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q-HERMITE POLYNOMIALS AND CLASSICAL ORTHOGONAL POLYNOMIALS

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ABSTRACT. We use generating functions to express orthogonality relations in the form of q-beta integrals. The integrand of such a q-beta integral is then used as a weight function for a new set of orthogonal or biorthogonal functions. This method is applied to the continuous q-Hermite polynomials, the Al-Salam-Carlitz polynomials, and the polynomials of Szegö and leads naturally to the Al-Salam-Chihara polynomials then to the Askey-Wilson polynomials, the big q-Jacobi polynomials and the biorthogonal rational functions of Al-Salam and Verma, and some recent biorthogonal functions of Al-Salam and Ismail.

1. **Introduction and preliminaries.** The q-Hermite polynomials seem to be at the bottom of a hierarchy of the classical q-orthogonal polynomials, [6]. They contain no parameters, other than q, and one can get them as special or limiting cases of other orthogonal polynomials.

The purpose of this work is to show how one can systematically build the classical q-orthogonal polynomials from the q-Hermite polynomials using a simple procedure of attaching generating functions to measures.

Let $\{p_n(x)\}$ be orthogonal polynomials with respect to a positive measure μ with moments of any order and infinite support such that

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)\,d\mu(x) = \zeta_n\delta_{m,n}. \tag{1.1}$$

Assume that we know a generating function for $\{p_n(x)\}$, that is we have

$$\sum_{n=0}^{\infty} p_n(x)t^n/c_n = G(x,t), \tag{1.2}$$

for a suitable numerical sequence of nonzero elements $\{c_n\}$. This implies that the orthogonality relation (1.1) is equivalent to

$$\int_{-\infty}^{\infty} G(x,t_1)G(x,t_2)\,d\mu(x) = \sum_{0}^{\infty}\zeta_n\frac{(t_1t_2)^n}{c_n^2}, \tag{1.3}$$

provided that we can justify the interchange of integration and sums.

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Our idea is to use $G(x,t_1)G(x,t_2)\,d\mu(x)$ as a new measure, the total mass of which is given by (1.3), and then look for a system of functions (preferably polynomials) orthogonal or biorthogonal with respect to it. If such a system is found one can then repeat the process. The generating function in (1.2) is assumed to be an elementary function, that is a quotient of products of powers and infinite products. It is clear that we cannot indefinitely continue this process. The form of the generating function will become too complicated at a certain level, and the process will then terminate. The referee wondered whether there is a principal reason which forbids that nice explicit (bi)orthogonal systems can be found with respect to measures which are not elementary. We do not know the answer to this question especially since in the case of associated orthogonal polynomials [11], [19], [28] the weight function involves the reciprocal of the square of the absolute value of a transcendental function. Part of the difficulty is that we do not have direct proofs of the orthogonality of the associated polynomials.

If μ has compact support it will often be the case that (1.2) converges uniformly for x in the support of μ and $|t|$ sufficiently small. In this case the justification is obvious.

We mention the following general result with no assumptions about the support of μ. For $0<\rho\le\infty$ we denote by $D(0,\rho)$ the set of $z\in\mathbf{C}$ with $|z|<\rho$.

PROPOSITION 1.1. *Assume that* (1.1) *holds and that the power series*

$$\sum_{n=0}^{\infty}\frac{\sqrt{\zeta_n}}{c_n}z^n \tag{1.4}$$

has a radius of convergence ρ with $0<\rho\le\infty$.

(i) Then there is a μ-null set $N\subseteq\mathbf{R}$ such that (1.2) converges absolutely for $|t|<\rho, x\in\mathbf{R}\setminus N$. Furthermore (1.2) converges in $L^2(\mu)$ for $|t|<\rho$, and (1.3) holds for $|t_1|,|t_2|<\rho$.

(ii) If μ is indeterminate then (1.2) converges absolutely and uniformly on compact subsets of $\Omega=\mathbf{C}\times D(0,\rho)$, and G is holomorphic in Ω.

PROOF. For $0<r_0<r<\rho$ there exists $C>0$ such that $(\sqrt{\zeta_n}/|c_n|)r^n\le C$ for $n\ge 0$, and we find

$$\left\|\sum_{n=0}^{N}|p_n(x)|\frac{r_0^n}{|c_n|}\right\|_{L^2(\mu)}\le\sum_{n=0}^{N}\frac{\sqrt{\zeta_n}}{|c_n|}r^n\left(\frac{r_0}{r}\right)^n\le C\sum_{n=0}^{\infty}\left(\frac{r_0}{r}\right)^n<\infty,$$

which by the monotone convergence theorem implies that

$$\sum_{n=0}^{\infty}|p_n(x)|\frac{r_0^n}{|c_n|}\in L^2(\mu),$$

and in particular the sum is finite for μ-almost all x. This implies that there is a μ-null set $N\subseteq\mathbf{R}$ such that $\sum p_n(x)(t^n/c_n)$ is absolutely convergent for $|t|<\rho$ and $x\in\mathbf{R}\setminus N$.

The series (1.2) can be considered as a power series with values in $L^2(\mu)$, and by assumption its radius of convergence is ρ. It follows that the series in (1.2) converges in $L^2(\mu)$ to some $G(x,t)$ for $|t|<\rho$, and (1.3) is a consequence of Parseval's formula.

If μ is indeterminate it is well known that $\sum |p_n(x)|^2/\zeta_n$ converges uniformly on compact subsets of $\mathbf{C}$, *cf.* [1], [26], and the assertion follows. ∎

In order to describe details of our work we will need to introduce some notations. There are three systems of q-Hermite polynomials. Two of them are orthogonal on compact subsets of the real line and the third is orthogonal on the unit circle. The two q-Hermite polynomials on the real line are the discrete q-Hermite polynomials $\{H_n(x:q)\}$ [13] and the continuous q-Hermite polynomials $\{H_n(x|q)\}$ of L. J. Rogers [8]. They are generated by

$$2xH_n(x|q) = H_{n+1}(x|q) + (1-q^n)H_{n-1}(x|q), \tag{1.5}$$

$$xH_n(x:q) = H_{n+1}(x:q) + q^{n-1}(1-q^n)H_{n-1}(x:q), \tag{1.6}$$

and the initial conditions

$$H_0(x|q) = H_0(x:q) = 1, \quad H_1(x|q) = 2x, \quad H_1(x:q) = x. \tag{1.7}$$

We will describe the q-Hermite polynomials on the unit circle later in the Introduction. The discrete and continuous q-Hermite polynomials have generating functions

$$\sum_0^\infty \frac{H_n(x:q)}{(q;q)_n} t^n = \frac{(t,-t;q)_\infty}{(xt;q)_\infty}, \tag{1.8}$$

and

$$\sum_0^\infty \frac{H_n(x|q)}{(q;q)_n} t^n = \frac{1}{(te^{i\theta}, te^{-i\theta};q)_\infty}, \quad x = \cos\theta, \tag{1.9}$$

respectively, where we used the notation in [16] for the q-shifted factorials

$$(a;q)_0 := 1, \ (a;q)_n := \prod_{k=1}^n (1-aq^{k-1}), \quad n = 1,2,\ldots, \text{or } \infty, \tag{1.10}$$

and the multiple q-shifted factorials

$$(a_1,a_2,\ldots,a_k;q)_n := \prod_{j=1}^k (a_j;q)_n. \tag{1.11}$$

A basic hypergeometric series is

$$\begin{aligned} {}_r\phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix} \,\middle|\, q,z\right) &= {}_r\phi_s(a_1,\ldots,a_r;b_1,\ldots,b_s;q,z) \\ &= \sum_{n=0}^\infty \frac{(a_1,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n} z^n \left((-1)^n q^{n(n-1)/2}\right)^{s+1-r}. \end{aligned} \tag{1.12}$$

In (1.9) $e^{\pm i\theta}$ is $x \pm \sqrt{x^2-1}$ and the square root is chosen so that $\sqrt{x^2-1} \approx x$ as $x \longrightarrow \infty$. This makes $|e^{-i\theta}| \leq |e^{i\theta}|$. It is clear that the right hand sides of (1.8) and (1.9) are analytic functions of the complex variable t for $|t| < 1/|x|$, $|t| < |e^{-i\theta}|$.

In Section 2 we apply the procedure outlined at the beginning of the Introduction to the continuous q-Hermite polynomials for $|q| < 1$ and we reach the Al-Salam-Chihara polynomials in the first step and the second step takes us to the Askey-Wilson polynomials. It is worth mentioning that the Askey-Wilson polynomials are the general classical orthogonal polynomials, [6]. As a byproduct we get a simple evaluation of the Askey-Wilson q-beta integral, [10]. This seems to be the end of the line in this direction. The case $q > 1$ will be studied in Section 5, see comments below. In Section 3 we apply the same procedure to the polynomials $\{U_n^{(a)}(x;q)\}$ and $\{V_n^{(a)}(x;q)\}$ of Al-Salam and Carlitz [2]. They are generated by the recurrences

$$(1.13)\qquad U_{n+1}^{(a)}(x;q) = [x-(1+a)q^n]U_n^{(a)}(x;q) + aq^{n-1}(1-q^n)U_{n-1}^{(a)}(x;q), \quad n>0,$$

$$(1.14)\qquad V_{n+1}^{(a)}(x;q) = [x-(1+a)q^{-n}]V_n^{(a)}(x;q) - aq^{1-2n}(1-q^n)V_{n-1}^{(a)}(x;q), \quad n>0,$$

and the initial conditions

$$(1.15)\qquad U_0^{(a)}(x;q) = V_0^{(a)}(x;q) = 1, \quad U_1^{(a)}(x;q) = V_1^{(a)}(x;q) = x-1-a,$$

[2], [14]. It is clear that $U_n^{(a)}(x;1/q) = V_n^{(a)}(x;q)$, so there is no loss of generality in assuming $0 < q < 1$ with appropriate restrictions on a. The U_n's provide a one parameter extension of the discrete q-Hermite polynomials when $0 < q < 1$ corresponding to $a = -1$. In Section 3 we show that our attachment procedure generates the big q-Jacobi polynomials from the U_n's. The big q-Jacobi polynomials were studied by Andrews and Askey in 1976. The application of our procedure to the V_n's does not lead to orthogonal polynomials but to a system of biorthogonal rational functions of Al-Salam and Verma [5].

The q-analogue of Hermite polynomials on the unit circle are the polynomials

$$(1.16)\qquad \mathcal{H}_n(z;q) = \sum_{k=0}^{n} \frac{(q;q)_n (q^{-1/2}z)^k}{(q;q)_k (q;q)_{n-k}}.$$

Szegö introduced these polynomials in [27] to illustrate his theory of polynomials orthogonal on the unit circle. Szegö used the Jacobi triple product identity to prove the orthogonality relation

$$(1.17)\qquad \frac{1}{2\pi}\int_0^{2\pi} \mathcal{H}_m(e^{i\theta};q)\,\overline{\mathcal{H}_n(e^{i\theta};q)}(q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta};q)_\infty\, d\theta = \frac{(q;q)_n q^{-n}}{(q;q)_\infty}\delta_{m,n}.$$

In Section 4 we show how generating functions transform (1.16) to a q-beta integral of Ramanujan. This explains the origin of the biorthogonal polynomials of Pastro [24] and the ${}_4\phi_3$ biorthogonal rational functions of Al-Salam and Ismail [4].

In Section 5 we consider the continuous q-Hermite polynomials for $q > 1$. They are orthogonal on the imaginary axis. For $0 < q < 1$ we put $h_n(x|q) = (-i)^n H_n(ix|1/q)$, and $\{h_n(x|q)\}$ are called the q^{-1}*-Hermite polynomials*. They correspond to an indeterminate moment problem considered in detail in [18]. Using a q-analogue of the Mehler formula for these polynomials we derive an analogue of the Askey-Wilson integral valid for all the solutions to the indeterminate moment problem. Our derivation, which is different from the one in [18[, is based on Parseval's formula.

The attachment procedure for the q^{-1}-Hermite polynomials leads to a special case of the Al-Salam-Chihara polynomials corresponding to $q > 1$, more precisely to the polynomials

$$u_n(x;t_1,t_2) = v_n\big(-2x;q,-(t_1+t_2)/q,t_1t_2q^{-2},-1\big), \tag{1.18}$$

cf. [9]. We prove that for any positive orthogonality measure μ for the q^{-1}-Hermite polynomials

$$d\nu_\mu(\sinh\xi;t_1,t_2) := \frac{(-t_1e^{\xi},t_1e^{-\xi},-t_2e^{\xi},t_2e^{-\xi};q)_\infty}{(-t_1t_2/q;q)_\infty}\,d\mu(\sinh\xi), \tag{1.19}$$

is an orthogonality measure for $\{u_n\}$.

The attachment procedure applied to $\{u_n\}$ leads to the biorthogonal rational functions

$$\varphi_n(\sinh\xi;t_1,t_2,t_3,t_4) := {}_4\phi_3\left(\begin{matrix} q^{-n},-t_1t_2q^{n-2},-t_1t_3/q,-t_1t_4/q \\ -t_1e^{\xi},t_1e^{-\xi},t_1t_2t_3t_4q^{-3}\end{matrix}\,\middle|\, q,q\right). \tag{1.20}$$

of Ismail and Masson [18] in the special case $t_3 = t_4 = 0$.

The referee raised the question of what determines the starting point in our process. In each case we used the polynomials with fewest possible parameters. In the cases of the polynomials in Sections 2, 4 and 5 we started with polynomials with no parameters, other than q. In Section 3 we used the 1-parameter family of Al-Salam-Carlitz polynomials. We cannot set $a = 0$ in the Al-Salam-Carlitz polynomials and maintain their orthogonality, so it seems that the full Al-Salam-Carlitz polynomials are the correct starting point. We do not have a canonical answer. The referee also remarked that the Askey scheme in [22] contains many q-polynomials at the lowest level of the classification and wondered if these q-polynomials are Hermite like and when we apply our procedure to such Hermite like polynomials we may obtain other results. We plan to investigate this point in a future work.

2. **The continuous q-Hermite ladder.** Here we assume $-1 < q < 1$. The orthogonality relation for the continuous q-Hermite polynomials is

$$\int_0^\pi H_m(\cos\theta|q)H_n(\cos\theta|q)(e^{2i\theta},e^{-2i\theta};q)_\infty\,d\theta = \frac{2\pi(q;q)_n}{(q;q)_\infty}\delta_{m,n}, \tag{2.1}$$

and follows easily from the Jacobi triple product identity [16]. The series in (1.9) converges for $|t| < 1$ uniformly in $\theta \in [0,\pi]$. The reason is that

$$H_n(\cos\theta|q) = \sum_{k=0}^{n}\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}e^{i(n-2k)\theta}$$

implies $|H_n(x|q)| \le H_n(1|q)$ for $x \in [-1,1]$ and (1.9) converges at $x = 1$. Thus (1.2), (1.3) and the generating function (1.9) imply

$$\int_0^\pi \frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(t_1e^{i\theta},t_1e^{-i\theta},t_2e^{i\theta},t_2e^{-i\theta};q)_\infty}\,d\theta = \frac{2\pi}{(q,t_1t_2;q)_\infty}, \quad |t_1|,|t_2| < 1, \tag{2.2}$$

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where we used the q-binomial theorem [16, (II.3)]

$$\sum_0^\infty \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{2.3}$$

with $a = 0$.

The next step is to find polynomials $\{p_n(x)\}$ orthogonal with respect to the weight function

$$w_1(x;t_1,t_2) := \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty}{(t_1e^{i\theta}, t_1e^{-i\theta}, t_2e^{i\theta}, t_2e^{-i\theta};q)_\infty} \frac{1}{\sqrt{1-x^2}}, \qquad x = \cos\theta, \tag{2.4}$$

which is positive for $t_1, t_2 \in (-1,1)$. Here we follow a clever technique of attachment which was used by Askey and Andrews and by Askey and Wilson in [10]. Write $\{p_n(x)\}$ in the form

$$p_n(x) = \sum_{k=0}^n \frac{(q^{-n}, t_1e^{i\theta}, t_1e^{-i\theta};q)_k}{(q;q)_k} a_{n,k}, \tag{2.5}$$

then determine $a_{n,k}$ such that $p_n(x)$ is orthogonal to $(t_2e^{i\theta}, t_2e^{-i\theta};q)_j$, $j = 0, 1, \ldots, n-1$. Note that $(ae^{i\theta}, ae^{-i\theta};q)_k$ is a polynomial in x of degree k, since

$$\begin{aligned}(ae^{i\theta}, ae^{-i\theta};q)_k &= \prod_{j=0}^{k-1}(1 - 2axq^j + a^2q^{2j}) \\ &= (-2a)^k q^{k(k-1)/2} x^k + \text{ lower order terms.}\end{aligned} \tag{2.6}$$

The reason for choosing the bases $\{(t_1e^{i\theta}, t_1e^{-i\theta};q)_k\}$ and $\{(t_2e^{i\theta}, t_2e^{-i\theta};q)_j\}$ is that they attach nicely to the weight function and (2.2) enables us to integrate $(t_1e^{i\theta}, t_1e^{-i\theta};q)_k(t_2e^{i\theta}, t_2e^{-i\theta};q)_j$ against the weight function $w_1(x;t_1,t_2)$. Indeed

$$(t_1e^{i\theta}, t_1e^{-i\theta};q)_k(t_2e^{i\theta}, t_2e^{-i\theta};q)_j w_1(x;t_1,t_2) = w_1(x;t_1q^k, t_2q^j).$$

Therefore

$$\begin{aligned}&\int_{-1}^1 (t_2e^{i\theta}, t_2e^{-i\theta};q)_j p_n(x) w_1(x;t_1,t_2)\,dx \\ &\quad = \sum_{k=0}^n \frac{(q^{-n};q)_k}{(q;q)_k} a_{n,k} \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty\,d\theta}{(t_1q^ke^{i\theta}, t_1q^ke^{-i\theta}, t_2q^je^{i\theta}, t_2q^je^{-i\theta};q)_\infty} \\ &\quad = \frac{2\pi}{(q;q)_\infty} \sum_{k=0}^n \frac{(q^{-n};q)_k a_{n,k}}{(q;q)_k(t_1t_2q^{k+j};q)_\infty} \\ &\quad = \frac{2\pi}{(q, t_1t_2q^j;q)_\infty} \sum_{k=0}^n \frac{(q^{-n}, t_1t_2q^j;q)_k}{(q;q)_k} a_{n,k}.\end{aligned}$$

At this stage we look for $a_{n,k}$ as a quotient of products of q-shifted factorials in order to make the above sum vanish for $0 \le j < n$. The q-Chu-Vandermonde sum [16, (II.6)]

$$_2\phi_1(q^{-n}, a; c; q, q) = \frac{(c/a;q)_n}{(c;q)_n} a^n \tag{2.7}$$

suggests

$$a_{n,k} = q^k/(t_1t_2;q)_k.$$

Therefore

$$\int_{-1}^{1}(t_2e^{i\theta},t_2e^{-i\theta};q)_j p_n(x)w_1(x;t_1,t_2)\,dx = \frac{2\pi(q^{-j};q)_n}{(q,t_1t_2q^j;q)_\infty(t_1t_2;q)_n}(t_1t_2q^j)^n.$$

It follows from (2.5) and (2.6) that the coefficient of x^n in $p_n(x)$ is

(2.8) $$(-2t_1)^n q^{n(n+1)/2}(q^{-n};q)_n/(q,t_1t_2;q)_n = (2t_1)^n/(t_1t_2;q)_n.$$

This leads to the orthogonality relation

(2.9) $$\int_{-1}^{1}p_m(x)p_n(x)w_1(x;t_1,t_2)\,dx = \frac{2\pi(q;q)_n t_1^{2n}}{(q,t_1t_2;q)_\infty(t_1t_2;q)_n}\delta_{m,n}.$$

Furthermore the polynomials are given by

(2.10) $$p_n(x) = {}_3\phi_2\left(\begin{matrix}q^{-n},t_1e^{i\theta},t_1e^{-i\theta}\\ t_1t_2,0\end{matrix}\,\middle|\,q,q\right).$$

The polynomials we have just found are the Al-Salam-Chihara polynomials and were first identified by W. Al-Salam and T. Chihara [3]. Their weight function was given in [9] and [10].

One might hope that it is possible to make other choices for the coefficients $a_{n,k}$ and possibly use summation theorems other than (2.7). We went through the summation theorems in [16] and found that (2.7) is the only summation theorem that works in the case at hand. It is also worth mentioning that the generating function (1.9) is the only elementary generating function for the q-Hermite polynomials known to us, [8].

Observe that the orthogonality relation (2.9) and the uniqueness of the polynomials orthogonal with respect to a positive measure show that $t_1^{-n}p_n(x)$ is symmetric in t_1 and t_2. This gives the known transformation

(2.11) $$ {}_3\phi_2\left(\begin{matrix}q^{-n},t_1e^{i\theta},t_1e^{-i\theta}\\ t_1t_2,0\end{matrix}\,\middle|\,q,q\right) = (t_1/t_2)^n\,{}_3\phi_2\left(\begin{matrix}q^{-n},t_2e^{i\theta},t_2e^{-i\theta}\\ t_1t_2,0\end{matrix}\,\middle|\,q,q\right)$$

as a byproduct of our analysis.

Our next task is to repeat the process with the Al-Salam-Chihara polynomials as our starting point. The representation (2.10) needs to be transformed to a form more amenable to generating functions. This can be done using an idea of Ismail and Wilson [21]. First write the ${}_3\phi_2$ as a sum over k then replace k by $n-k$. Applying

(2.12) $$(a;q)_{n-k} = \frac{(a;q)_n}{(q^{1-n}/a;q)_k}(-q/a)^k q^{-kn+k(k-1)/2}$$

we obtain

(2.13)
$$p_n(x) = \frac{(t_1e^{i\theta},t_1e^{-i\theta};q)_n}{(t_1t_2;q)_n}q^{-n(n-1)/2}(-1)^n\sum_{k=0}^{n}\frac{(-t_2/t_1)^k(q^{-n},q^{1-n}/t_1t_2;q)_k}{(q,q^{1-n}e^{i\theta}/t_1,q^{1-n}e^{-i\theta}/t_1;q)_k}q^{k(k+1)/2}.$$

Applying the q-analogue of the Pfaff-Kummer transformation [16, (III.4)]

$$\sum_{n=0}^{\infty} \frac{(A, C/B; q)_n}{(q, C, Az; q)_n} q^{n(n-1)/2}(-Bz)^n = \frac{(z;q)_\infty}{(Az;q)_\infty} {}_2\phi_1(A, B; C; q, z), \tag{2.14}$$

with

$$A = q^{-n}, \quad B = t_2 e^{i\theta}, \quad C = q^{1-n}e^{i\theta}/t_1, \quad z = qe^{-i\theta}/t_1$$

to (2.13), we obtain the representation

$$p_n(x) = \frac{(t_1 e^{-i\theta}; q)_n t_1^n e^{in\theta}}{(t_1 t_2; q)_n} {}_2\phi_1 \left(\begin{matrix} q^{-n}, t_2 e^{i\theta} \\ q^{1-n} e^{i\theta}/t_1 \end{matrix} \,\middle|\, q, qe^{-i\theta}/t_1 \right).$$

Using (2.12) we express a multiple of p_n as a Cauchy product of two sequences. The result is

$$p_n(x) = \frac{(q;q)_n t_1^n}{(t_1 t_2; q)_n} \sum_{k=0}^{n} \frac{(t_2 e^{i\theta}; q)_k}{(q;q)_k} e^{-ik\theta} \frac{(t_1 e^{-i\theta}; q)_{n-k}}{(q;q)_{n-k}} e^{i(n-k)\theta}.$$

This and the q-binomial theorem (2.3) establish the generating function

$$\sum_{n=0}^{\infty} \frac{(t_1 t_2; q)_n}{(q;q)_n} p_n(x)(t/t_1)^n = \frac{(tt_1, tt_2; q)_\infty}{(te^{-i\theta}, te^{i\theta}; q)_\infty}. \tag{2.15}$$

The orthogonality relation (2.9), the q-binomial theorem and the generating function (2.15) imply the Askey-Wilson q-beta integral, [10], [16].

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^4 (t_j e^{i\theta}, t_j e^{-i\theta}; q)_\infty} \, d\theta = \frac{2\pi (t_1 t_2 t_3 t_4; q)_\infty}{(q;q)_\infty \prod_{1 \le j < k \le 4} (t_j t_k; q)_\infty}. \tag{2.16}$$

The polynomials orthogonal with respect to the weight function whose total mass is given by (2.16) are the Askey-Wilson polynomials. Their explicit representation and orthogonality follow from (2.16) and the q-analogue of the Pfaff-Saalschütz theorem, [16, (II.12)]. The details of this calculation are in [10]. The polynomials are

$$p_n(x; t_1, t_2, t_3, t_4 | q) = t_1^{-n} (t_1 t_2, t_1 t_3, t_1 t_4; q)_n {}_4\phi_3 \left(\begin{matrix} q^{-n}, t_1 t_2 t_3 t_4 q^{n-1}, t_1 e^{i\theta}, t_1 e^{-i\theta} \\ t_1 t_2, t_1 t_3, t_1 t_4 \end{matrix} \,\middle|\, q, q \right). \tag{2.17}$$

The orthogonality relation of the Askey-Wilson polynomials is [10, (2.3)–(2.5)]

$$\begin{aligned} &\int_0^\pi p_m(\cos\theta; t_1, t_2, t_3, t_4 | q) p_n(\cos\theta; t_1, t_2, t_3, t_4 | q) w(\cos\theta; t_1, t_2, t_3, t_4) \, d\theta \\ &\qquad = \frac{2\pi (t_1 t_2 t_3 t_4 q^{2n}; q)_\infty (t_1 t_2 t_3 t_4 q^{n-1}; q)_n}{(q^{n+1}; q)_\infty \prod_{1 \le j < k \le 4} (t_j t_k q^n; q)_\infty} \delta_{m,n}, \end{aligned} \tag{2.18}$$

for $\max\{|t_1|, |t_2, |t_3|, |t_4|\} < 1$, and the weight function is given by

$$w(\cos\theta; t_1, t_2, t_3, t_4) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^4 (t_j e^{i\theta}, t_j e^{-i\theta}; q)_\infty}. \tag{2.19}$$

Observe that the weight function in (2.19) and the right-hand side of (2.18) are symmetric functions of t_1, t_2, t_3, t_4. The weight function in (2.18) is positive when

$\max\{|t_1|,|t_2|,|t_3|,|t_4|\} < 1$, and the uniqueness of the polynomials orthogonal with respect to a positive measure shows that the Askey-Wilson polynomials are symmetric in the four parameters t_1, t_2, t_3, t_4. This symmetry is the Sears transformation [16, (III.15)], a fundamental transformation in the theory of basic hypergeometric functions. The Sears transformation may be stated in the form

$$(2.20)\qquad {}_4\phi_3\left(\begin{matrix} q^{-n}, a, b, c \\ d, e, f \end{matrix}\,\middle|\, q, q\right) = \left(\frac{bc}{d}\right)^n \frac{(\frac{de}{bc}, \frac{df}{bc}; q)_n}{(e, f; q)_n}\, {}_4\phi_3\left(\begin{matrix} q^{-n}, a, d/b, d/c \\ d, \frac{de}{bc}, \frac{df}{bc} \end{matrix}\,\middle|\, q, q\right),$$

where $abc = defq^{n-1}$.

Ismail and Wilson [21] used the Sears transformation to establish the generating function

$$(2.21)\qquad \begin{aligned} &\sum_{n=0}^{\infty} \frac{p_n(\cos\theta; t_1, t_2, t_3, t_4|q)}{(q, t_1t_2, t_3t_4; q)_n} t^n \\ &\quad = {}_2\phi_1\left(t_1e^{i\theta}, t_2e^{i\theta}t_1t_2 \,\middle|\, q, te^{-i\theta}\right) {}_2\phi_1\left(t_3e^{-i\theta}, t_4e^{-i\theta}t_3t_4 \,\middle|\, q, te^{i\theta}\right). \end{aligned}$$

Thus (2.18) leads to the evaluation of the following integral

$$(2.22)\qquad \begin{aligned} &\int_0^{\pi} \prod_{j=5}^{6} {}_2\phi_1\left(t_1e^{i\theta}, t_2e^{i\theta}t_1t_2 \,\middle|\, q, t_je^{-i\theta}\right) {}_2\phi_1\left(t_3e^{-i\theta}, t_4e^{-i\theta}t_3t_4 \,\middle|\, q, t_je^{i\theta}\right) \\ &\qquad \times \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^{4}(t_je^{i\theta}, t_je^{-i\theta}; q)_\infty}\, d\theta \\ &= \frac{2\pi(t_1t_2t_3t_4; q)_\infty}{(q;q)_\infty \prod_{1\le j<k\le 4}(t_jt_k; q)_\infty} \\ &\qquad \times {}_6\phi_5\left(\begin{matrix} \sqrt{t_1t_2t_3t_4/q}, -\sqrt{t_1t_2t_3t_4/q}, t_1t_3, t_1t_4, t_2t_3, t_2t_4 \\ \sqrt{t_1t_2t_3t_4q}, -\sqrt{t_1t_2t_3t_4q}, t_1t_2, t_3t_4, t_1t_2t_3t_4/q \end{matrix}\,\middle|\, q, t_5t_6\right), \end{aligned}$$

valid for $\max\{|t_1|,|t_2|,|t_3|,|t_4|,|t_5|,|t_6|\} < 1$.

In [20] the Askey-Wilson integral (2.16) was evaluated using Rogers's linearization formula of products of continuous q-Hermite polynomials, [8], without going through the Al-Salam-Chihara polynomials. The approach in this work is different and does not use Rogers's formula. In fact Rogers's linearization formula is the special case $t_4 = 0$ of (2.16).

3. **The discrete q-Hermite ladder.** Here we assume $0 < q < 1$. Instead of using the discrete q-Hermite polynomials directly we will use the Al-Salam-Carlitz q-polynomials which are a one parameter generalization of the discrete q-Hermite polynomials. The Al-Salam-Carlitz polynomials $\{U_n^{(a)}(x;q)\}$ have the generating function [2], [14].

$$(3.1)\qquad G(x;t) := \sum_{n=0}^{\infty} U_n^{(a)}(x;q)\frac{t^n}{(q;q)_n} = \frac{(t, at; q)_\infty}{(tx; q)_\infty}, \quad a < 0,\ 0 < q < 1,$$

and satisfy the orthogonality relation

(3.2) $$\int_{-\infty}^{\infty} U_m^{(a)}(x;q)U_n^{(a)}(x;q)\,d\mu^{(a)}(x) = (-a)^n q^{n(n-1)/2}(q;q)_n\delta_{m,n},$$

with $\mu^{(a)}$ a discrete probability measure on $[a, 1]$ given by

(3.3) $$\mu^{(a)} = \sum_{n=0}^{\infty}\left[\frac{q^n}{(q,q/a;q)_n(a;q)_\infty}\varepsilon_{q^n} + \frac{q^n}{(q,aq;q)_n(1/a;q)_\infty}\varepsilon_{aq^n}\right].$$

In (3.3) ε_y denotes a unit mass supported at y. The form of the orthogonality relation (3.2)–(3.3) given in [2] and [14] contained a complicated form of a normalization constant. The value of the constant was simplified in [17]. Since the radius of convergence of (1.4) is $\rho = \infty$ we can apply Proposition 1.1 and get

(3.4)
$$\int_{-\infty}^{\infty} G(x;t_1)\,G(x;t_2)\,d\mu^{(a)}(x) = \sum_{n=0}^{\infty}\frac{(-at_1t_2)^n}{(q;q)_n}q^{n(n-1)/2} = (at_1t_2;q)_\infty, \quad t_1,t_2 \in \mathbf{C},$$

where the last expression follows by Euler's theorem [16, (II.2)]

(3.5) $$\sum_{n=0}^{\infty} z^n q^{n(n-1)/2}/(q;q)_n = (-z;q)_\infty.$$

This establishes the integral

(3.6) $$\int_{-\infty}^{\infty}\frac{d\mu^{(a)}(x)}{(xt_1,xt_2;q)_\infty} = \frac{(at_1t_2;q)_\infty}{(t_1,t_2,at_1,at_2;q)_\infty}.$$

When we substitute for $\mu^{(a)}$ from (3.3) in (3.4) or (3.6) we discover the nonterminating Chu-Vandermonde sum, [16, (II.23)],

(3.7)
$$\frac{(Aq/C,Bq/C;q)_\infty}{(q/C;q)_\infty}{}_2\phi_1\left(\begin{matrix}A,B\\C\end{matrix}\,\middle|\,q,q\right) + \frac{(A,B;q)_\infty}{(C/q;q)_\infty}{}_2\phi_1\left(\begin{matrix}Aq/C,Bq/C\\q^2/C\end{matrix}\,\middle|\,q,q\right) = (ABq/C;q)_\infty.$$

We now restrict the attention to $t_1, t_2 \in (a^{-1}, 1)$ in the case of which $1/(xt_1, xt_2; q)_\infty$ is a positive weight function on $[a, 1]$. The next step is to find polynomials orthogonal with respect to $d\mu^{(a)}(x)/(xt_1,xt_2;q)_\infty$. Define $P_n(x)$ by

(3.8) $$P_n(x) = \sum_{k=0}^{n}\frac{(q^{-n},xt_1;q)_k}{(q;q)_k}q^k a_{n,k}$$

where $a_{n,k}$ will be chosen later. Using (3.6) it is easy to see that

$$\int_{-\infty}^{\infty} P_n(x)\frac{(xt_2;q)_m}{(xt_1,xt_2;q)_\infty}\,d\mu^{(a)}(x) = \sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}q^k a_{n,k}\frac{(at_1t_2q^{k+m};q)_\infty}{(t_1q^k,at_1q^k,t_2q^m,at_2q^m;q)_\infty}$$
$$= \frac{(at_1t_2q^m;q)_\infty}{(t_1,at_1,t_2q^m,at_2q^m;q)_\infty}\sum_{k=0}^{n}\frac{(q^{-n},t_1,at_1;q)_k}{(q,at_1t_2q^m;q)_k}a_{n,k}q^k.$$

The choice $a_{n,k} = (\lambda;q)_k/(t_1, at_1;q)_k$ allows us to apply the q-Chu-Vandermonde sum (2.7). The choice $\lambda = at_1t_2q^{n-1}$ leads to

(3.9)
$$\int_{-\infty}^{\infty} P_n(x)\frac{(xt_2;q)_m}{(xt_1,xt_2;q)_\infty}\,d\mu^{(a)}(x) = \frac{(at_1t_2q^m;q)_\infty(q^{m+1-n};q)_n\,(at_1t_2q^{n-1})^n}{(t_1,at_1,t_2q^m,at_2q^m;q)_\infty(at_1t_2q^m;q)_n}.$$

The right-hand side of (3.9) vanishes for $0 \le m < n$. The coefficient of x^n in $P_n(x)$ is

$$\frac{(q^{-n},at_1t_2q^{n-1};q)_n}{(q,t_1,at_1;q)_n}(-t_1)^nq^{n(n+1)/2} = \frac{(at_1t_2q^{n-1};q)_n}{(t_1,at_1;q)_n}t_1^n.$$

Therefore

(3.10)
$$P_n(x) = \varphi_n(x;a,t_1,t_2) = {}_3\phi_2\left(\begin{matrix} q^{-n},at_1t_2q^{n-1},xt_1 \\ t_1,at_1\end{matrix}\,\middle|\, q,q\right),$$

satisfies the orthogonality relation

(3.11)
$$\int_{-\infty}^{\infty}\varphi_m(x;a,t_1,t_2)\,\varphi_n(x;a,t_1,t_2)\frac{d\mu^{(a)}(x)}{(xt_1,xt_2;q)_\infty} = \frac{(q,t_2,at_2,at_1t_2q^{n-1};q)_n(at_1t_2q^{2n};q)_\infty}{(t_1,at_1,t_2,at_2;q)_\infty(t_1,at_1;q)_n}(-at_1^2)^nq^{n(n-1)/2}\delta_{m,n}.$$

The polynomials $\{\varphi_n(x;a,t_1,t_2)\}$ are the big q-Jacobi polynomials of Andrews and Askey [6] in a different normalization. The Andrews-Askey normalization is

(3.12)
$$P_n(x;\alpha,\beta,\gamma:q) = {}_3\phi_2\left(\begin{matrix} q^{-n},\alpha\beta q^{n+1},x \\ \alpha q,\gamma q\end{matrix}\,\middle|\, q,q\right).$$

Note that we may rewrite the orthogonality relation (3.11) in the form

(3.13)
$$\int_{-\infty}^{\infty} t_1^{-m}(t_1,at_1;q)_m\,\varphi_m(x;a,t_1,t_2)t_1^{-n}(t_1,at_1;q)_n\,\varphi_n(x;a,t_1,t_2)\frac{d\mu^{(a)}(x)}{(xt_1,xt_2;q)_\infty} = \frac{(q,t_1,at_1,t_2,at_2,at_1t_2q^{n-1};q)_n(at_1t_2q^{2n};q)_\infty}{(t_1,at_1,t_2,at_2;q)_\infty}(-a)^nq^{n(n-1)/2}\delta_{m,n}.$$

Since $d\mu^{(a)}(x)/(xt_1,xt_2;q)_\infty$ and the right-hand side of (3.13) are symmetric in t_1 and t_2, then

$$t_1^{-n}(t_1,at_1;q)_n\,\varphi_n(x;a,t_1,t_2)$$

must be symmetric in t_1 and t_2. This gives the known ${}_3\phi_2$ transformation

(3.14)
$${}_3\phi_2\left(\begin{matrix} q^{-n},at_1t_2q^{n-1},xt_1 \\ t_1,at_1\end{matrix}\,\middle|\, q,q\right) = \frac{t_1^n(t_2,at_2;q)_n}{t_2^n(t_1,at_1;q)_n}{}_3\phi_2\left(\begin{matrix} q^{-n},at_1t_2q^{n-1},xt_2 \\ t_2,at_2\end{matrix}\,\middle|\, q,q\right).$$

We now consider the polynomials $\{V_n^{(a)}(x;q)\}$ and restrict the parameters to $0 < a$, $0 < q < 1$, in which case the polynomials are orthogonal with respect to a positive measure, *cf.* [14, VI.10]. The corresponding moment problem is determinate if and only if $0 < a \le q$ or $1/q \le a$. In the first case the unique solution is

$$m^{(a)} = (aq;q)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q,aq;q)_n} \varepsilon_{q^{-n}}, \tag{3.15}$$

and in the second case it is

$$\sigma^{(a)} = (q/a;q)_\infty \sum_{n=0}^{\infty} \frac{a^{-n} q^{n^2}}{(q,q/a;q)_n} \varepsilon_{aq^{-n}}, \tag{3.16}$$

cf. [12]. The total mass of these measures was evaluated to 1 in [17].

If $q < a < 1/q$ the problem is indeterminate and both measures are solutions. In [12] the following one-parameter family of solutions with an analytic density was found

$$\nu(x;a,q,\gamma) = \frac{\gamma|a-1|(q,aq,q/a;q)_\infty}{\pi a[(x/a;q)_\infty^2 + \gamma^2 (x;q)_\infty^2]}, \quad \gamma > 0. \tag{3.17}$$

In the above $a = 1$ has to be excluded. For a similar formula when $a = 1$ see [12].

If μ is one of the solutions of the moment problem we have the orthogonality relation

$$\int_{-\infty}^{\infty} V_m^{(a)}(x;q) V_n^{(a)}(x;q)\, d\mu(x) = a^n q^{-n^2} (q;q)_n \delta_{m,n}. \tag{3.18}$$

The polynomials have the generating function [2], [14]

$$V(x;t) := \sum_{n=0}^{\infty} V_n^{(a)}(x;q) \frac{q^{n(n-1)/2}}{(q;q)_n} (-t)^n = \frac{(xt;q)_\infty}{(t,at;q)_\infty}, \quad |t| < \min(1,1/a), \tag{3.19}$$

and $\zeta_n = q^{-n^2} a^n (q;q)_n$.

The power series (1.4) has the radius of convergence $\sqrt{q/a}$, and therefore (1.3) becomes

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{(xt_1,xt_2;q)_\infty \, d\mu(x)}{(t_1,at_1,t_2,at_2;q)_\infty} &= \int_{-\infty}^{\infty} V(x,t_1)\, V(x,t_2)\, d\mu(x) \\ &= \sum_{n=0}^{\infty} \frac{(at_1t_2/q)^n}{(q;q)_n} \\ &= \frac{1}{(at_1t_2/q;q)_\infty}, \quad |t_1|, |t_2| < \sqrt{q/a}. \end{aligned}$$

This identity with $\mu = m^{(a)}$ or $\mu = \sigma^{(a)}$ is nothing but the q-analogue of the Gauss theorem,

$${}_2\phi_1(a,b;c;q,c/ab) = \frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}, \quad |c| < |ab|, \tag{3.20}$$

[16, (II.8)].

Specializing to the density (3.17) we get

$$\int_{-\infty}^{\infty} \frac{(xt_1,xt_2;q)_\infty \, dx}{(x/a;q)_\infty^2 + \gamma^2(x;q)_\infty^2} = \frac{\pi a(t_1,at_1,t_2,at_2;q)_\infty}{|a-1|\gamma(q,aq,q/a,at_1t_2/q;q)_\infty}, \tag{3.21}$$

valid for $q < a < 1/q, a \neq 1, \gamma > 0$. Formula (3.21) seems to be new.

We now seek polynomials or rational functions that are orthogonal with respect to the measure

$$d\nu(x) = (xt_1, xt_2; q)_\infty \, d\mu(x), \tag{3.22}$$

where μ satisfies (3.18). It is clear that we can integrate $1/[(xt_1;q)_k(xt_2;q)_j]$ with respect to the measure ν. Set

$$\psi_n(x; a, t_1, t_2) := \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} \frac{q^k a_{n,k}}{(xt_1;q)_k}. \tag{3.23}$$

The rest of the analysis is similar to our treatment of the U_n's. We get

$$\int_{-\infty}^{\infty} \frac{\psi_n(x; a, t_1, t_2)}{(xt_2;q)_m} \, d\nu(x) = \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} q^k a_{n,k} \int_{-\infty}^{\infty} (xt_1 q^k, xt_2 q^m; q)_\infty \, d\mu(x),$$

and if we choose $a_{n,k} = (t_1, at_1; q)_k / (at_1 t_2/q; q)_k$ the above expression is equal to

$$\frac{(t_1, at_1, t_2 q^m, at_2 q^m; q)_\infty}{(at_1 t_2 q^{m-1}; q)_\infty} {}_2\phi_1\left(\begin{matrix} q^{-n}, at_1 t_2 q^{m-1} \\ at_1 t_2/q \end{matrix} \middle| q, q\right)$$
$$= \frac{(t_1, at_1, t_2 q^m, at_2 q^m; q)_\infty (q^{-m};q)_n}{(at_1 t_2 q^{m-1}; q)_\infty (at_1 t_2/q; q)_n} (at_1 t_2 q^{m-1})^n,$$

which is 0 for $m < n$. We have used the Chu-Vandermonde sum (2.7). Since ν is symmetric in t_1, t_2, this leads to the biorthogonality relation

(3.24)
$$\int_{-\infty}^{\infty} \psi_m(x; a, t_2, t_1) \, \psi_n(x; a, t_1, t_2) \, d\nu(x) = \frac{(t_1, at_1, t_2, at_2; q)_\infty (q;q)_n}{(at_1 t_2/q; q)_\infty (at_1 t_2/q; q)_n} (at_1 t_2/q)^n \delta_{m,n}.$$

The ψ_n's are given by

$$\psi_n(x; a, t_1, t_2) = {}_3\phi_2\left(\begin{matrix} q^{-n}, t_1, at_1 \\ xt_1, at_1 t_2/q \end{matrix} \middle| q, q\right). \tag{3.25}$$

They are essentially the rational functions studied by Al-Salam and Verma in [5]. Al-Salam and Verma used the notation

$$R_n(x; \alpha, \beta, \gamma, \delta; q) = {}_3\phi_2\left(\begin{matrix} \beta, \alpha\gamma/\delta, q^{-n} \\ \beta\gamma/q, \alpha q x \end{matrix} \middle| q, q\right). \tag{3.26}$$

The translation between the two notations is

$$\psi_n(x; a, t_1, t_2) = R_n(\beta x q^{-1}/\alpha; \alpha, \beta, \gamma, \delta; q), \tag{3.27}$$

with

$$t_1 = \beta, \quad t_2 = \beta\delta/q\alpha, \quad a = \alpha\gamma/\beta\delta. \tag{3.28}$$

Note that R_n has only three free variables since one of the parameters α, β, γ, δ can be absorbed by scaling the independent variable.

4. **The Szegö ladder.** Here we assume $0 < q < 1$. As already mentioned in the introduction Szegö [27] used the Jacobi triple product identity to prove (1.17). The explicit form (1.16) and the q-binomial theorem (2.3) give

$$\mathcal{H}(z,t) := \sum_{n=0}^{\infty} \mathcal{H}_n(z;q)\frac{t^n}{(q;q)_n} = 1/(t, tzq^{-1/2};q)_\infty, \tag{4.1}$$

for $|t| < 1$, $|tz| < q^{1/2}$. From (1.16) it follows that $|\mathcal{H}_n(z;q)| \leq \mathcal{H}_n(|z|;q)$. Furthermore Darboux's method [23] and (4.1) give

$$\mathcal{H}_n(1;q) \approx q^{-n/2}/(q^{-1/2};q)_\infty.$$

Thus (1.17) and (4.1) imply the Ramanujan q-beta integral [16]

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{(q^{1/2}z, q^{1/2}/z;q)_\infty}{(t_1q^{-1/2}z, t_2q^{-1/2}/z;q)_\infty}\frac{dz}{z} = \frac{(t_1,t_2;q)_\infty}{(q,t_1t_2/q;q)_\infty}, \tag{4.2}$$

for $|t_1| < q^{1/2}$, $|t_2| < q^{1/2}$, since (1.4) has radius of convergence $q^{1/2}$ and the series in (4.1) converges uniformly in z for z on the unit circle. This can be proved by estimating $\mathcal{H}_n(z;q)$ directly from (1.16).

Putting

$$\Omega(z) = \frac{(q, t_1t_2q, q^{1/2}z, q^{1/2}/z;q)_\infty}{(t_1q, t_2q, t_1q^{1/2}z, t_2q^{1/2}/z;q)_\infty} \tag{4.3}$$

and applying the attachment technique to (4.2) we find that the polynomials

$$\tilde{p}_n(z,t_1,t_2) := {}_3\phi_2\left(\begin{matrix} q^{-n}, t_1q^{1/2}z, t_1q \\ 0, t_1t_2q \end{matrix}\,\middle|\, q,q\right). \tag{4.4}$$

satisfy the biorthogonality relation

$$\frac{1}{2\pi i}\int_{|z|=1} \tilde{p}_m(z,t_1,t_2)\overline{\tilde{p}_n(z,\overline{t_2},\overline{t_1})}\Omega(z)\frac{dz}{z} = \frac{(q;q)_n}{(t_1t_2q;q)_n}(t_1t_2q)^n\delta_{m,n}. \tag{4.5}$$

Using the transformation [16, (III.7)] we see that

$$\tilde{p}_n(z,t_1,t_2) = \frac{(q;q)_n}{(t_1t_2q;q)_n}(t_1q)^n p_n(z,t_1,t_2), \tag{4.6}$$

where

(4.7)

$$p_n(z,a,b) = \frac{(b;q)_n}{(q;q)_n}{}_2\phi_1(q^{-n}, aq; q^{1-n}/b; q, q^{1/2}z/b) = \sum_{k=0}^{n}\frac{(aq;q)_k(b;q)_{n-k}}{(q;q)_k(q;q)_{n-k}}(q^{-1/2}z)^k,$$

are the polynomials considered by Pastro [24] and for which the biorthogonality relation reads

$$\frac{1}{2\pi i}\int_{|z|=1} p_m(z,t_1,t_2)\overline{p_n(z,\overline{t_2},\overline{t_1})}\Omega(z)\frac{dz}{z} = \frac{(t_1t_2q;q)_n}{(q;q)_n}q^{-n}\delta_{m,n}. \tag{4.8}$$

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The special case when a and b are real was considered by Askey in his comments on [27] in Szegö's Collected Papers. Al-Salam and Ismail [4] used (4.8) and the generating function

(4.9)
$$\sum_{n=0}^{\infty} p_n(z;a,b)t^n = \frac{(atzq^{1/2},bt;q)_\infty}{(tzq^{-1/2},t;q)_\infty}$$

to establish a q-beta integral and found the rational functions biorthogonal to its integrand. The interested reader is referred to [4] for details.

5. **The q^{-1}-Hermite ladder.** When $q > 1$ in (1.5) the polynomials $\{H_n(x|q)\}$ become orthogonal on the imaginary axis. The result of replacing x by ix and q by $1/q$ put the orthogonality on the real line and the new q is now in $(0,1)$, [7]. Denote $(-i)^n H_n(ix|1/q)$ by $h_n(x|q)$. In this new notation the recurrence relation (1.5) and the initial conditions (1.7) become

(5.1)
$$h_{n+1}(x|q) = 2xh_n(x|q) - q^{-n}(1-q^n)h_{n-1}(x|q), n > 0,$$

(5.2)
$$h_0(x|q) = 1, \quad h_1(x|q) = 2x.$$

The polynomials $\{h_n(x|q)\}$ are called the *q^{-1}-Hermite polynomials*, [18]. The corresponding moment problem is indeterminate. Let $\mathcal{V}_q$ be the set of probability measures which solve the problem. For any $\mu \in \mathcal{V}_q$ we have

(5.3)
$$\int_{-\infty}^{\infty} h_m(x|q)h_n(x|q)\,d\mu(x) = q^{-n(n+1)/2}(q;q)_n\delta_{m,n}.$$

The h_n's have the generating function, [18],

(5.4)
$$\sum_{n=0}^{\infty} h_n(x|q)\frac{t^n}{(q;q)_n}q^{n(n-1)/2} = (-te^{\xi},te^{-\xi};q)_\infty, \quad x = \sinh\xi,\ t,\xi \in \mathbf{C}.$$

By Proposition 1.1 it is clear that (5.3) and (5.4) imply

(5.5)
$$\int_{-\infty}^{\infty}(-t_1e^{\xi},t_1e^{-\xi},-t_2e^{\xi},t_2e^{-\xi};q)_\infty\,d\mu(x) = (-t_1t_2/q;q)_\infty, \quad t_1,t_2 \in \mathbf{C},\ \mu \in \mathcal{V}_q$$

since the power series (1.4) has radius of convergence $\rho = \infty$. Incidentally the function

(5.6)
$$\chi_t(x) = \left(-te^{\xi},te^{-\xi};q\right)_\infty = \left(-t(\sqrt{x^2+1}+x),t(\sqrt{x^2+1}-x);q\right)_\infty$$

belongs to $L^2(\mu)$ for any $\mu \in \mathcal{V}_q$ and any $t \in \mathbf{C}$. Therefore the complex measure $\nu_\mu(t_1,t_2)$ defined by

(5.7)
$$d\nu_\mu(x;t_1,t_2) := \frac{\chi_{t_1}(x)\chi_{t_2}(x)}{(-t_1t_2/q;q)_\infty}\,d\mu(x), \quad \mu \in \mathcal{V}_q,\ t_1,t_2 \in \mathbf{C},\ t_1t_2 \neq -q^{1-k},\ k \geq 0$$

has total mass 1, and it is non-negative if $t_1 = \overline{t_2}$.

Note that $(-te^{\xi},te^{-\xi};q)_k$ is a polynomial of degree k in $x = \sinh\xi$ for each fixed $t \neq 0$. Since

$$(-te^{\xi}/q^k,te^{-\xi}/q^k;q)_k\chi_t(x) = \chi_{t/q^k}(x),$$

we see that the non-negative polynomial $|(-te^{\xi}/q^k, te^{-\xi}/q^k; q)_k|^2$ of degree $2k$ is $\nu_\mu(t,\bar{t})$-integrable. This implies that $\nu_\mu(t,\bar{t})$ has moments of any order, and by the Cauchy-Schwarz inequality every polynomial is $\nu_\mu(t_1,t_2)$-integrable for all $t_1,t_2 \in \mathbf{C}$, $\mu \in \mathcal{V}_q$.

Introducing the orthonormal polynomials

$$\tilde{h}_n(x|q) = \frac{h_n(x|q)}{\sqrt{(q;q)_n}} q^{n(n+1)/4} \tag{5.8}$$

the q-Mehler formula, *cf.* [18], reads

$$\sum_{n=0}^{\infty} \tilde{h}_n(\sinh\xi|q)\tilde{h}_n(\sinh\eta|q)z^n = \frac{(-zqe^{\xi+\eta}, -zqe^{-\xi-\eta}, zqe^{\xi-\eta}, zqe^{-\xi+\eta}; q)_\infty}{(z^2q;q)_\infty} \tag{5.9}$$

and is valid for $\xi,\eta \in \mathbf{C}$, $|z| < 1/\sqrt{q}$.

Applying the Darboux method [23] to (5.9) Ismail and Masson [18] found the asymptotic behavior of $\tilde{h}_n(\sinh\eta|q)$, and from their result it follows that $(\tilde{h}_n(\sinh\eta|q)z^n) \in l^2$ for $|z| < q^{-1/4}, \eta \in \mathbf{C}$. In this case the right-hand side of (5.9) belongs to $L^2(\mu)$ as a function of $x = \sinh\xi$ and the formula is its orthogonal expansion. Putting $t = qze^{\eta}$, $s = -qze^{-\eta}$, we have $z^2 = -stq^{-2}$, so if $|st| < q^{3/2}$ we have $|z| < q^{-1/4}$ and the right-hand side of (5.9) becomes $\chi_t(\sinh\xi)\chi_s(\sinh\xi)/(-st/q;q)_\infty$, which belongs to $L^2(\mu)$. Using this observation we can give a simple proof of the following formula from [18].

PROPOSITION 5.1. *Let $\mu \in \mathcal{V}_q$ and let $t_i \in \mathbf{C}$, $i = 1,\ldots,4$ satisfy $|t_1t_3|, |t_2t_4| < q^{3/2}$. (This holds in particular if $|t_i| < q^{3/4}$, $i = 1,\ldots,4$). Then $\prod_{i=1}^4 \chi_{t_i} \in L^1(\mu)$ and*

$$\int \prod_{i=1}^{4} \chi_{t_i}\, d\mu = \frac{\prod_{1\le j<k\le 4}(-t_jt_k/q;q)_\infty}{(t_1t_2t_3t_4q^{-3};q)_\infty}. \tag{5.10}$$

PROOF. We write

$$qz_1e^{\eta_1} = t_1, \quad qz_2e^{\eta_2} = t_2, \quad -qz_1e^{-\eta_1} = t_3, \quad -qz_2e^{-\eta_2} = t_4,$$

noting that $z_1^2 = -t_1t_3q^{-2}, z_2^2 = -t_2t_4q^{-2}$, so the equations have solutions z_i, η_i, $i = 1,2$ if $t_i \neq 0$ for $i = 1,\ldots,4$. We next apply Parseval's formula to the two $L^2(\mu)$-functions $\chi_{t_1}\chi_{t_3}$, $\chi_{t_2}\chi_{t_4}$ and get

$$\int \prod_{i=1}^{4} \chi_{t_i}\, d\mu = (-t_1t_3/q, -t_2t_4/q; q)_\infty \sum_{n=0}^{\infty} \tilde{h}_n(\sinh\eta_1|q)\tilde{h}_n(\sinh\eta_2|q)(z_1z_2)^n,$$

which by the q-Mehler formula gives the right-hand side of (5.10).

If $t_1 = 0$ and $t_2t_3t_4 \neq 0$ we apply Parseval's formula to χ_{t_3} and $\chi_{t_2}\chi_{t_4}$, and if two of the parameters are zero the formula reduces to (5.5). ■

We shall now look at orthogonal polynomials for the measures $\nu_\mu(t_1,t_2)$. When $q > 1$ the Al-Salam-Chihara polynomials are orthogonal on $(-\infty,\infty)$ and their moment problem may be indeterminate [9], [3], [15]. If one replaces q by $1/q$ in the Al-Salam-Chihara polynomials, they can be renormalized to polynomials $\{v_n(x;q,a,b,c)\}$ satisfying

$$(1-q^{n+1})v_{n+1}(x;q,a,b,c) = (a - xq^n)v_n(x;q,a,b,c) - (b - cq^{n-1})v_{n-1}(x;q,a,b,c), \tag{5.11}$$

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where $0 < q < 1$, and a, b, c are complex parameters. We now consider the special case

$$u_n(x;t_1,t_2) = v_n\big(-2x;q,-(t_1+t_2)/q,t_1t_2q^{-2},-1\big), \tag{5.12}$$

where t_1, t_2 are complex parameters. The corresponding monic polynomials $\hat{u}_n(x)$ satisfy the recurrence relation determined from (5.11)

$$x\hat{u}_n(x) = \hat{u}_{n+1}(x) + \frac{1}{2}(t_1+t_2)q^{-n-1}\hat{u}_n(x) + \frac{1}{4}(t_1t_2q^{-2n-1}+q^{-n})(1-q^n)\hat{u}_{n-1}(x), \tag{5.13}$$

so by Favard's theorem, *cf.* [14], $\{u_n(x;t_1,t_2)\}$ are orthogonal with respect to a complex measure $\alpha(t_1,t_2)$ if and only if $t_1t_2 \neq -q^{n+1}$, $n \geq 1$, and with respect to a probability measure $\alpha(t_1,t_2)$ if and only if $t_2 = \overline{t_1} \in \mathbf{C}\setminus\mathbf{R}$ or $t_1,t_2 \in \mathbf{R}$ and $t_1t_2 \geq 0$. In the latter case

$$\int_{-\infty}^{\infty} u_m(x;t_1,t_2)u_n(x;t_1,t_2)\,d\alpha(x;t_1,t_2) = \frac{q^{n(n-3)/2}}{(q;q)_n}(-t_1t_2q^{-n-1};q)_n\delta_{m,n}. \tag{5.14}$$

It follows from (3.77) in [9] that the moment problem is indeterminate if $t_1 = \overline{t_2}$. If t_1, t_2 are real, different and $t_1t_2 \geq 0$, we can assume $|t_1| < |t_2|$ without loss of generality, and in this case the moment problem is indeterminate if and only if $|t_1/t_2| > q$.

The generating function (3.70) in [9] takes the form ($x = \sinh\xi$)

$$\sum_{n=0}^{\infty} u_n(x;t_1,t_2)t^n = \frac{(-te^{\xi},te^{-\xi};q)_\infty}{(-t_1t/q,-t_2t/q;q)_\infty}, \quad \text{for } |t| < \min\{q/|t_1|,q/|t_2|\}. \tag{5.15}$$

By the method used in [9] we can derive formulas for u_n in the following way: Use the q-binomial theorem to write the right-hand side of (5.15) as a product of two power series in t and equate coefficients of t^n to get

$$u_n(x;t_1,t_2) = \sum_{k=0}^{n} \frac{(qe^{\xi}/t_1;q)_k}{(q;q)_k}(-t_1/q)^k\frac{(-qe^{-\xi}/t_2;q)_{n-k}}{(q;q)_{n-k}}(-t_2/q)^{n-k}. \tag{5.16}$$

Application of the identity (I.11) in [16] gives the explicit representation

$$u_n(x;t_1,t_2) = (-t_2/q)^n\frac{(-qe^{-\xi}/t_2;q)_n}{(q;q)_n}\,{}_2\phi_1\left(\begin{matrix} q^{-n},qe^{\xi}/t_1 \\ -t_2e^{\xi}/q^n \end{matrix}\,\middle|\, q,-t_1e^{\xi}\right), \tag{5.17}$$

which by (III.8) in [16] can be transformed to

(5.18)
$$u_n(x;t_1,t_2) = \frac{\big(-q^2/(t_1t_2);q\big)_n}{(q;q)_n}(-t_2/q)^n\,{}_3\phi_1\left(\begin{matrix} q^{-n},qe^{\xi}/t_1,-qe^{-\xi}/t_1 \\ -q^2/(t_1t_2) \end{matrix}\,\middle|\, q,(t_1/t_2)q^n\right).$$

Writing the ${}_3\phi_1$ as a finite sum and applying the formula

$$(a;q)_k = (q^{1-k}/a;q)_k(-a)^kq^{k(k-1)/2},$$

we see that (5.18) can be transformed to

(5.19)
$$u_n(x;t_1,t_2) = (-1/t_1)^n\frac{(-t_1t_2/q^{n+1};q)_n}{(q;q)_n}q^{n(n+1)/2}\sum_{k=0}^{n}\frac{(q^{-n},-t_1e^{\xi}/q^k,t_1e^{-\xi}/q^k;q)_k}{(q,-t_1t_2/q^{k+1};q)_k}q^{nk}.$$

By symmetry of t_1, t_2 a similar formula holds for t_1 and t_2 interchanged.

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THEOREM 5.2. *For $\mu \in \mathcal{V}_q$ and $t_1, t_2 \in \mathbf{C}$ such that $t_1 t_2 \neq -q^{n+1}$, $n \geq 1$ the Al-Salam-Chihara polynomials $\{u_n(x; t_1, t_2)\}$ are orthogonal with respect to the complex measure $\nu_\mu(t_1, t_2)$ given by (5.7).*

PROOF. It follows by (5.13) that $\hat{u}_n(x; 0, 0) = 2^{-n} h_n(x|q)$ so the assertion is clear for $t_1 = t_2 = 0$.

Assume now that $t_1 \neq 0$. By the three term recurrence relation it suffices to prove that

$$\int u_n(x; t_1, t_2)\, d\nu_\mu(x; t_1, t_2) = 0 \quad \text{for } n \geq 1. \tag{5.20}$$

By (5.19) and (5.7) we get

$$\int u_n(x; t_1, t_2)\, d\nu_\mu(x; t_1, t_2)$$
$$= (-1/t_1)^n \frac{(-t_1 t_2/q^{n+1}; q)_n}{(q; q)_n} q^{n(n+1)/2} \sum_{k=0}^{n} q^{nk} \frac{(q^{-n}; q)_k}{(q; q)_k} \int \frac{\chi_{t_1/q^k}(x) \chi_{t_2}(x)}{(-t_1 t_2/q^{k+1}; q)_\infty}\, d\mu(x).$$

By (5.5) the integral is 1, and the sum is equal to ${}_1\phi_0(q^{-n}; -; q, q^n)$, which is equal to 0 for $n \geq 1$ by (II.4) in [16]. ∎

In particular, if $t_2 = \overline{t_1}$ then $\nu_\mu(t_1, \overline{t_1})$ is a positive measure and the Al-Salam-Chihara moment problem corresponding to $\{u_n(x; t_1, \overline{t_1})\}$ is indeterminate. The set

$$\{\nu_\mu(t_1, \overline{t_1}) \mid \mu \in \mathcal{V}_q\}$$

is a compact convex subset of the full set $\mathcal{C}(t_1)$ of solutions to the $\{u_n(x; t_1, \overline{t_1})\}$-moment problem.

If $t_1 = t_2 \in \mathbf{R}$ then

$$\{\nu_\mu(t_1, \overline{t_1}) \mid \mu \in \mathcal{V}_q\} \neq \mathcal{C}(t_1)$$

since the measures on the left can have no mass at the zeros of $\chi_{t_1}(x)$.

If $t_1 = t \in (q, 1)$ and $t_2 = 0$ then the Al-Salam-Chihara moment problem is determinate and the set $\{\nu_\mu(t, 0) \mid \mu \in \mathcal{V}_q\}$ contains exactly one positive measure namely the one coming from $\mu \in \mathcal{V}_q$ being the N-extremal solution corresponding to the choice $a = q/t$ in (6.27) and (6.30) of [18], *i.e.*, μ is the discrete measure with mass m_n at x_n for $n \in \mathbf{Z}$, where

$$x_n = \frac{1}{2}\left(\frac{t}{q^{n+1}} - \frac{q^{n+1}}{t}\right)$$

and

$$m_n = \frac{(q/t)^{4n} q^{n(2n-1)}(1 + q^{2n+2}/t^2)}{(-q^2/t^2, -t^2/q, q; q)_\infty}.$$

The function $\chi_t(x)$ vanishes for $x = x_n$ when $n < 0$ and we get

$$\nu_\mu(t, 0) = \sum_{n=0}^{\infty} c_n \varepsilon_{x_n},$$

where

$$c_n = \frac{q^{3n(n+1)/2}(1 + q^{2n+2}/t^2)(-q^2/t^2; q)_n}{t^{2n}(q; q)_n(-q^2/t^2; q)_\infty}.$$

We now go back to (5.5) and integrate $1/(-t_1e^{\xi}, t_1e^{-\xi}; q)_k$ against the integrand in (5.5). Here again the attachment method works and we see that

$$\varphi_n(\sinh\xi; t_1, t_2) := {}_3\phi_2\left(\begin{matrix} q^{-n}, -t_1t_2q^{n-2}, 0 \\ -t_1e^{\xi}, t_1e^{-\xi} \end{matrix} \,\middle|\, q, q\right) \tag{5.21}$$

satisfies the biorthogonality relation

$$\begin{aligned} &\int_{-\infty}^{\infty} \varphi_m(x; t_1, t_2)\, \varphi_n(x; t_2, t_1)\chi_{t_1}(x)\chi_{t_2}(x)\, d\mu(x) \\ &\qquad = \frac{1 + t_1t_2q^{n-2}}{1 + t_1t_2q^{2n-2}}(-t_1t_2q^{n-1}; q)_\infty(q; q)_n q^{n(n-3)/2}(t_1t_2)^n\delta_{m,n}. \end{aligned} \tag{5.22}$$

The biorthogonal rational functions (5.21) are the special case $t_3 = t_4 = 0$ of the biorthogonal rational functions

$$\varphi_n(\sinh\xi; t_1, t_2, t_3, t_4) := {}_4\phi_3\left(\begin{matrix} q^{-n}, -t_1t_2q^{n-2}, -t_1t_3/q, -t_1t_4/q \\ -t_1e^{\xi}, t_1e^{-\xi}, t_1t_2t_3t_4q^{-3} \end{matrix} \,\middle|\, q, q\right). \tag{5.23}$$

of Ismail and Masson [18]. We have not been able to apply a generating function technique to (5.21) because we have not been able to find a suitable generating function for the rational functions (5.21).

We now return to the Al-Salam-Chihara polynomials $\{u_n(x; t_1, t_2)\}$ in the positive definite case and reconsider the generating function (5.15). The radius of convergence of (1.4) is $\rho = q^{3/2}/\sqrt{t_1t_2}$, and we get by Proposition 1.1, (5.14) and the q-binomial theorem

$$\int_{-\infty}^{\infty} \chi_{t_3}(x)\chi_{t_4}(x)\, d\alpha(x; t_1, t_2) = \frac{(-t_1t_3/q, -t_1t_4/q, -t_2t_3/q, -t_2t_4/q, -t_3t_4/q; q)_\infty}{(t_1t_2t_3t_4q^{-3}; q)_\infty}, \tag{5.24}$$

valid for $|t_3|, |t_4| < \rho$.

Applying this to the measures $\alpha(t_1, t_2) = \nu_\mu(t_1, t_2)$, we get a new proof of (5.10), now under slightly different assumptions on $t_1, \ldots, t_4$.

The attachment procedure works in this case, and we prove the biorthogonality relation of [18] under the same assumptions as in Proposition 5.1:

$$\begin{aligned} &\int_{-\infty}^{\infty} \varphi_m(x; t_1, t_2, t_3, t_4)\, \varphi_n(x; t_2, t_1, t_3, t_4) \prod_{i=1}^{4} \chi_{t_i}(x)\, d\mu(x) \\ &\qquad = \frac{1 + t_1t_2q^{n-2}}{1 + t_1t_2q^{2n-2}} \frac{(t_1t_2t_3t_4q^{-3})^n(q, -q^2/t_3t_4; q)_n(-t_1t_2q^{n-1}; q)_\infty}{(t_1t_2t_3t_4q^{-3}; q)_n} \\ &\qquad\quad \times \frac{\prod_{1\le j<k\le 4}(-t_jt_k/q; q)_\infty}{(t_1t_2t_3t_4q^{-3}; q)_\infty}\delta_{m,n}. \end{aligned} \tag{5.25}$$

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