Proceedings of the Edinburgh Mathematical Society (1997) **40**, 99–110 ©

MODULARITY* IN LIE ALGEBRAS

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(Received 1st February 1995)

A subalgebra U of a Lie algebra L over a field F is called modular* in L if U satisfies the dual of the modular identities in the lattice of subalgebras of L. Our aim is the study of the influence of the modular* identities in the structure of the algebra. First we prove that if the modular* conditions are imposed on an ideal of L then every element of L acts as an scalar on this ideal and if they are imposed on a non-ideal subalgebra U of L then the largest ideal of L contained in U also satisfies the modular* identities. We determine Lie algebras having a subalgebra which satisfies both the modular and modular* identities, provided that either L is solvable or char$(F) \neq 2, 3$. As immediate consequences of this result we obtain that the existence of a co-atom satisfying the modular* identities in the lattice $\mathcal{L}(L)$ forces that the lattice $\mathcal{L}(L)$ is modular and that the modular* identities on any subalgebra U forces that U is quasi-abelian. In the case when L is supersolvable we obtain that the modular* conditions on any non-ideal of L are enough to guarantee that $\mathcal{L}(L)$ is modular. For arbitrary fields and any Lie algebra L, we prove that the modular* conditions on every co-atom of the lattice $\mathcal{L}(L)$ guarantee that $\mathcal{L}(L)$ is modular.

1991 *Mathematics subject classification:* 17B05.

1. Introduction

Throughout L will denote a finite dimensional Lie algebra over a field F. We say that L is *almost-abelian* if L has a basis $a_1, \ldots, a_n, x$ with product given by $[a_i, a_j] = 0$ and $[a_i, x] = a_i$ for every i, j. If L is either abelian or almost-abelian we say that L is quasi-abelian. The *core* of a subalgebra S of L, denoted by S_L, is the largest ideal of L contained in S. If $S_L = 0$ we say S is core-free in L.

A subalgebra U of L is called *modular* in L if it is a modular element in the lattice of subalgebras of L; that is

$$\langle U, B\rangle \cap C = \langle B, U \cap C\rangle \text{ for all subalgebras } B \leq C$$

and

$$\langle U, B\rangle \cap C = \langle B \cap C, U\rangle \text{ for all subalgebras } U \leq C.$$

Here $\langle X, Y\rangle$ denotes the subalgebra of L generated by X and Y. (The easiest example of modular subalgebras are the subalgebras Q of L such that $[Q, V] \subseteq Q + V$ for every subspace V of L, such subalgebras are called quasi-ideals). Quasi-ideals have been studied in **[1]**, **[5]** and **[15]**. The only known examples of modular subalgebras which

https://doi.org/10.1017/S0013091500023464 Published online by Cambridge University Press

are not quasi-ideals are the one dimensional subalgebras of the three dimensional non-split simple Lie algebras and the standard maximal subalgebra of the hamiltonian Lie algebra $\mathcal{H}(2 : \underline{1} : \Phi(\gamma))^2$ (see [**16**]).

Although many properties of modular subalgebras have been obtained (see [**2**, **6**, **7**, **13**, **14**, **16**]) the general problem of determining the Lie algebras with modular subalgebras is still open. One of the known results is that every modular subalgebra U of L is a quasi-ideal of L, except when L is three dimensional non-split simple, provided that either L is solvable or $\mathrm{char}(F) = 0$ or L is restricted and F is algebraically closed of characteristic $p > 7$ (see [**16**]). So, under the cited conditions, if U is core-free and modular in L then L is either almost-abelian, three-dimensional non-split simple or a Zassenhaus algebra (see [**1**]). For perfect fields of characteristic different from two or three, it is also known that if a minimal subalgebra A of L is modular then either $A \triangleleft L$ or L is either almost-abelian or three dimensional non-split simple (see [**13**]) and that if a rank one simple subalgebra of L is modular then it must be an ideal of L (see [**15**]).

The modular condition can be dualised to give, a subalgebra U of L is called *modular** in L if

$$\langle U, B\rangle \cap C = \langle B, U \cap C\rangle \text{ for all subalgebras } B \leq C$$

and

$$\langle U \cap B, C\rangle = \langle B, C\rangle \cap U \text{ for all subalgebras } C \leq U.$$

Modular* subalgebras have been introduced in [3]. Note that if U is a maximal subalgebra of L and modular* in L then U is modular in L. Dually, every minimal subalgebra of L which is modular in L is modular*.

The easiest example of subalgebras satisfying the modular* identities are the subalgebras U such that $\langle u, S\rangle = ((u)) + S$ for every subalgebra S of L and $u \in L$, where $((u))$ denotes the subspace of L generated by u; in other words, the subalgebras U such that $((u))$ is a quasi-ideal of L for every $u \in U$ (these subalgebras will be called strong quasi-ideals). We will say that an ideal N of L is a strong ideal if $((a))$ is an ideal of L for every $a \in N$. We have the following chains of implications:

$$\begin{array}{ccccc} \text{strong ideal} & \Longrightarrow & \text{strong quasi-ideal} & \Longrightarrow & \text{modular*} \\ \Downarrow & & \Downarrow & & \\ \text{ideal} & \Longrightarrow & \text{quasi-ideal} & \Longrightarrow & \text{modular} \end{array}$$

In this paper, first we characterize quasi-ideals which are modular* and cores of modular* subalgebras. We show that if a quasi-ideal is modular* in L then either it is a strong ideal or L is almost-abelian. We prove that the core of a modular* subalgebra of L is also modular* and it is a strong ideal. We are able to characterize Lie algebras having subalgebras which are both modular and modular*, in the cases when either

https://doi.org/10.1017/S0013091500023464 Published online by Cambridge University Press

char$(F) \neq 2, 3$ or L is solvable. Then, as maximal and modular* subalgebras are modular, Lie algebras having a maximal and modular* subalgebra will be determined. After that, we will be able to obtain that every modular* subalgebra must be quasi-abelian, in those cases. In the general case, we obtain that the modular* conditions on every maximal subalgebra of L force that the $\mathcal{L}(L)$ is modular. Some similar questions have been looked at by Towers in [**11**], however the approach is quite different.

It is known that for fields of characteristic different from two or three, the lattice $\mathcal{L}(L)$ is modular if and only if L is either quasi-abelian or a μ-algebra (this means that every proper subalgebra of L is one dimensional), see [**9**] or [**18**, Corollary 5]. If F is perfect and char$(F) \neq 2, 3$ then every μ-algebra is three dimensional non-split simple (see [**6**, Proposition 1]).

Finally in this section we give two easy lemmas that will be useful throughout this paper.

Lemma 1.1. *If N is a proper subalgebra of L for which $[n, x] \in ((n))$ for some $n \in N$ and all $x \in L - N$, then $((n))$ is an ideal of L.*

Proof. Let $m \in N$. Then $m + x \notin N$, so $[n, m + x] \in ((n))$, whence $[n, m] \in ((n))$. □

Lemma 1.2. *If Q is a proper subalgebra of L for which $[q, x] \in ((q)) + ((x))$ for some $q \in Q$ and all $x \in L - Q$, then $((q))$ is a quasi-ideal of L.*

Proof. Let $r \in Q$. Then $r + x \notin Q$, so $[q, r + x] \in ((q)) + ((r + x))$, whence $[q, r] \in \{((q)) + ((r)) + ((x))\} \cap Q = ((q)) + ((r))$. □

2. Quasi-ideals and modular* subalgebras

We begin studying relationships between quasi-ideals and modular* subalgebras. Examples of modular* which are not strong quasi-ideals are the following: (i) every proper subalgebra of a three dimensional non-split simple Lie algebra, (ii) every one dimensional subalgebra of L contained in A, being $L = A + ((x))$ where A is an abelian minimal ideal of L. We note that the subalgebras cited in (i) and (ii) above are not quasi-ideals.

Lemma 2.1. *An ideal N of a Lie algebra L is a strong ideal if and only if N is abelian and each element of L acts as a scalar on N.*

Proof. Suppose that N is a strong ideal. We may assume dim $N > 1$. Let x, x' be linearly independent elements of N. We see that $[x, x'] \in ((x)) \cap ((x')) = 0$. This yields that N is abelian. Moreover, for each element $y \in L - N$, we have $[x, y] = \lambda x, x'y = \mu x'$ and $[x + x', y] = \alpha(x + x')$, where $\lambda, \mu, \alpha \in F$. This yields that $\lambda = \mu$. Therefore each element of $L - N$ acts a scalar on N. The converse is clear. □

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Lemma 2.2. *If Q is a strong quasi-ideal of L, then either Q is a strong ideal or L is almost-abelian.*

Proof. Assume Q is a strong quasi-ideal and that there exists $q \in Q$ such that $((q))$ is not an ideal of L. Then, from Theorem 3.6 of [1] it follows that L is almost-abelian. Then assume $((q)) \triangleleft L$ for every $q \in Q$. Clearly, Q is an abelian ideal of L. Hence Q is a strong ideal of L. □

Proposition 2.3. *Let Q be a proper quasi-ideal of a Lie algebra L which is modular* in L. Then, Q is a strong quasi-ideal and so either Q is a strong ideal or L is almost-abelian.*

Proof. Assume Q is modular*. Let $q \in Q$ and $x \in L - Q$. We have $[q, x] \in [Q, x] = Q + ((x))$. So, $[q, x] - \lambda(q, x)x \in Q$ for some $\lambda(q, x) \in F$. Put $C = ((q))$ and $B = ((x))$. We have $\langle Q \cap B, C \rangle = C = ((q))$, and $[q, x] - \lambda(q, x)x \in \langle B, C \rangle \cap Q$. Now, by using the second modular* identity we obtain $q' \in ((q))$ thus $[q, x] \in ((q)) + ((x))$. Thus $((q))$ is a quasi-ideal of L for every $q \in Q$ by Lemma 1.2. Hence, Q is a strong quasi-ideal. The last assertion in the proposition follows from Lemma 2.2. □

Proposition 2.4. *Let U be a proper modular* subalgebra of L. Then the following holds:*

(i) *U_L is a strong ideal of L.*

(ii) *$U_L = \{u \in U \mid ((u)) \trianglelefteq L\}$.*

Proof. (i) Let $u \in U_L$ and $x \in L - U$. Put $C = ((u))$ and $B = ((x))$. From the second modular* identity we have $((u)) = C = \langle B, C \rangle \cap U = \langle u, x \rangle \cap U$. As U_L is an ideal of L, we have $[u, x] \in U_L \cap \langle u, x \rangle \cap U$. Therefore, $[u, x] \in ((u))$. Therefore by Lemma 1.1, $((u)) \trianglelefteq L$ for every $u \in U_L$. So that U_L is a strong ideal of L.

(ii) Write $K = \{u \in U \mid ((u)) \trianglelefteq L\}$. From (i) it follows $U_L \leq K$. Let $u, v \in K$. Clearly, $[u, v] = 0$ and $\lambda u \in K$ for every $\lambda \in F$. Now we prove that $u + v \in K$. To do that take $x \in L - K$. By the second modular* identity, we find $\langle u + v, x \rangle \cap U = ((u + v))$. Since, $[u + v, x] \in U \cap \langle u + v, x \rangle$, we get $[u + v, x] \in ((u + v))$. So that $((u + v)) \trianglelefteq L$. Therefore, K is an ideal of L. Thus $K \leq U_L$. This completes the proof. □

We say that an ideal N of L is *supersolvably immersed* in L if every chief factor of L below N is one dimensional.

Corollary 2.5. *Let U be a proper modular* subalgebra of L. Then the following holds:*

(i) *$U_L + ((y))$ is quasi-abelian for every $y \in L$.*

(ii) *U_L is supersolvably immersed in L.*

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3. The solvable case

In this section we determine modular and modular* subalgebras in solvable Lie algebras L over any field. First we consider the case when L is supersolvable. In this case we obtain that if a non-ideal of L is modular* then L is almost-abelian. Then, by using results of the previous section we will be able to prove that if a solvable Lie algebra L has a non-ideal, modular and modular* subalgebra then it is almost-abelian. From this it is easy to obtain that if a maximal subalgebra of L satisfies the modular* identities then L is almost-abelian and that every modular* subalgebra of any solvable Lie algebra is quasi-abelian. The results in this section will be used in the next section.

We begin with the following

Lemma 3.1. *Let L be any Lie algebra and let U be modular* in L. Then for every $x \in L - U$ and $0 \neq u \in U$, the subalgebra $((x))$ is maximal in $\langle u, x\rangle$.*

Proof. Let M be a maximal subalgebra of $\langle u, x\rangle$ containing $((x))$. By using the second modular* identity, we obtain

$$((u)) = \langle U \cap ((x)), u\rangle = \langle u, x\rangle \cap U.$$

This yields, $U \cap M \leq U \cap \langle u, x\rangle = ((x))$. Since $M \neq \langle u, x\rangle$, it follows $u \notin M$. Hence, $U \cap M = 0$. Now, by using the first modular* identity, we obtain

$$\langle U, x\rangle \cap M = M = \langle x, U \cap M\rangle = ((x)).$$

Hence, $((x))$ is maximal in $\langle u, x\rangle$. □

Theorem 3.2. *Let L be supersolvable over any field and let U be a proper modular* subalgebra of L. Then, either U is a strong ideal of L or L is almost-abelian.*

Proof. We want to prove that U is a quasi-ideal of L. First take $u \in U$ and $x \in L - U$. By Lemma 3.1, $((x))$ is maximal in $\langle u, x\rangle$. As $\langle u, x\rangle$ is supersolvable, we have that $((x))$ has codimension one in $\langle u, x\rangle$. Therefore $[u, x] \in ((u)) + ((x))$. Therefore U is a quasi-ideal of L by Lemma 1.2. Then the result follows from Proposition 2.3. □

Now we can determine the solvable Lie algebras having a modular and modular* subalgebra.

Theorem 3.3. *Let L be solvable over any field F. Let U be a proper subalgebra of L which is modular and modular* in L. Then either U is a strong ideal or L is almost-abelian.*

Proof. If $U \triangleleft L$, then U is a strong ideal by Proposition 2.4. Suppose then U is not an ideal of L. Since U is modular in L, by Corollary 1.2 of **[14]** it follows that

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L/U_L is almost-abelian. As U is modular* in L, by Corollary 2.5(ii) it follows that U_L is supersolvably immersed in L. Then we have that every chief factor of L has dimension one. So, L is supersolvable. Therefore, by Theorem 3.2 it follows that L is almost-abelian. □

Next we determine the solvable Lie algebras having a maximal subalgebra which is modular*.

Corollary 3.4. *Let L be solvable over any field. Then the following are equivalent:*
(i) *L has a maximal subalgebra which is modular* in L.*
(ii) *The lattice $\mathcal{L}(L)$ of all subalgebras of L is modular.*
(iii) *L is quasi-abelian.*

Proof. (i) implies (iii): Suppose that M is maximal and modular* in L. Then, we have that M is also modular in L. Therefore, by Theorem 3.3, either M is a strong ideal or L is almost-abelian. In the former case, we have $\dim L/M = 1$ and by Corollary 2.5 it follows that L must be quasi-abelian. Clearly, (iii) implies (ii). If $\mathcal{L}(L)$ is modular, then every subalgebra of L is modular and modular* in L. So, (ii) implies (i). The proof is complete. □

As an immediate consequence of the above corollary we have

Corollary 3.5. *Let L be solvable over any field. Then every proper subalgebra of L which is modular* in L is quasi-abelian.*

Proof. Suppose that $U < L$ is modular* in L. Take a subalgebra M of L containing U such that U is maximal in M. Then, by Corollary 3.4, M is quasi-abelian. Therefore, U is quasi-abelian too. □

4. The non-solvable case

In this section we consider the case when the Lie algebra L is nonsolvable over a field F with $\mathrm{char}(F) \neq 2, 3$. We obtain that if L has a non-ideal, modular and modular* subalgebra then every proper subalgebra of L is one dimensional (such a Lie algebra is called a *μ-algebra*). Then, by using results of the previous sections, it is easy to prove that if L has a maximal subalgebra satisfying the modular* identities then L must be a μ-algebra and that every modular* subalgebra of any Lie algebra is quasi-abelian.

The proof of the main result depends heavily on results on supersimple Lie algebras which appear in **[12]** and **[18]**. A Lie algebra L is said to be *supersimple* if every subalgebra of L of dimension greater than one is simple. Every μ-algebra is supersimple. More generally, a Lie algebra of dimension greater than one is supersimple if and only if it has no two dimensional subalgebras (Proposition 3.2 of **[12]**). If $\mathrm{char}(F) \neq 2, 3$ and if L is supersimple, then for each $0 \neq x \in L$ there exists

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$y \in L$ such that $L = \langle x, y \rangle$ (Theorem 4 of **[18]**). We recall that for perfect fields of characteristic different from two or three, the only supersimple Lie algebras are the three dimensional non-split simple (it follows from Proposition 1 of **[6]**).

First we consider core-free, modular and modular* subalgebras of a nonsolvable Lie algebra L. More generally, we give the following

Lemma 4.1. *Let L be a Lie algebra over an arbitrary field F. Let U be a core-free subalgebra of L such that $\langle u, z \rangle$ is either two dimensional or a μ-algebra for every $0 \neq u \in U$ and $z \in L - U$. Then one of the following holds:*

(i) *L is almost-abelian.*

(ii) *$\langle u, z \rangle$ is a μ-algebra for every $0 \neq u \in U$ and $z \in L - U$.*

Proof. Pick $0 \neq u \in U$. Suppose that $\langle u, z \rangle$ is two dimensional for every $z \in L - U$. Then $[u, z] \in ((u)) + ((z))$. Hence $((u))$ is a quasi-ideal of L by Lemma 1.2. As U is core-free, $((u))$ is not an ideal of L. So, by **[1]** it follows that L is almost-abelian. Now suppose that there exist $x, z \in L - U$ such that $\langle x, u \rangle$ is two dimensional and $\langle u, z \rangle$ is a μ-algebra. Write $Q = \langle x, u \rangle$ and $T = \langle u, z \rangle$. We have $Q \cap T = ((u))$. Let us first suppose $((u)) \lhd Q$. Then $[x, u] = \alpha u$ where $\alpha \in F$. Since $[x + z, u] = \alpha u + [z, u]$ and since $[x + z, u] \in \langle x + z, u \rangle$, we have $[z, u] \in \langle x + z, u \rangle$. So, $\langle u, [z, u] \rangle = T \leq \langle x + z, u \rangle$. This yields that $\langle x + z, u \rangle$ has dimension greater than two. Then, $\langle x + z, u \rangle$ must be a μ-algebra. Therefore, $T = \langle x + z, u \rangle$. But then we have $x \in T$ and so $Q \leq T$, which is a contradiction. Now suppose that $((u))$ is not an ideal of Q. Let $q \in Q$ such that $((q)) \lhd Q$. Then $[q, u] = \beta q$ where $\beta \in F$ and $((u)) \neq ((q))$. We have $[q + z, u] = \beta q + [z, u] \in \langle q + z, u \rangle$. This yields

$$-\beta z - [u, z] = -\beta z + [z, u] = \beta q + [z, u] - \beta(q + z) \in \langle q + z, u \rangle.$$

So that, $0 \neq \beta z + [u, z] \in \langle q + z, u \rangle$. Since $\beta z + [u, z] \notin ((u))$, we have $T = \langle \beta z + [u, z], u \rangle \leq \langle q + z, u \rangle$. Therefore, $\langle q + z, u \rangle$ has dimension greater than 2. It follows that $\langle q + z, u \rangle$ must be a μ-algebra. Hence $T = \langle q + z, u \rangle$. This yields, $q \in T \cap Q = ((u))$, which is a contradiction. We deduce that either L is almost-abelian or $\langle u, z \rangle$ is a μ-algebra for every $0 \neq u \in U$ and $z \in L - U$. □

Proposition 4.2. *Let F be any field of characteristic different from two or three and let L be a non-solvable Lie algebra. Assume that U is a proper core-free, modular and modular* subalgebra of L. Then L is a μ-algebra.*

Proof. Let us first suppose $\dim U = 1$. Write $U = ((u))$. In the case when F is perfect, then Theorem 2.2 of **[13]** applies and L is three dimensional non-split. In the case when F is not perfect we need to make a slight variation of the proof of that theorem by using results of **[17]** and **[18]**. In the proof of Theorem 2.2 of **[13]** it is proved that L has no two dimensional abelian subalgebras. Now we prove that L has no two dimensional nonabelian subalgebras either. Before that we claim that U satisfies the conditions in Lemma 4.1. Let $0 \neq u \in U$ and $z \in L - U$. Since U is

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modular in L we have that U is maximal and modular in $\langle u, z\rangle$ (see [**2**]). Then we get that $\langle u, z\rangle$ is either two dimensional or a μ-subalgebra (see the proof of Theorem 2.2 of [**13**]). Thus Lemma 4.1 applies and $\langle u, z\rangle$ is a μ-algebra for every $0 \neq u \in U$ and $z \in L - U$. Suppose that S is a two dimensional nonabelian subalgebra of L. Let $x \in S$ such that $((x)) \triangleleft S$. Since U is self-normalizing, $U \neq ((x))$. Then $\langle u, x\rangle$ is a μ-algebra and $U < \langle u, x\rangle < \langle u, S\rangle$. We can suppose without loss of generality that $\langle u, S\rangle = L$. Let $0 \neq N$ be an ideal of L. Pick $0 \neq y \in N$. We have that $\langle U, y\rangle \cap N$ is a nonzero ideal of $\langle U, y\rangle$. Since $\langle U, y\rangle$ is simple, it follows $U \leq N$. But since U is self-normalizing in L (see [**2**]), we get $N = L$. Therefore, L is simple. By the modularity of U it follows that S is maximal in L. Since $\langle u, x\rangle$ is a μ-algebra we have that x is not ad-nilpotent. This yields that S coincides with the Fitting null component of ad x, which contradicts Proposition 1.9 of [**17**]. Therefore L has no two dimensional subalgebras and hence L is supersimple (Proposition 3.2 of [**12**]). As char$(F) \neq 2, 3$, by Theorem 4 of [**18**] there exists $z \in L$ such that $L = \langle u, z\rangle$. We conclude that L is a μ-algebra.

Next we prove that U must be one dimensional. Assume $\dim U > 1$. Let L be a counterexample of minimal dimension. By Proposition 2.3 and Lemma 2.1, U is self-normalizing. So, by the minimality of L we have that U is a maximal subalgebra of L. Now we claim that U is supersimple. To do that we prove that U has no two dimensional subalgebras. Suppose that S is a two dimensional subalgebra of U and let u, u' be a basis for S with $[u, u'] = \alpha u$ where $\alpha \in F$. Let $z \in L - U$. By the second modular* identity we have $((u)) = U \cap \langle u, z\rangle$. Now from the modularity of U it follows that $((u))$ is modular in $\langle u, z\rangle$. So, by Theorem 2.2 of [13] it follows that $\langle u, z\rangle$ is two dimensional or a μ-algebra. So, by Lemma 4.1, $\langle u, z\rangle$ is a μ-algebra for every $0 \neq u \in U$ and $z \in L - U$. On the other hand, we have

$$[u, u' + z] = [u, u'] + [u, z] = \alpha u + [u, z] \in \langle u, u' + z\rangle.$$

This yields $\langle u, [u, z]\rangle \leq \langle u, u' + z\rangle$. As $\langle u, z\rangle$ and $\langle u, u' + z\rangle$ are both μ-algebras, we have $\langle u, z\rangle = \langle u, [u, z]\rangle = \langle u, u' + z\rangle$. This yields $u' \in \langle u, z\rangle \cap U = ((u))$, which is a contradiction. Therefore, U has no two dimensional subalgebras. The claim is proved.

For perfect fields of characteristic different from two or three, every supersimple subalgebra is three dimensional non-split simple. So by Theorem 1.5 of [**15**] we have $U \triangleleft L$ which is a contradiction. For arbitrary fields, we need to work more.

Next we prove that L is also supersimple. Assume not. Then by [**12**], L has a subalgebra Q of dimension two. Since U is supersimple, S is not contained in U. Since U is maximal and modular, we have $U \cap Q \neq 0$. Take $0 \neq u \in U \cap Q$ and $x \in Q, x \notin U$. We have that $\langle u, x\rangle$ is two dimensional. By Lemma 1, it follows that L is almost-abelian which is a contradiction. Now take any $0 \neq u \in U$. As char$(F) = 2, 3$ and L is supersimple, Theorem 4 of [**18**] applies and there exist an element $v \in L - U$ such that $L = \langle u, v\rangle$. But then we have $\langle U \cap ((v)), u\rangle = ((u))$ whereas $\langle v, u\rangle \cap U = L$ and so the second modular* identity does not hold for U. This contradiction shows that $\dim U = 1$ and hence the result. □

Theorem 4.3. *Let F be any field of characteristic different from two or three. Let U*

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be a proper modular and modular subalgebra of a non-solvable Lie algebra L. Then either U is a strong ideal or L is a μ-algebra.*

Proof. If U is an ideal of L, then U is a strong ideal of L by Proposition 2.4. Then suppose that U is not an ideal of L. We have that U/U_L is a core-free, modular and modular* subalgebra of L/U_L. Then, by Proposition 4.2, it follows that L/U_L is a μ-algebra. Assume $U_L \neq 0$. By Corollary 2.5, U_L is abelian and it is supersolvably immersed in L. Thus we may suppose without loss of generality that $\dim U_L = 1$. Put $U_L = ((a))$. Let $u \in U - ((a))$ and $z \in L - U$. By the second modular* identity we have $((u)) = U \cap \langle u, z\rangle$. This yields $U_L \not\leq \langle u, z\rangle$. Since L/U_L has no proper subalgebras of dimension greater than one, $L = U_L + \langle u, z\rangle$. Therefore, $\langle u, z\rangle$ is a μ-algebra for every $u \in U - ((a))$ and $z \in L - U$. On the other hand, we have that u, z and $[u, z]$ act as scalars on U_L by Proposition 2.4 and Lemma 2.1. Thus $[u, a] = \lambda a, [z, a] = \theta a$ and $[[u, z], a] = \rho a$ where $\lambda, \theta, \rho \in F$. By the Jacobi identity, we have $\rho a = [[u, z], a] = [u, [z, a]] + [[u, a], z] = \theta[u, a] + \lambda[a, z] = 0$. Put $y = [u, z]$. We see that $\langle u, z\rangle = \langle u, y\rangle$ and that $\langle a + u, y\rangle$ is a μ-algebra. Moreover we have $[u, y] = [a + u, y] \in \langle y, a + u\rangle$. This yields, $\langle u, y\rangle = \langle y, [y, u]\rangle \leq \langle a + u, y\rangle$ and hence $\langle a + u, y\rangle = \langle u, y\rangle$. So, $a \in \langle u, y\rangle = \langle u, z\rangle$, which is a contradiction. Thus $U_L = 0$ and L is a μ-algebra. The proof is complete. □

Corollary 4.4. *Let F be a perfect field of characteristic $\neq 2, 3$. Let U be a modular and modular* subalgebra of a non-solvable Lie algebra L. Then either U is a strong ideal or L is three dimensional non-split simple.*

Now we are able to determine the nonsolvable Lie algebras having a maximal subalgebra which is modular*.

Corollary 4.5. *Let F be of characteristic $\neq 2, 3$. For a nonsolvable Lie algebra L the following are equivalent:*

(i) *L has a maximal subalgebra which is modular* in L.*

(ii) *L is a μ-algebra.*

Proof. (i) implies (ii): Suppose M is a maximal subalgebra of L which is modular* in L. We have that M is also modular in L. So, by Theorem 4.3 it follows that either M is a strong ideal, or L is a μ-algebra. In the former case, since M is maximal we have that L is quasi-abelian by Corollary 2.5, which is a contradiction.

Clearly, (ii) implies (i). □

As an immediate consequence of the above corollary and Corollary 3.5 we have

Corollary 4.6. *Let F be of characteristic $\neq 2, 3$. Then every proper subalgebra of a Lie algebra L which is modular* must be quasi-abelian.*

Proof. Let $U < L$ be modular* in L. Take a subalgebra S of L containing U such that U is maximal in S. If S is solvable, then by Corollary 3.5 we have that U is

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quasi-abelian. If S is nonsolvable, then by Corollary 4.5 it follows that S is a μ-algebra and so $\dim U = 1$. The proof is complete. □

5. Lie Algebras all of whose maximal subalgebras are modular*

In this section the ground field is an arbitrary field.

Theorem 5.1. *Let L be a Lie algebra over any field. The following are equivalent:*
(i) *every maximal subalgebra of L is modular* in L,*
(ii) *every subalgebra of L is modular* in L,*
(iii) *every subalgebra of L is modular in L,*
(iv) *the lattice $\mathcal{L}(L)$ of all subalgebras of L is modular.*

Proof. (i) implies (ii): By hypothesis, every maximal subalgebra of L is modular* in L. By the *height* $h(S)$ of a subalgebra S of L, we mean the minimum length h of chains

$$S = M_h < M_{h-1} < \ldots < M_1 < M_0 = L$$

such that M_i is maximal in M_{i-1} for every $1 \leq i \leq h$. If $h(S) = 1$, then U is maximal in L. So U is modular* in L. We proceed by induction on h. Then suppose that every subalgebra of L having height less than or equal to $h-1$ is modular* in L and let S be a subalgebra of L such that $h(S) = h$. Pick $x \in M_{h-2}, x \notin M_{h-1}$. Put $B = ((x))$, $C = S$ and $U = M_{h-1}$. We have $U \cap B = 0$ and so $\langle U \cap B, C\rangle = C = S$. Also, we have $h(U) \leq h-1$ and so U is modular* in L, by the inductive hypothesis. By the second modular* identity, it follows that $\langle B, C\rangle \cap U = S$. Now take a maximal subalgebra T of M_{h-2} containing $\langle B, C\rangle$. We see that $h(T) \leq h-1$ and so T is modular* in L. Moreover, we have $S \leq T \cap M_{h-1} \leq M_{h-1}$. Since S is maximal in M_{h-1}, this yields either $S = T \cap M_{h-1}$ or $M_{h-1} \leq T \leq M_{h-2}$. In the former case, we find that S is modular* in L since so are T and M_{h-1} ([3, Lemma 2.3]). In the latter case, we have $T = M_{h-1}$ since M_{h-1} is maximal in M_{h-2} and $T \neq M_{h-2}$. But then we get $x \in T \leq M_{h-1}$, which is a contradiction.

(ii) implies (iii): Let $U \leq L$. To show that U is modular in L, we only need to prove that the second modular identity for U holds. To do that, take a subalgebra C of L containing U and any subalgebra B of L. By hypothesis, C is modular* in L. By the second modular* identity for C, it follows

$$\langle C \cap B, U\rangle = \langle B, U\rangle \cap C.$$

This is just what we needed to prove.

(iii) implies (ii): It is the dual of (ii) implies (iii). Obviously, (iii) and (iv) are equivalent and (ii) implies (i). The proof is complete. □

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6. Lie Algebras all of whose minimal subalgebras are modular*

We recall that in lattice $\mathcal{L}$ it said that Y covers X if $X < Y$ and moreover $X < Z < Y$ is not satisfied by any Z. A lattice $\mathcal{L}$ is called upper semimodular if X covers $X \cap Y$ then $X \vee Y$ covers Y.

By Lemma 2.12 of [3], a minimal subalgebra A of L is modular* if and only if S is maximal in $\langle S, A\rangle$ for every subalgebra S of L such that $A \cap S = 0$. It follows that if every minimal subalgebra of L is modular* the the lattice $\mathcal{L}(L)$ is upper semimodular, by a known result of lattice theory (see [10]).

When $\mathrm{char}(F) \neq 2, 3$, the subalgebra lattice of a Lie algebra is upper semimodular if and only if it is modular (see [18]). For fields of characteristic three, the above result is not true in general. Indeed, the example given by Gein in [6, Example 2] shows a Lie algebra over a certain perfect field of characteristic three whose subalgebra lattice is upper semimodular but not modular.

Acknowledgements. The authors would like to thank the referee for his suggestions and comments.

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https://doi.org/10.1017/S0013091500023464 Published online by Cambridge University Press