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SURFACES EMBEDDED IN $M^2 \times S^1$

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1. Introduction. In this paper we study incompressible and injective (see § 2 for definitions) surfaces embedded in $M^2 \times S^1$, where M^2 is a surface and S^1 is the 1-sphere. We are able to characterize embeddings which are incompressible in $M^2 \times S^1$ when M^2 is closed and orientable. Namely, *a necessary and sufficient condition for the closed surface F to be incompressible in $M^2 \times S^1$, where M^2 is closed and orientable, is that there exists an ambient isotopy h_t, $0 \leqq t \leqq 1$, of $M^2 \times S^1$ onto itself so that either*

(i) *there is a non-trivial simple closed curve $J \subset M^2$ and $h_1(F) = J \times S^1$, or*

(ii) *$p|h_1(F)$ is a covering projection of $h_1(F)$ onto M^2, where p is the natural projection of $M^2 \times S^1$ onto M^2.*

This theorem is used to give an alternate proof for the classification of non-orientable, closed surfaces which can be embedded in $M^2 \times S^1$, where M^2 is closed and orientable. See Corollaries 5.4 and 5.5. These latter results were first obtained by Bredon and Wood [**1**, Theorem 4.8].

We show in § 6 that 3-manifolds fibred over S^1 with fibre a surface F do not determine the fibre F uniquely. In fact, for *M^2 a surface and $\chi(M^2) \leqq 0$, we see that for any integer $k > 0$, $M^2 \times S^1$ can be fibred over S^1 with fibre a surface F and $\chi(F) = k\chi(M^2)$.*

In § 4, and assuming no more about M^2 than that it is a surface, we give sufficient conditions for a proper embedding of a surface F in $M^2 \times S^1$ to be injective in $M^2 \times S^1$.

2. Definitions and notation. The term *surface* is used to mean a compact 2-manifold with or without boundary. If we wish to emphasize that a surface M^2 does not have boundary, we say that M^2 is a *closed* surface.

D^n and S^n are used to denote the n-cell and the n-sphere, respectively. We also use P^2 to denote real projective 2-space.

A manifold N^k is said to be *properly embedded* in the manifold M^n, $n > k$, if $N^k \cap \text{Bd}\, M^n = \text{Bd}\, N^k$. If the surface F is properly embedded in the 3-manifold M^3, we say F is *injective* in M^3 if exactly one of the following cases holds:

(i) If $F = S^2$, then F does not bound a 3-cell in M^3;

(ii) If $F = D^2$, then either $\text{Bd}\, D^2$ does not bound a disk in $\text{Bd}\, M^3$ or whenever $\text{Bd}\, D^2$ does bound a disk $D_1{}^2$ in $\text{Bd}\, M^3$, then the 2-sphere $D^2 \cup D_1{}^2$ is injective in M^3;

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(iii) If $F \neq S^2$ or D^2, then

$$\ker(i_*: \pi_1(F) \to \pi_1(M^3))$$

is trivial, where i_* is induced by inclusion.

If the surface F is properly embedded in the 3-manifold M^3, then we say that F is *incompressible* in M^3 if exactly one of the following cases holds:

(i) If $F = S^2$ or D^2, then F is injective in M^3;

(ii) If $F \neq S^2$ or D^2, then there is no disk D in M^3 where $D \cap F = \text{Bd}\, D$ and $\text{Bd}\, D$ is not contractible in F.

If F is injective in M^3, then F is incompressible in M^3; however, the converse is not true in general. See [**13**] and the remark in this paper following Proposition 4.4. If F is two-sided in M^3, then F is injective in M^3 if and only if F is incompressible in M^3.

We say that a simple closed curve J in the space X is *trivial* in X if J can be contracted to a point in X. Otherwise, we say that J is *non-trivial* in X. A 3-manifold M is called *irreducible* if it contains no injective polyhedral 2-spheres.

The combinatorial terminology is consistent with that used in [**17**]. However, we use the term regular enlargement of a polyhedron P in a manifold M^n along with that of a regular neighbourhood of a polyhedron P in a manifold M^n. The submanifold N^n is called a *regular enlargement* of the polyhedron P in M^n if N^n is a polyhedron in M^n and for some subdivision of N^n and some subdivision of P, N^n collapses to P (see [**17**]).

Let F denote a surface which is properly embedded in the 3-manifold M^3. Suppose that D is a disk in M^3 so that $D \cap F = \text{Bd}\, D$. Then $\text{Bd}\, D$ is a two-sided simple closed curve in F. Furthermore, there is a 3-cell B (not unique) which is a regular enlargement of D in M^3 where

$$B \cap F = \text{Bd}\, B \cap F = A,$$

an annulus, which is a regular neighbourhood of $\text{Bd}\, D$ in F. Let D_1 and D_2 denote the closures of the components of $\text{Bd}\, B - A$. The resultant (either one or two surfaces) of replacing A by $D_1 \cup D_2$ is called an *elementary surgery on F along D*.

We use the term *map* to mean continuous function. A map f of X into Y is said to be *essential* if and only if f is not homotopic to a constant map. Otherwise, f is *inessential*. A map f of X onto Y is called a *covering projection* if for each $y \in Y$ there is an open set U of Y with $y \in U$ and $f^{-1}(U)$ can be written as a mutually exclusive collection of open sets $\{U_\alpha\}$, where f/U_α is a homeomorphism of U_α onto U for each α. If $f: X \to Y$ is a covering projection, we say that X *covers* Y.

Using the terminology of [**2**; **14**], we say that the 3-manifold M^3 is *fibred over* S^1 with *fibre* a surface F if M^3 is the identification space obtained from $F \times I$ by identifying $F \times 0$ and $F \times 1$ with a homeomorphism η of F onto itself. Generally, if M^3 is fibred over S^1 with fibre a surface F, then M^3 is written $F \times I/\eta$.

We use the notation $\chi(K)$ to stand for the Euler characteristic of a complex K. If M^2 is a closed and orientable surface, we use $g(M^2)$ to denote the *genus* of M^2.

The following lemmas are well known.

LEMMA 2.1. *If $M^2 \neq S^2$ and M^3 can be fibred over S^1 with fibre M^2, then M^3 is irreducible.*

LEMMA 2.2. *If M^3 can be fibred over S^1 with fibre S^2, then a polyhedral 2-sphere S in M^3 is injective if and only if S does not separate M^3.*

3. Surfaces separating products. Let $M^3 = M^2 \times I$, where M^2 is a surface. Let $p: M^3 \rightarrow M^2$ denote the natural projection of M^3 onto the factor M^2.

PROPOSITION 3.1. *Suppose that F is an incompressible surface in M^3 with* Bd $F \subset M^2 \times \{0\}$. *Then there is an ambient isotopy h_t, $0 \leqq t \leqq 1$, of M^3 so that for each t the map h_t is fixed on* Bd M^3 *and $p|h_1(F)$ is a homeomorphism into M^2.*

Proof. Since Bd F is contained in $M^2 \times \{0\}$, each component of Bd F is a two-sided curve in $M^2 \times \{0\}$. The proof now follows directly from the techniques of Waldhausen in proving [**16**, Proposition 3.1].

PROPOSITION 3.2. *Let M^2 denote a closed surface. If F is a closed surface in $M^2 \times I$ separating $M^2 \times 0$ from $M^2 \times 1$, then $\chi(F) \leqq \chi(M^2)$.*

Proof. The conclusion follows vacuously if $M^2 = S^2$. Hence, assume that $M^2 \neq S^2$. Since $M^2 \times I$ is irreducible and F separates $M^2 \times 0$ from $M^2 \times 1$, the surface $F \neq S^2$. It will be shown that there is an injective (hence, incompressible) closed surface $G \subset M^2 \times I$ with $\chi(F) \leqq \chi(G)$.

Suppose that G' is a closed surface in $M^2 \times I$ separating $M^2 \times 0$ from $M^2 \times 1$ and $\chi(F) \leqq \chi(G')$. If

$$\ker(\pi_1(G') \rightarrow \pi_1(M^2 \times I)) = \{1\},$$

then let $G = G'$. Otherwise, there is a disk $D \subset M^2 \times I$ so that $D \cap G' = \text{Bd}\, D$ and Bd D is not trivial in G' (see [**13**, § 6]).

Perform an elementary surgery on G' along the disk D. If Bd D separates G', we obtain two surfaces G_1' and G_2', where $\chi(G') < \chi(G_i')$, $i = 1, 2$. Furthermore, either G_1' or G_2' separates $M^2 \times 0$ from $M^2 \times 1$. If Bd D does not separate G', we obtain a surface G'' which separates $M^2 \times 0$ from $M^2 \times 1$ and $\chi(G') < \chi(G'')$.

In either case, there is a closed surface G'' in $M^2 \times I$ separating $M^2 \times 0$ from $M^2 \times 1$ and $\chi(G') < \chi(G'')$. Since there is an upper bound on the Euler characteristic of a closed surface and $G'' \neq S^2$ (G'' separates $M^2 \times 0$ from $M^2 \times 1$), the desired injective surface G may be obtained.

To complete the proof apply Proposition 3.1 with G the F of that proposition. Since G is closed, G is homeomorphic to M^2 and therefore

$$\chi(F) \leqq \chi(G) = \chi(M^2).$$

The next lemma is a technical lemma which is used later. Its proof is straightforward.

LEMMA 3.3. *Suppose that $J_1, \ldots, J_k$ is a mutually exclusive collection of simple closed curves in* Bd $D^2 \times I \subset D^2 \times I$. *Then there is a mutually exclusive collection of* 2-*cells $D_1^2, \ldots, D_k^2$ in $D^2 \times I$ so that for each $i = 1, \ldots, k$,*

$$D_i^2 \cap \operatorname{Bd}(D^2 \times I) = \operatorname{Bd} D_i^2 \cap (\operatorname{Bd} D^2 \times I) = J_i.$$

COROLLARY 3.4. *If M^2 is a surface and F is a surface in $M^2 \times I$ separating $M^2 \times 0$ from $M^2 \times 1$, then $\chi(F) \leqq \chi(M^2)$.*

Proof. If Bd $M^2 = \emptyset$, then this is just Proposition 3.2. Hence, assume that Bd $M^2 \neq \emptyset$. Let $k \geqq 1$ denote the number of components of Bd M^2. Let M_+^2 denote the closed surface obtained from M^2 by attaching a copy of D^2 to each component of Bd M^2. Then $M^2 \times I \subset M_+^2 \times I$.

Let k' denote the number of boundary components of F. Since F separates $M^2 \times 0$ from $M^2 \times 1$ in $M^2 \times I$, we have $k' \geqq k$. Each component of Bd F is contained in (Bd $M^2 \times I$). Applying Lemma 3.3, the surface F may be expanded to a closed surface F_+ which separates $M_+^2 \times 0$ from $M_+^2 \times 1$ and $\chi(F_+) = \chi(F) + k'$.

From Proposition 3.2 it follows that $\chi(F_+) \leqq \chi(M_+^2)$. Hence,

$$\chi(F) \leqq \chi(F) + (k' - k) \leqq \chi(M^2).$$

PROPOSITION 3.5. *Let M^2 denote a surface. If F is injective in $M^2 \times S^1$ and $\chi(F) \neq 0$, then F does not separate $M^2 \times S^1$.*

Proof. Case 1. $\chi(F) = 2$. Then $F = S^2$ and $M^2 = S^2$. If F separates $S^2 \times S^1$, then F is not injective.

Case 2. $\chi(F) = 1$. Then either $F = D^2$ or $F = P^2$. If $F = D^2$, then $M^2 \neq S^2$ and therefore, $M^2 \times S^1$ is irreducible. If F separates $M^2 \times S^1$, then by van Kampen's Theorem [**9**], $\pi_1(M^2 \times S^1)$ can be expressed as a non-trivial free product [**8**]. This is a contradiction to $\pi_1(M^2 \times S^1)$ having non-trivial centre.

If $F = P^2$, then $M^2 = P^2$. Hence, F does not separate since P^2 does not bound a 3-manifold.

Case 3. $\chi(F) < 0$. If F does separate $M^2 \times S^1$, then it follows from van Kampen's theorem that

$$\pi_1(M^2 \times S^1) \approx G_1 \underset{\pi_1(F)}{*} G_2$$

is a non-trivial free product with amalgamation along $\pi_1(F)$. But $\chi(F) < 0$

implies that $\pi_1(F)$ does not have centre [**3**; **6**]. Thus $\pi_1(M^2 \times S^1)$ could not have centre [**8**, Vol. II, p. 32]. This is a contradiction to $\pi_1(M^2 \times S^1)$ having an infinite cyclic group in its centre.

Remarks. (1) If M^2 is a closed surface and $\chi(M^2) < 0$, then there is an injective surface $F \subset M^2 \times S^1$, where $\chi(F) = 0$ and F separates $M^2 \times S^1$. (See Proposition 4.4.)

(2) It would seem natural to expect that a surface F in $M^2 \times S^1$ which does not separate $M^2 \times S^1$ to have the property $\chi(F) = 0$ or $\chi(F) \leqq \chi(M^2)$. However, this is not the case. In fact, if $\chi(M) \leqq 0$, there is a non-separating surface F in $M^2 \times S^1$ with $\chi(F) = -2k$ for any $k \geqq 0$.

4. Existence of injective surfaces in products.

LEMMA 4.1. *Let F denote a surface different from the Klein bottle or the torus. If G is a group and $\pi_1(F)$ embeds in $G \times Z$, then $\pi_1(F)$ embeds in G or in Z.*

Proof. If F is a surface different from the Klein bottle or the torus, x and y are elements of $\pi_1(F)$ and $xy = yx$, then there is an element $z \in \pi_1(F)$ and integers m and n so that $x = z^m$, $y = z^n$ (see [**3**; **6**] for the closed case; otherwise $\pi_1(F)$ is a free group).

Consider the diagram

$$\pi_1(F) \xrightarrow{i_*} G \times Z \quad \begin{array}{l} \nearrow^{\rho_1}\ G \\ \searrow_{\rho_2}\ Z \end{array}$$

where i_* is injective and ρ_1, ρ_2 are the natural projections. If $\ker(\rho_2 i_*) = \{1\}$, then our proof is complete. Hence, assume that $\ker(\rho_2 i_*) \neq \{1\}$.

Suppose that $x \in \ker(\rho_1 i_*)$. Let $y \in \ker(\rho_2 i_*)$ be chosen so that $y \neq 1$. It follows that $i_*(x) \in \ker(\rho_1)$ and $i_*(y) \in \ker(\rho_2)$. Let $(1, x')$ and $(y', 1) \in G \times Z$ represent $i_*(x)$ and $i_*(y)$, respectively. Thus $i_*(xy) = i_*(yx)$. Since the homomorphism i_* is injective, $xy = yx$. Let integers m, n be chosen so that $x = z^m$, $y = z^n$.

We shall show that $x = 1$; hence, $\ker(\rho_1 i_*) = \{1\}$. Let $(z_1, z_2) = i_*(z)$. Then $z_1^m = 1$ and $z_2^n = 1$. It follows that $z_2 = 1$ since $y \neq 1$ and i_* injective imply that $n \neq 0$. Thus

$$i_*(x) = i_*(z^m) = (z_1^m, 1) = (1, 1)$$

and i_* injective yields $x = 1$.

PROPOSITION 4.2. *Let M^2 denote a surface. If F is injective in $M^2 \times S^1$, then there is a $k \geqq 0$ such that $\chi(F) = k\chi(M^2)$.*

Proof. Case 1. $\chi(F) = 2$. Whenever $M^2 \neq S^2$, $M^2 \times S^1$ is irreducible; hence, both $M^2 = S^2$ and $F = S^2$. Let $k = 1$.

Case 2. $\chi(F) = 1$. Then $F = D^2$ or $F = P^2$.

Suppose that $F = D^2$. Since $M^2 \times S^1$ is irreducible ($M^2 \neq S^2$ by Bd $M^2 \neq \emptyset$) and each component of $\text{Bd}(M^2 \times S^1)$ is a torus, $M^2 = D^2$. Let $k = 1$.

Suppose that $F = P^2$. If $M^2 \neq P^2$, then no element of $\pi_1(M^2) \times Z$ is of finite order. Hence, $M^2 = P^2$. Let $k = 1$.

Case 3. $\chi(F) = 0$. Let $k = 0$.

Case 4. $\chi(F) < 0$. Then by Lemma 4.1, $\pi_1(F)$ embeds in $\pi_1(M^2)$. It follows that F covers M^2 and $\chi(F) = k\chi(M^2)$ for some $k \geqq 1$.

Remark. In Theorem 5.2 we will prove that if M^2 is closed and orientable and F is incompressible in $M^2 \times S^1$, then F is orientable and there is an integer $k \geqq 0$ such that

$$g(F) = k(g(M^2) - 1) + 1.$$

PROPOSITION 4.3. *Let M^2 denote a closed surface (orientable or not) different from the Klein bottle. Then there is no injective embedding of the Klein bottle in $M^2 \times S^1$.*

Proof. Suppose that $F \subset M^2 \times S^1$ is injective, where F is the Klein bottle. There are elements $x \neq 1$, $y \neq 1$ in $\pi_1(F)$ such that $x^2y^2 = 1$ in $\pi_1(F)$ and $x \neq y^{-1}$. Let (x_1, x_2) and (y_1, y_2) denote the representations of x and y, respectively, in $\pi_1(M^2 \times S^1) \approx \pi_1(M^2) \times Z$, where $x_1, y_1 \in \pi_1(M^2)$ and $x_2, y_2 \in Z$. It follows that

$$({x_1}^2{y_1}^2, (x_2y_2)^2) = (1, 1).$$

Thus ${x_1}^2{y_1}^2 = 1$ and $x_2y_2 = 1$. This states that $x_2 = {y_2}^{-1}$.

We wish to obtain a contradiction to the choice of $y \neq x^{-1}$ by showing that $x_1 = {y_1}^{-1}$. There are two subcases to consider.

The first subcase is when $\chi(M^2) < 0$. Consider the group G generated by x_1, y_1 in $\pi_1(M^2)$. Then G is a free subgroup of $\pi_1(M^2)$ [**6**, Corollary 2]. If $G = 1$, then $x_1 = {y_1}^{-1}$, and the desired contradiction is obtained. Otherwise, G is free on x_1 and y_1 or G is infinite cyclic. The former does not occur since ${x_1}^2{y_1}^2 = 1$. Hence, there is a $z \in \pi_1(M^2)$ and integers m and n such that $z^m = x_1$ and $z^n = y_1$. That is, $z^{2m+2n} = 1$. Hence, if $m \neq 0$, then $m = -n$ and $z^{-n} = x_1$ or ${y_1}^{-1} = (z^n)^{-1} = x_1$. If $m = 0$, then $n = 0$ and $x_1 = 1 = {y_1}^{-1}$.

The second subcase is when $\chi(M^2) \geqq 0$. In this case $\pi_1(M^2) \times Z$ is Abelian and does not admit an embedding of $\pi_1(F)$.

PROPOSITION 4.4. *Let M^2 denote a surface distinct from P^2. If J is a non-trivial simple closed curve in M^2, then $J \times S^1$ is injective in $M^2 \times S^1$.*

Proof. This follows from the fact that a non-trivial element of $\pi_1(M^2)$ has infinite order.

Remark. In the case $M^2 = P^2$ and J is a non-trivial simple closed curve in M^2, we see that $J \times S^1$ is *not* injective in $M^2 \times S^1$; however $J \times S^1$ is incompressible in $M^2 \times S^1$. This case offers another counterexample to a conjectured extension of the Loop Theorem (see [**13**, p. 18]).

PROPOSITION 4.5. *Let M^2 denote a surface distinct from D^2 and let p denote the natural projection of $M^2 \times S^1$ onto M^2. If F is a surface in $M^2 \times S^1$ and $p|F$ is a covering projection of F onto M^2, then F is injective in $M^2 \times S^1$.*

Proof. Case 1. $M^2 \neq S^2$ or P^2. Then $F \neq S^2$ or D^2 since neither can cover any manifold distinct from D^2, S^2, and P^2. Hence it is sufficient to show that

$$\ker(\pi_1(F) \to \pi_1(M^2 \times S^1))$$

is trivial.

Suppose that

$$x \in \ker(\pi_1(F) \to \pi_1(M^2 \times S^1)).$$

Then $p|x$ is a trivial loop in $\pi_1(M^2)$. Since $p|F$ is a covering projection of F *onto* M^2, this contraction can be lifted to F; i.e. x is trivial in $\pi_1(F)$.

Case 2. $M^2 = S^2$. Then $F = S^2$. The 2-sphere F is not injective in $S^2 \times S^1$ if and only if F separates $S^2 \times S^1$. In this case there is an ambient isotopy h_t, $0 \leqq t \leqq 1$, of $S^2 \times S^1$ such that $p|h_1(F)$ is *not* onto S^2. Hence, the map $p|F$ is inessential from F to S^2. This contradicts $p|F$ is a covering projection.

Case 3. $M^2 = P^2$. Then $F \neq S^2$. For $p|F$ is an essential map of F onto P^2 and since $P^2 \times S^1$ is irreducible there would be a natural extension of $p|F$ to a 3-cell if $F = S^2$. The proof that F is injective in $P^2 \times S^1$ now follows as in Case 2.

The following theorem shows that in a sense Proposition 4.2 was a best possible result.

PROPOSITION 4.6. *Let M^2 denote a surface where $\chi(M^2) \leqq 0$. Then for each integer $k \geqq 0$, there is a two-sided surface F in $M^2 \times S^1$ such that F is injective in $M^2 \times S^1$ and $\chi(F) = k\chi(M^2)$.*

Proof. Consider $M^2 \times S^1$ as the identification space [**4**] obtained from $M^2 \times I$ by setting $(x, 0)$ in $M^2 \times 0$ equal to $(x, 1)$ in $M^2 \times 1$.

Case 1. $k = 0$. Since $\chi(M^2) \leqq 0$, there is a non-trivial simple closed curve $J \subset M^2$. By Proposition 4.4, $F = J \times S^1$ is injective in $M^2 \times S^1$. Furthermore, $\chi(F) = 0$.

Case 2. $k > 0$. There are two situations to consider. The first is when M^2 does *not* contain a two-sided, non-separating simple closed curve. In this situation there is an arc α in M^2 such that $\alpha \cap \mathrm{Bd}\, M = \mathrm{Bd}\, \alpha$ and $M^2 - \alpha$ is connected. Let A denote a regular neighbourhood of α in M^2. Then A is a disk; and if $\tilde{M}^2$ is the closure of $M^2 - A$, then $\tilde{M}^2 \cap A = A^{-1} \cup A^1$ where ${}^jA$ si an arc for $j = -1$ or 1 and $A^{-1} \cap A^1 = \emptyset$. It follows that $\chi(\tilde{M}^2) = \chi(M^2) + 1$.

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For $1 \leqq n \leqq k$, define $F_n = \tilde{M}^2 \times n/(k+1)$. Let

$$\theta: \alpha \times [-1, 1] \to A$$

be a parametrization of A such that $\theta|\alpha \times 0$ is the identity and for $j = -1$ or 1, $\theta|\alpha \times j$ is a homeomorphism onto A^j.

For $1 \leqq n < k$, let

$$\theta_n: [-1, 1] \to [n/(k+1), (n+1)/(k+1)]$$

be the linear function

$$\theta_n(t) = \frac{t + 2n + 1}{2(k+1)}.$$

Define

$$A_n = \{(\theta(x, t), \theta_n(t)): (x, t) \in \alpha \times [-1, 1]\}.$$

Notice that A_n is a disk in

$$M^2 \times [n/(k+1), (n+1)/(k+1)] \subset M^2 \times I$$

and A_n meets F_n in $A^{-1} \times n/(k+1)$ while A_n meets F_{n+1} in

$$A^1 \times (n+1)/(k+1).$$

For $i = 0, 1$, let

$$\theta_i: [-i, 1-i] \to \left[\frac{ni}{k+1}, \frac{ni+1}{k+1}\right]$$

be the linear function

$$\theta_i(t) = \frac{ni + t}{k+1};$$

and define

$$A_{ik} = \{(\theta(x, t), \theta_i(t)): (x, t) \in \alpha \times [-i, 1-i]\}.$$

Notice that A_{ik} is a disk in

$$M^2 \times \left[\frac{in}{k+1}, \frac{in+1}{k+1}\right] \subset M^2 \times I$$

and A_{ik} meets F_{in} in

$$\theta(\alpha \times (1-i)) \times \frac{in+1}{k+1}.$$

Let F be the image in $M^2 \times S^1$ of the natural projection [**4**] of the surface

$$\left(\bigcup_{n=1}^{k} F_n\right) \cup \left(\bigcup_{n=0}^{k} A_n\right).$$

Then by Proposition 4.5, F is an injective surface in $M^2 \times S^1$ (see Figure 1). It follows that

$$\chi(F) = k(\chi(M^2) + 1) + (k+1) - 2(k+1) + 1 = k\chi(M^2).$$

Now consider those surfaces M^2 which contain a two-sided, non-separating simple closed curve J. Let A denote a regular neighbourhood of J in M^2.

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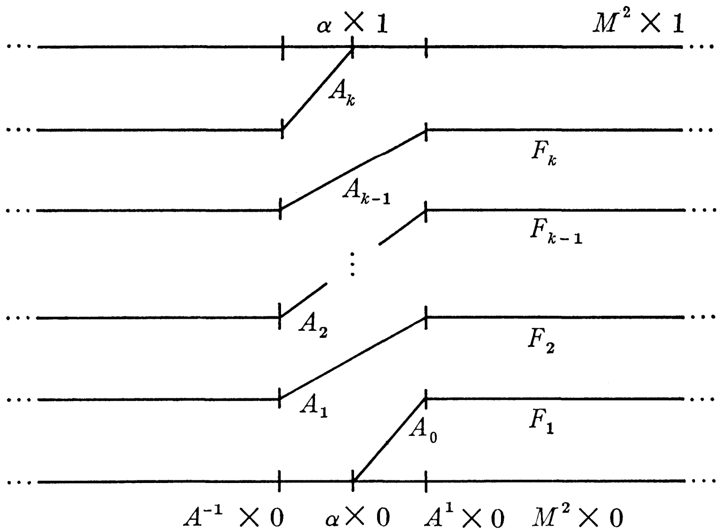

FIGURE 1

Then A is an annulus and if $\tilde{M}^2$ is the closure of $M^2 - A$, then $\tilde{M}^2 \cap A = J^{-1} \cup J^1$ where J^j is a simple closed curve for $j = -1$ or 1 and $J^{-1} \cap J^1 = \emptyset$. Also, $\chi(\tilde{M}^2) = \chi(M^2)$.

The construction of F in this situation is analogous to the construction of F above; only, here A is an annulus rather than a disk. The equation for $\chi(F)$ in this situation turns out to be

$$\chi(F) = k\chi(\tilde{M}^2) = k\chi(M^2).$$

COROLLARY 4.7. *If M^2 is a closed and orientable surface distinct from S^2, then for each integer $k \geqq 0$ there is an injective surface F in $M^2 \times S^1$ with $g(F) = k(g(M^2) - 1) + 1$.*

5. Necessary and sufficient conditions for incompressible surfaces. In this section the conclusions of § 4 are improved for the case that M^2 is a closed and orientable surface.

LEMMA 5.1. *Let G denote an orientable surface. Suppose that $\{G_1, \ldots, G_n\}$ is a mutually exclusive collection of incompressible surfaces in $G \times I$ such that*

(a) *for each $i = 1, \ldots, n$,* Bd $G_i \subset G \times 0 \cup G \times 1$, *and*

(b) *if K_i is a component of* Bd G_i, *K_j is a component of* Bd G_j, *and $p(K_i) \cap p(K_j) \neq \emptyset$, then $p(K_i) = p(K_j)$, where p is the natural projection of $G \times I$ onto G.*

Then there is an isotopy h_t, $0 \leqq t \leqq 1$, of $G \times I$ onto itself, h_t is fixed on Bd$(G \times I)$ *for each t and for $i = 1, \ldots, n$ either*

(i) *There is a non-trivial simple closed curve* $J_i \subset G$ *and* $h_1(G_i) = J_i \times I$

or

(ii) $p|h_1(G_i)$ *is a local homeomorphism of* $h_1(G_i)$ *into* G.

Proof. The proof of this lemma parallels the proof in [**16**, p. 65, proof of Proposition 3.1]. There are, however, three noteworthy observations.

The first observation is that the theorem is true for $G = S^2$ and in fact $p|h_1(G_i)$ is a homeomorphism of $h_1(G_i)$ onto G. The second observation is that the point set $\bigcup_i p(\text{Bd } G_i)$ is either void or a mutually exclusive collection of simple closed curves in G. This enables the considerations of Waldhausen in the case that G is a disk, annulus, or 2-sphere with three holes. Furthermore, in the general case, it enables a curve to be found in G so that the induction hypothesis of Waldhausen goes through.

The third observation is that in the situation of Lemma 5.1 the best possible result is that either h_1G_1 is vertical, i.e. $h_1(G_i) = p^{-1}p(h_1(G_i))$, or $p|h_1(G_i)$ is a local homeomorphism. This is due to $h_1(G_i)$ possibly having boundary on both $G \times 0$ and $G \times 1$.

THEOREM 5.2. *Let* M^2 *denote a closed, orientable surface. The surface* F *is incompressible in* $M^2 \times S^1$ *if and only if there is an isotopy* h_t, $0 \leqq t \leqq 1$, *of* $M^2 \times S^1$ *onto itself such that either*

(i) *there is a non-trivial simple closed curve* $J \subset M^2$ *and* $h_1(F) = J \times S^1$

or

(ii) $p|h_1(F)$ *is a covering projection of* $h_1(F)$ *onto* M^2, *where* p *is the natural projection of* $M^2 \times S^1$ *onto* M^2.

Proof. That conditions (i) and (ii) are sufficient for F to be incompressible (in fact, injective) in $M^2 \times S^1$ follows from Propositions 4.4 and 4.5. Hence, we shall show that conditions (i) and (ii) are also necessary.

Suppose that F is incompressible in $M^2 \times S^1$. Consider $M^2 \times S^1$ as the identification space $M^2 \times I/\eta$ obtained from $M^2 \times I$ by the homeomorphism $\eta: M^2 \to M^2$ so that $\eta(x) = x$ and η reverses orientation on M^2. Let $\rho: M^2 \times I \to M^2 \times S^1$ be the identification projection.

With an isotopy g_t, $0 \leqq t \leqq 1$, of $M^2 \times S^1$, make $g_1(F)$ in general position with $\rho(M^2 \times 0)$ and $g_1(F) \cap \rho(M^2 \times 0)$ minimal. Let $G_1, \ldots, G_n$ denote the components of $\rho^{-1}(g_1(F))$ in $M^2 \times I$. If $M^2 = S^2$, then $F = S^2$ and $\rho^{-1}(g_1(F))$ is a 2-sphere separating $M^2 \times 0$ from $M^2 \times 1$. If $M^2 \neq S^2$, then $F \neq S^2$ and no component of $\rho^{-1}(g_1(F))$ is the 2-sphere. Hence, in any case each component of $\rho^{-1}(g_1(F))$ is incompressible in $M^2 \times I$.

By Lemma 5.1, there is an isotopy h_t', $0 \leqq t \leqq 1$, of $M^2 \times I$ onto itself with h_t' fixed on $M^2 \times 0 \cup M^2 \times 1$ and either

(i) there is a non-trivial simple closed curve $J_i \subset M^2$ and $h_1'(G_i) = J_i \times I$

or

(ii) if p' is the projection of $M^2 \times I$ onto M^2, then $p'|h_1'(G_i)$ is a local homeomorphism of $h_1'(G_i)$ into M^2.

Since h_t' is fixed on $M^2 \times 0 \cup M^2 \times 1$, it induces an isotopy $\bar{h}_t'$, $0 \leqq t \leqq 1$, on $M^2 \times S^1$ so that the diagram

$$\begin{array}{ccc} M^2 \times I & \xrightarrow{h_t'} & M^2 \times I \\ \rho \downarrow & & \downarrow \rho \\ M^2 \times S^1 & \xrightarrow{\bar{h}_t'} & M^2 \times S^1 \end{array}$$

commutes for each t. Define

$$h_t = \begin{cases} g_{2t}, & 0 \leqq t \leqq 1/2, \\ \bar{h}'_{2t-1}, & 1/2 \leqq t \leqq 1. \end{cases}$$

It needs to be shown that h_t, $0 \leqq t \leqq 1$, satisfies the conclusions of Theorem 5.2.

Suppose that there is a non-trivial simple closed curve $J_i \subset M^2$ and $h_1'(G_i) = J_i \times I$. Then $\rho h_1'(G_i) = J_i \times S^1$ and thus $\rho(G_i) = h_1(F)$. It follows that $h_1(F) = J_i \times S^1$. Having made this observation, it may be assumed that for no i is $h_1'(G_i)$ vertical; i.e.

$$h_1'(G_i) = (p')^{-1}p'(h_1'(G_i)).$$

Case 1. $g(M^2) = 0$. Then $F = S^2$ and $\rho^{-1}(g_1(F)) = S^2$. Furthermore, $\rho^{-1}(G_1(F))$ separates $M^2 \times 0$ from $M^2 \times 1$ in $M^2 \times I$. It follows that $p|h_1(F)$ is actually a homeomorphism of $h_1(F)$ onto M^2.

Case 2. $g(M^2) = 1$. Then either $\rho^{-1}(g_1(F))$ is a torus and thus $p|h_1(F)$ is a homeomorphism onto M^2 [**16**, Corollary 3.2] or each component G_i of $\rho^{-1}(g_1(F))$ is an annulus having one component of Bd G_i in $M^2 \times 0$ and the other in $M^2 \times 1$. (In general, an incompressible surface in $M^2 \times I$ need not be orientable; however in this case G_i is an annulus. Again by [**16**, Corollary 3.2], if Bd G_i is contained in $M^2 \times 0$, then $g_1(F) \cap \rho(M^2 \times 0)$ is not minimal. Similarly if Bd G_i is contained in $M^2 \times 1$.)

If $p|h_1(F)$ is not a covering projection, then there is a simple closed curve $J \subset M^2$ such that $J \times 0$ is a component of Bd G_i for some i and $J \times 1$ is a component of Bd G_j for some j (j may be equal to i) and p fails to be a local homeomorphism at each point of $\rho(J \times 0)$.

Let $U(J)$ denote a regular neighbourhood of J in M^2 such that for each component of $h_1'(G_i) \cap (U(J) \times I)$ and each component of

$$h_1'(G_j) \cap (U(J) \times I)$$

the projection onto M^2 is a homeomorphism. The simple closed curve J separates $U(J)$ into two components. Denote the closures of these as $U^+(J)$ and $U^-(J)$. It follows that both the component of $h_1'(G_i) \cap (U(J) \times I)$ containing $J \times 0$ and the component of $h_1'(G_j) \cap (U(J) \times I)$ containing $J \times 1$ are contained in (say) $U^+(J) \times I$.

Suppose that $i = j$. Each component common to $h_1'(G_i)$ and the closure of $M^2 \times I - (U^+(J) \times I)$ is an annulus with a boundary component on each component of $\mathrm{Bd}(U^+(J) \times I)$. An analysis of the way that the boundary of these components would have to be spanned in $U^+(J) \times I$ shows that this situation cannot happen.

Suppose that $i \neq j$. Then an analysis like that above for $i = j$ shows that the projection of neither $h_1'(G_i)$ nor $h_1'(G_j)$ is onto M^2. By the way that $h_1'(G_i)$ and $h_1'(G_j)$ meet $U^+(J) \times I$, it follows that either the projection of $h_1'(G_i)$ is contained in the projection of $h_1'(G_j)$ or vice versa. Suppose that the projection of $h_1'(G_i)$ is contained in the projection of $h_1'(G_j)$. Then an analysis shows that either $h_1'(G_j)$ cannot have boundary on $M^2 \times 0$ or the projection of $h_1'(G_j)$ into M^2 is not a local homeomorphism. Both of these conclusions give rise to a contradiction.

Case 3. $g(M^2) > 1$. In this case either $\rho^{-1}(g_1(F))$ is a closed surface with genus equal to $g(M^2)$ and $p|h_1(F)$ is a homeomorphism onto M^2 or each component G_i of $\rho^{-1}(g_1(F))$ has boundary and $\mathrm{Bd}\, G_i$ meets both $M^2 \times 0$ and $M^2 \times 1$.

If $p|h_1(F)$ is not a covering projection, then there is a simple closed curve $J \subset M^2$ and components G_i and G_j as in Case 2. The component C of $p'(h_1'(G_i)) \cap p'(h_1'(G_j))$ containing J is a surface in M^2. To see this there are three considerations to make. If $x \in C$ and x is in

$$p'(\mathrm{Int}\, h_1'(G_i)) \cap p'(\mathrm{Int}\, h_1'(G_j)),$$

then $x \in \mathrm{Int}\, C$. If $x \in C$ and x is in either

$$p'(\mathrm{Int}\, h_1'(G_i)) \cap p'(\mathrm{Bd}\, h_1'(G_j))$$

or

$$p'(\mathrm{Bd}\, h_1'(G_i)) \cap p'(\mathrm{Int}\, h_1'(G_j))$$

but does not satisfy the first consideration, then $x \in \mathrm{Bd}\, C$. If $x \in C$ and x is in

$$p'(\mathrm{Bd}\, h_1'(G_i)) \cap p'(\mathrm{Bd}\, h_1'(G_j))$$

and x does not satisfy either the first or second consideration, then $x \in \mathrm{Bd}\, C$. Notice in the last situation that $x \in J'$, a simple closed curve in M^2 and $J' \times 0$ along with $J' \times 1$ are boundary components of $h_1'(G_{i'})$ and $h_1'(G_{j'})$, respectively, where $p|\rho(J' \times 0)$ is not a local homeomorphism when considered as a map of $h_1(F)$ into M^2.

Let C' denote the component of C containing $J \times 0$ and complementary to

$$p'(\mathrm{Bd}\, h_1'(G_i) - J \times 0) \cup p'(\mathrm{Bd}\, h_1'(G_j) - J \times 1).$$

Since $g(M^2) > 1$, there is a $J \subset M^2$ and G_i, G_j as before such that C' is not an annulus. Hence there is a non-trivial simple closed curve l in C' based on J and l is not homotopic to J in C'.

Since p' is a local homeomorphism on each component of $h_1'\rho^{-1}(g_1(F))$ and misses

$$p'(\text{Bd}\, h_1'(G_i)) \cup p'(\text{Bd}\, h_1'(G_j)),$$

except for J, the simple closed curve l lifts to a loop l_0 in $h_1'(G_i)$ based at $J \times 0$ and a loop l_1 in $h_1'(G_j)$ based at $J \times 1$.

Consider the loop $\rho(l_0 l_1^{-1})$ in $h_1(F)$. The loop $\rho(l_0 l_1^{-1})$ is trivial in $M^2 \times S^1$ since $\rho(l_0) \sim \rho(l) \sim \rho(l_1)$ in $M^2 \times S^1$. There is a simple closed curve homotopic to $\rho(l_0 l_1^{-1})$ in $h_1(F)$ which bounds a disk D in $M^2 \times S^1$, where $D \cap h_1(F) = \text{Bd}\, D$. Since $h_1(F)$ is incompressible in $M^2 \times S^1$, the loop $\rho(l_0 l_1^{-1})$ is trivial in $h_1(F)$. By choosing l neither trivial in C' nor homotopic to J in C', this leads to a contradiction. The projection p of the contraction $\rho(l_0 l_1^{-1})$ in $h_1(F)$ gives rise to either a contraction of l in C' or a homotopy of l and J in C'.

The proof of Theorem 5.2 will be complete if whenever $h_1(F) \neq J \times S^1$ for some J, then $p|h_1(F)$ is *onto* M^2. However, it has been shown that in this case $p|h_1(F)$ is indeed a local homeomorphism. Thus by invariance of domain for manifolds [**5**], the image of the projection $p|h_1(F)$ is both open and closed in M^2. It follows that $p|h_1(F)$ is *onto* M^2.

COROLLARY 5.3. *Let M^2 denote a closed and orientable surface. The closed surface F is injective in $M^2 \times S^1$ if and only if F is incompressible in $M^2 \times S^1$.*

The next two corollaries have also been obtained by Bredon and Wood [**1**] using different techniques.

COROLLARY 5.4. *Let M^2 denote a closed and orientable surface different from S^2. The closed, non-orientable surface F can be embedded in $M^2 \times S^1$ if and only if $\chi(F)$ is even and F is not the Klein bottle.*

Proof. It is easy to see how to embed non-orientable surfaces with even, non-zero, Euler characteristic in $M^2 \times S^1$. Namely, the surface $M^2 \neq S^2$ has a non-separating simple closed curve J. Any simple closed curve meeting $J \times S^1$ in a single "piercing point" will guide a non-orientable handle for attachment on $J \times S^1$. Such an operation lowers the Euler characteristic by two.

For F a non-orientable surface, let $\bar{g}(F)$ denote the maximal number of two-sided simple closed curves in F the union of which does not separate F. If $\bar{g}(F) = n$, then $\chi(F) = 2 - 2n$ or $1 - 2n$.

If F is non-orientable and F can be embedded in $M^2 \times S^1$, then F is not incompressible in $M^2 \times S^1$. We shall show that if F is non-orientable and $F \subset M^2 \times S^1$, then $\bar{g}(F) \neq 0$ or 1.

If $\bar{g}(F) = 0$, then $F = P^2$. But each embedding of P^2 in a 3-manifold must be incompressible. If $\bar{g}(F) = 1$, then F is either the Klein bottle or a non-orientable surface with $\chi(F) = -1$. If F were the Klein bottle, then F admits an elementary surgery along some disk D in $M^2 \times S^1$. Since $M^2 \times S^1$ is irreducible ($M^2 \neq S^2$), the result of such a surgery would lead to an embedding of the solid Klein bottle in $M^2 \times S^1$. This would contradict $M^2 \times S^1$ being orientable. If $\chi(F) = -1$, then F admits an elementary surgery along a disk D in $M^2 \times S^1$. The result of such a surgery would lead to an embedding of P^2 in $M^2 \times S^1$. Hence, again we arrive at a contradiction.

The proof will proceed by an induction on $\bar{g}(F)$; namely, if $\bar{g}(F) = k$, $k \geqq 2$, and F can be embedded in $M^2 \times S^1$, then $\chi(F) = 2 - 2k$.

If $\bar{g}(F) = 2$, then $\chi(F) \neq -3$. If this were true, then by an elementary surgery on F along a disk D in $M^2 \times S^1$, there would result a closed surface F', where $\chi(F') = 1$ or -1. We have seen that this cannot happen.

If $\bar{g}(F) = k + 1$, then by an elementary surgery on F along a disk D in $M^2 \times S^1$, there would result either one closed surface F' with $\bar{g}(F') \leqq k$ or two closed surfaces F_1 and F_2 with $\bar{g}(F_i) \leqq k$, $i = 1, 2$. In the former, $\chi(F')$ is even and hence, $\chi(F)$ is even. In the latter, $\chi(F_i)$ is even; hence, $\chi(F)$ is even.

COROLLARY 5.5. *The closed non-orientable surface F can be embedded in $S^2 \times S^1$ if and only if $\chi(F)$ is even.*

Proof. This proof is analogous to the proof of Corollary 5.4. However, since $S^2 \times S^1$ is not irreducible, it admits an embedding of the Klein bottle. Such an embedding can be obtained from a non-separating 2-sphere S in $S^2 \times S^1$ by adding a non-orientable handle guided by a simple closed curve "piercing" S at precisely one point.

6. Non-unique fiberings over S^1.

THEOREM 6.1. *Let F denote an incompressible, two-sided surface in $M^2 \times S^1$ where $\chi(F) < 0$. Then there is a retraction r of $M^2 \times S^1$ onto a simple closed curve J in $M^2 \times S^1$ and*

$$\ker(r_*: \pi_1(M^2 \times S^1) \to Z)$$

is $\pi_1(F)$.

Proof. It follows from Proposition 3.5 that F does not separate $M^2 \times S^1$. Hence, there is a simple closed curve $J \subset M^2 \times S^1$ and J meets F in a single point $q \in F$. Furthermore, locally about q the simple closed curve J is in different sides of F. Let $U(F)$ denote a regular neighbourhood of F in $M^2 \times S^1$ meeting J in a subarc A of J, where $q \in A$.

The Tietze Extension Theorem now yields a retraction of $U(F)$ onto A. This retraction may be extended to a retraction r of $M^2 \times S^1$ onto J by again applying the Tietze Extension Theorem to retract the closure of $M^2 \times S^1 - U(F)$ onto the closure of $J - A$ in J (see [**7**] for similar techniques of building retractions).

The infinite cyclic covering space corresponding to the non-separating surface F and constructed in the fashion of Neuwirth [**10**] has as its fundamental group $\ker(r_*)$. Since $\chi(F) < 0$, the group $\pi_1(F)$ does not have centre [**3**; **6**]. Since $\pi_1(M^2 \times S^1)$ has an infinite cyclic subgroup in its centre, an argument like that in [**15**, the proof of Lemma 4.4] shows that $\pi_1(F) \approx \ker(r_*)$.

THEOREM 6.2. *Let M^2 denote a surface where $\chi(M^2) \leqq 0$. Then for any integer $k > 0$, $M^2 \times S^1$ can be fibred over S^1 with fibre a surface F and $\chi(F) = k\chi(M^2)$.*

Proof. Case 1. $\chi(M^2) = 0$. Then $F = M^2$ satisfies the theorem.

Case 2. $\chi(M^2) < 0$. By Proposition 4.6, there is a two-sided surface F which is injective in $M^2 \times S^1$ and $\chi(F) = k\chi(M^2)$. By Theorem 6.1, there is a retraction r of $M^2 \times S^1$ onto a simple closed curve J so that the sequence

$$1 \rightarrow \pi_1(F) \xrightarrow{i_*} \pi_1(M^2 \times S^1) \xrightarrow{r_*} \pi_1(J) \rightarrow 1$$

is exact, where i_* is induced by inclusion. It now follows by [**14**] and the fact that $M^2 \times S^1$ is irreducible that $M^2 \times S^1$ can be fibred over S^1 with fibre the surface F. This completes the proof of the theorem.

COROLLARY 6.3. *If M^2 is a closed, orientable surface distinct from S^2, then $M^2 \times S^1$ admits a fibration over S^1 with fibre F a closed, orientable surface and $g(F) = k(g(M^2) - 1) + 1$, where $k > 0$.*

It is now clear that a result similar to Proposition 4.2 for $M^2 \times S^1$ is not true for 3-manifolds which are non-trivial fibrations over S^1 with fibre a surface F; that is, we have the following.

COROLLARY 6.4. *If M is fibred over S^1 with fibre a surface F, then for F' injective in M it is not necessarily true that $\chi(F') \leqq \chi(F)$.*

Proof. Let F' be a closed orientable surface with $g(F') = 2$. Then $F' \times S^1$ can be fibred over S^1 with fibre a surface F where $g(F) = k \geqq 2$. If $k > 2$, then $\chi(F') \nleqq \chi(F)$; yet, F' is injective in $F' \times S^1$.

Let f and g denote embeddings of the space X into the space Y. If there is a homeomorphism h of Y onto itself such that $hf = g$, then f and g are said to be *equivalent.*

COROLLARY 6.5. *There is a 3-manifold M fibred over S^1 with fibre a surface F, where $\chi(F) < 0$ and a non-separating embedding $f: F \rightarrow M$ such that $f(F)$ is not equivalent to any injective embedding of F into M.*

Proof. Let $M = F' \times S^1$, where $g(F') = 2$. Let F denote a surface in $F' \times S^1$ so that $g(F) = 3$ and M can be fibred over S^1 with fibre the surface F.

If $f(F)$ is the embedding of F in M obtained by adding a small handle to F' in M, then $f(F)$ is not equivalent to an injective surface in M; in particular, $f(F)$ is not equivalent to F.

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