism corresponding to the sequence $0 \to \mathbf{Z}_p \to \mathbf{Z}_{p^2} \to \mathbf{Z}_p \to 0$ and μ generates $H^3(\tilde{E}_1, \partial\tilde{E}_1; \mathbf{Z}_p)$ (see, for example [**10**, p. 225]).

We have a commutative diagram (Figure 4)

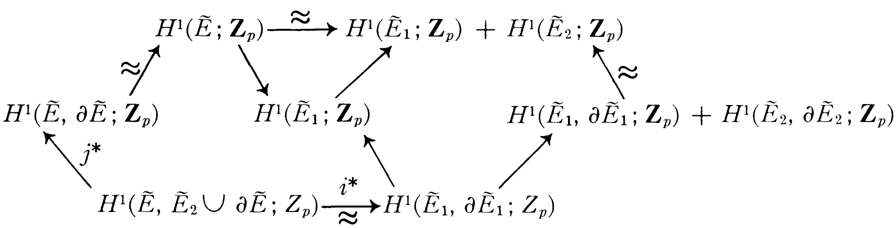

FIGURE 4

in which all arrows have the obvious meaning.

If we write $\bar{x} = j^*(i^*)^{-1}(x)$, then $\tilde{h}^*(\bar{x})$ is an element of the form $r\bar{x} + \bar{y}$ which is the image of an element

$$(rx, y) \in H^1(\tilde{E}_1, \partial\tilde{E}_1; \mathbf{Z}_p) + H^1(\tilde{E}_2, \partial\tilde{E}_2; \mathbf{Z}_p)$$

by the arrows in the upper part of the diagram. Notice that

$$\beta(\tilde{h}^*(\bar{x})) = \beta(r\bar{x} + \bar{y}) = r\beta(\bar{x}),$$

since $\beta(y) = 0$.

We have $H^3(\tilde{E}_2; \mathbf{Z}_p) \approx \mathrm{Hom}\,(H_3(\tilde{E}_2); \mathbf{Z}_p) + \mathrm{Ext}\,(H_2(\tilde{E}_2), \mathbf{Z}_p)$. Using [**9**, § 63], we obtain $H^3(\tilde{E}_2; \mathbf{Z}_p) = 0$ because $H_3(\tilde{E}_2) = 0$ and $H_2(\tilde{E}_2)$ has no **Z**-torsion [**17**]. Using the exact sequence of the triple $(\tilde{E}, \tilde{E}_2 \cup \partial\tilde{E}, \partial\tilde{E})$ and the fact that

$$H^3(\tilde{E}_2 \cup \partial\tilde{E}, \partial\tilde{E}; \mathbf{Z}_p) \approx H^3(\tilde{E}_2, \partial\tilde{E}_2 \cap \partial\tilde{E}_1; \mathbf{Z}_p) \approx H^3(\tilde{E}_2; \mathbf{Z}_p) \approx 0$$

we conclude that $j^*: H^3(\tilde{E}, \tilde{E}_2 \cup \partial\tilde{E}; \mathbf{Z}_p) \rightarrow H^3(\tilde{E}, \partial\tilde{E}; \mathbf{Z}_p)$ is an isomorphism. Therefore, if $i^*: H^3(\tilde{E}, \tilde{E}_2 \cup \partial\tilde{E}; \mathbf{Z}_p) \rightarrow H^3(\tilde{E}_1, \partial\tilde{E}_1; \mathbf{Z}_p)$ is the inclusion induced isomorphism, then $\bar{\mu} = j^*(i^*)^{-1}(\mu)$ is a generator of $H^3(\tilde{E}, \partial\tilde{E}; \mathbf{Z}_p)$ and we have the relation $\bar{x} \cup \beta(\bar{x}) = q\bar{\mu}$. Hence

$$\begin{aligned} q\tilde{h}^*(\bar{\mu}) = \tilde{h}^*(q\bar{\mu}) = \tilde{h}^*(\bar{x}) \cup \beta\tilde{h}^*(\bar{x}) &= (r\bar{x} + \bar{y}) \cup r\beta(\bar{x}) \\ &= r^2\bar{x} \cup \beta(\bar{x}) = r^2q\bar{\mu}, \end{aligned}$$

that is, $\tilde{h}^*(\bar{\mu}) \neq -\bar{\mu}$ because $r^2 \not\equiv -1 \bmod p$. Therefore $\tilde{h}^*: H^3(\tilde{E}, \partial\tilde{E}) \rightarrow H^3(\tilde{E}, \partial\tilde{E})$ is the identity and, using the isomorphisms

$$H^3(\tilde{E}, \partial\tilde{E}) \underset{\approx}{\overset{\delta}{\longleftarrow}} H^2(\partial\tilde{E}) \underset{\approx}{\overset{\eta^*}{\longleftarrow}} H^2(\partial E) \underset{\approx}{\longleftarrow} H^2(T) \underset{\approx}{\longrightarrow} H^2(S^2),$$

we conclude that h preserves the orientation of S^2. This proves that (S^4, S^2) is not $-$amphicheiral.

Finally, to complete the proof of Theorem 1, we give examples of knots satisfying i) and ii) above. Knots whose group has a presentation of deficiency one have the property that the first homology module of the infinite cyclic cover of its complement can be presented by a square matrix [**13**, p. 107] and therefore this module has no torsion [**6**]. If $\Delta(t)$ is a (not necessarily symmetric) polynomial satisfying $\Delta(1) = \pm 1$, there are smooth 2-knots whose groups can be presented with two generators and one relation, with Alexander polynomial $\Delta(t)$ [**18**]. Another knot satisfying i) and ii) is the Cappell-Kirby-Akbulut knot ([**3**] and [**1**]); thus there are smooth fibered 2-knots which are neither amphicheiral nor invertible.

4. Remarks.

Remark 1. Notice that the existence of non-amphicheiral knots is not implied, in principle, by the existence of knots which are not $-$amphicheiral and knots which are not +amphicheiral. Kinoshita essentially produces knots in every dimension which are not +amphicheiral (see [**8**, problem 35]). Farber [**7**, Theorem 4] gives a necessary condition for an n-knot, with $n \equiv 2$ (4), to be −amphicheiral (not "amphicheiral" as Farber erroneously states). Using this condition he shows that there are 2-knots which are non-invertible and are not −amphicheiral.

Remark 2. The proof of Case II can be adapted to get fibered $2q$-knots, that satisfy the theorem, using a generalization of the linking pairing to arbitrary smooth knots (see [**17**] and [**7**]). We note that the Blanchfield pairing (see [**14**]) can be used to obtain fibered knots, that satisfy the theorem for n odd.

Finally we mention some examples of amphicheiral knots.

a) 2-ribbon knots [**23**] are clearly −amphicheiral.

b) Let (S^{n+2}, S^n) be a fibered knot with monodromy $\alpha : P \to P$, where P is the closure of the fiber, such that α is isotopic rel ∂P to $h^{-1}\alpha^{-1}h$ for some orientation preserving homeomorphism $h : P \to P$. Then there is an orientation reversing homeomorphism of S^{n+2} which is the identity on the closure of some fiber. For example 2-twist spun knots satisfy this condition (with h = identity). Also the r-twist spun of the torus knot (p, q) satisfies this condition, h being isotopic to an involution.

c) Let (S^{n+2}, S^n) be a knot as in b). Construct another knot (S^{n+2}, S_1^n) by performing r spherical modifications on 0-spheres and then r spherical modifications on 1-spheres, contained in the complement of a fixed fiber. Then (S^{n+2}, S_1^n) is +amphicheiral.

d) Let (S^{n+2}, S^n) be a fibered knot with monodromy $\alpha : P \to P$ such that α is isotopic rel ∂P to $h^{-1}\alpha h$ for some orientation reversing homeomorphism $h : P \to P$. Then the knot is −amphicheiral. The Cappell-Akbulut-Kirby knot ([**3**], [**1**]) is an example.

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Instituto de Matemáticas de la U.N.A.M.,
Mexico;
The Institute for Advanced Study,
Princeton, New Jersey