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A CHARACTERIZATION OF THE WEAK RADON–NIKODÝM PROPERTY BY FINITELY ADDITIVE INTERVAL FUNCTIONS

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Abstract

A characterization of Banach spaces possessing the weak Radon–Nikodým property is given in terms of finitely additive interval functions. Due to that characterization several Banach space valued set functions that are only finitely additive can be represented as integrals.

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1. Introduction

It has been proven in [4] that a Banach space X has the weak Radon–Nikodým property (WRNP) if and only if for every measure $\nu\colon \mathcal{L} \to X$ (where $\mathcal{L}$ is the σ-algebra of all Lebesgue measurable subsets of $[0, 1]$) of σ-finite variation, which is absolutely continuous with respect to the Lebesgue measure, there exists a Pettis integrable function $f\colon [0, 1] \to X$ such that

$$\nu(E) = \int_E f(t)\, dt \quad \text{for every set } E \in \mathcal{L}. \tag{1.1}$$

There are other characterizations of Banach spaces possessing the WRNP (see [5, 6, 8]). Those based on measure theory always involve countably additive Banach space valued measures.

In this paper we prove that a Banach space X has the WRNP if and only if for every finitely additive X-valued set function ν defined on intervals that has absolutely continuous variational measure (see Definition 3.1) there is a Henstock–Kurzweil–Pettis integrable function $f\colon [0, 1] \to X$ such that (1.1) holds true for every interval $E \subset [0, 1]$ (but now the integral is the Henstock–Kurzweil–Pettis integral).

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We think that it is quite surprising that the WRNP can be completely described in terms of finitely additive interval functions.

We also obtain other interesting characterizations of Banach spaces possessing the WRNP, in terms of the differentiation of functions whose variational measure is absolutely continuous with respect to the Lebesgue measure (Theorem 4.5). This form is similar to the classical characterization of the Radon–Nikodým property (RNP) via differentiation of absolutely continuous functions (see [1, p. 217]) but now, as the WRNP is not separably determined, it is based on the notion of pseudo-differentiation (see Definition 2.3).

2. Preliminaries

Let $[0, 1]$ be the unit interval of the real line $\mathbb{R}$ equipped with the usual topology and the Lebesgue measure λ. We denote by $\mathcal{I}$ the family of all nontrivial closed subintervals of $[0, 1]$ and by $\mathcal{L}$ the family of all Lebesgue measurable subsets of $[0, 1]$.

If $E \subset [0, 1]$, we denote by $|E|_e$ its outer Lebesgue measure and by $|E|$ its Lebesgue measure, in the case where $E \in \mathcal{L}$. Throughout this paper X is a Banach space with dual X^*. The closed unit ball of X^* is denoted by $B(X^*)$. If μ is an outer measure on $[0, 1]$, then by $\mu \ll \lambda$ we mean that $|E| = 0$ implies $\mu(E) = 0$. A mapping $\nu : \mathcal{L} \to X$ is said to be an X-*valued measure* if ν is countable additive in the norm topology of X. The mapping ν is said to be λ-*continuous* if $|E| = 0$ implies $\nu(E) = 0$. The *variation* of an X-valued measure ν is denoted by $|\nu|$. A function $f : [0, 1] \to X$ is said to be *scalarly measurable* if for all $x^* \in X^*$ the real function $x^* f$ is measurable.

In the sequel the symbol $\int f \, d\lambda$ denotes the Lebesgue integral of f, if f is a scalar function.

A *tagged partition in* $[0, 1]$, or simply a *partition in* $[0, 1]$ is a finite collection of pairs $\mathcal{P} = \{(I_1, t_1), \ldots, (I_p, t_p)\}$, where $I_1, \ldots, I_p$ are nonoverlapping subintervals of $[0, 1]$ and $t_i \in I_i$, $i = 1, \ldots, p$. Given a subset E of $[0, 1]$, we say that the partition $\mathcal{P}$ is *anchored on* E if $t_i \in E$ for all $i = 1, \ldots, p$. If $\bigcup_{i=1}^{p} I_i = [0, 1]$ we say that $\mathcal{P}$ is a *partition of* $[0, 1]$. A *gauge* on $E \subset [0, 1]$ is a positive function on E. For a given gauge δ, we say that a partition $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ is δ-*fine* if $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, $i = 1, \ldots, p$.

DEFINITION 2.1. A function $f : [0, 1] \to \mathbb{R}$ is said to be *Henstock–Kurzweil integrable*, or simply HK-*integrable*, on $[0, 1]$, if there exists $w \in \mathbb{R}$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on $[0, 1]$ such that

$$\left| \sum_{i=1}^{p} f(t_i)|I_i| - w \right| < \varepsilon,$$

for all δ-fine partitions $\mathcal{P} = \{(I_1, t_1), \ldots, (I_p, t_p)\}$ of $[0, 1]$.

We set $w := (\mathrm{HK}) \int_0^1 f \, d\lambda$.

It is known that if $f : [0, 1] \to \mathbb{R}$ is HK-integrable on $[0, 1]$ and $I \in \mathcal{I}$, then $f \chi_I$ is also HK-integrable on $[0, 1]$. We say in such a case that f is HK-integrable on I. We call the additive interval function $F(I) := (\mathrm{HK}) \int_I f \, d\lambda$ the HK-*primitive* of f.

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DEFINITION 2.2. A function $f: [0, 1] \to X$ is said to be *scalarly Henstock–Kurzweil integrable* if for all $x^* \in X^*$ the function $x^* f$ is Henstock–Kurzweil integrable. A scalarly Henstock–Kurzweil integrable function f is said to be *Henstock–Kurzweil–Pettis integrable* (or simply HKP-*integrable*) if for all $I \in \mathcal{I}$ there exists $w_I \in X$ such that

$$\langle x^*, w_I \rangle = (\mathrm{HK}) \int_I x^* f \, d\lambda \quad \text{for every } x^* \in X^*.$$

We call w_I the *Henstock–Kurzweil–Pettis integral* of f over I and we write $w_I := (\mathrm{HKP}) \int_I f d\lambda$. If $I = [a, b]$, then we write $(\mathrm{HKP}) \int_a^b f d\lambda$ instead of $(\mathrm{HKP}) \int_{[a,b]} f \, d\lambda$. We call the additive interval function $F(I) := (\mathrm{HKP}) \int_I f \, d\lambda$ the HKP-*primitive* of f.

We denote by HKP([0, 1], X) the set of all X-valued Henstock–Kurzweil–Pettis integrable functions on [0, 1] (functions that are scalarly equivalent are identified). More information on HKP-integrable functions can be found in [3].

It is known that the HK-primitive (HKP-primitive) F of a function f is continuous (weakly continuous, that is, $x^* F$ is continuous for every $x^* \in X^*$).

In the following, the symbol $\langle a, b \rangle$ stands for $[\min\{a, b\}, \max\{a, b\}]$.

DEFINITION 2.3. Let $F : [0, 1] \to X$ be a function and G be a nonempty subset of [0, 1]. If there is a function $F'_p : G \to X$ such that for all $x^* \in X^*$,

$$\lim_{h \to 0} \frac{x^*(F\langle t, t+h \rangle)}{|h|} = x^*(F'_p(t)),$$

for almost all $t \in G$, then F is said to be *pseudo-differentiable on* G (the exceptional sets depend on x^*), with a *pseudo-derivative* F'_p.

3. Variational measures

Throughout this paper the letter Φ will denote an arbitrary additive interval function $\Phi : \mathcal{I} \to X$. Even if it is not explicitly written, we identify an interval function Φ with the point function $\Phi(t) = \Phi([0, t])$, $t \in [0, 1]$; and conversely we identify a point function $\Phi: [0, 1] \to X$ with the interval function $\Phi([a, b]) = \Phi(b) - \Phi(a)$, $[a, b] \in \mathcal{I}$.

DEFINITION 3.1. Given $\Phi : \mathcal{I} \to X$, a gauge δ and a set $E \subset [0, 1]$, we define

$$\mathrm{Var}(\Phi, \delta, E) := \sup \left\{ \sum_{i=1}^{p} \|\Phi(I_i)\| : \begin{array}{c} \{(I_i, t_i) : i = 1, \ldots, p\} \ \delta\text{-fine} \\ \text{partition anchored on } E \end{array} \right\}.$$

Then we set

$$V_\Phi(E) := \inf\{\mathrm{Var}(\Phi, \delta, E) : \delta \text{ a gauge on } E\}.$$

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We call V_Φ the *variational measure generated by* Φ. It is known that V_Φ is a metric outer measure on [0, 1] (see [9]). In particular, V_Φ is a measure over all Borel sets of [0, 1].

If Φ is continuous, then $V_\Phi(I) \le |\Phi|(I)$ for every $I \in \mathcal{I}$, where

$$|\Phi|(I) := \sup\left\{\sum_i \|\Phi(I_i)\| : \ I_i \text{ are nonoverlapping subintervals of } I\right\}.$$

DEFINITION 3.2. We say that the variational measure V_Φ is σ*-finite* if there is a sequence of (pairwise disjoint) sets F_n covering [0, 1] and such that $V_\Phi(F_n) < \infty$, for every $n \in \mathbb{N}$.

Thomson (see [9, Theorem 3.15]) proved that V_Φ has the so-called measurable cover property; that is, if $A \subset [0, 1]$, then there exists $B \in \mathcal{L}$ such that $B \supset A$ and $V_\Phi(B) = V_\Phi(A)$. It follows from this that the sets F_n in the previous definition can be taken from $\mathcal{L}$.

PROPOSITION 3.3. *If $V_\Phi \ll \lambda$, then Φ is continuous on* [0, 1] *and V_Φ is σ-finite.*

PROOF. Let t_0 be a point in [0, 1]. As $V_\Phi(\{t_0\}) = 0$, for every $\varepsilon > 0$ there is a gauge δ such that for all δ-fine intervals I containing t_0 we have $\|\Phi(I)\| < \varepsilon$ and so Φ is continuous in t_0.

By repeating, with obvious changes, the proof of [2, Theorem 1] we infer that V_Φ is σ-finite. □

We recall that a function $\Phi : [0, 1] \to X$ is said to be BV_* on a set $E \subseteq [0, 1]$ if $\sup \sum_{i=1}^{n} \omega(\Phi(J_i)) < +\infty$, where the supremum is taken over all finite collections $\{J_1, \ldots, J_n\}$ of nonoverlapping intervals from $\mathcal{I}$ with end-points in E, and the symbol $\omega(\Phi(J))$ stands for $\sup\{\|\Phi(u) - \Phi(z)\| : u, \ z \in J\}$. The function Φ is said to be BVG_* on [0, 1] if $[0, 1] = \bigcup_n E_n$ and Φ is BV_* on each E_n.

PROPOSITION 3.4. *V_Φ is σ-finite if and only if Φ is BVG_* on* [0, 1].

In case of a continuous real-valued function Φ, this proposition has been proven in [9, Theorem 7.8]. For the sake of completeness, at the end of this paper, we give a new and direct proof for the vector-valued case.

4. The main result

To prove our main result we need a few lemmas.

LEMMA 4.1. *Let X be a Banach space and let $\nu : \mathcal{L} \to X$ be a λ-continuous measure of finite variation. If $\Phi : \mathcal{I} \to X$ is defined by $\Phi(I) := \nu(I)$, for each $I \in \mathcal{I}$, then V_Φ is finite, $V_\Phi \ll \lambda$ and $V_\Phi(E) \le |\nu|(E)$, whenever $E \in \mathcal{L}$.*

PROOF. Since ν is an X-valued measure and λ is finite, the λ-continuity of ν implies that $\lim_{\lambda(A)\to 0}\|\nu(A)\| = 0$. It follows that the function Φ is continuous. The σ-additivity of $|\nu|$ yields $V_\Phi(I) \le |\nu|(I)$, for $I \in \mathcal{I}$, and then $V_\Phi(U) \le |\nu|(U)$, if U is open. Now let $E \in \mathcal{L}$ and let $U \supseteq E$ be open; then $V_\Phi(E) \le V_\Phi(U) \le |\nu|(U)$. By the outer regularity of $|\nu|$, then, $V_\Phi(E) \le |\nu|(E)$. This easily implies that $V_\Phi(X) < \infty$. The relation $V_\Phi \ll \lambda$ is obvious. □

LEMMA 4.2. *If X has the WRNP and $h : [0, 1] \to X$ is a Lipschitz function, then h is pseudo-differentiable.*

PROOF. Without loss of generality we may assume that $h(0) = 0$. Define $\Phi : \mathcal{I} \to X$ by setting $\Phi[a, b] := h(b) - h(a)$. If M is the Lipschitz constant, then we have $\|\Phi(I)\| \le M|I|$ for every $I \in \mathcal{I}$. It easily follows from this that if $\widetilde{\Phi}$ is the unique additive extension of Φ to the algebra $\alpha(\mathcal{I})$ generated by $\mathcal{I}$, then $\|\widetilde{\Phi}(A)\| \le M|A|$, for all $A \in \alpha(\mathcal{I})$. Hence $\widetilde{\Phi}$ is absolutely continuous with respect to the Lebesgue measure. It follows that $\widetilde{\Phi}$ has the unique extension to an X-valued measure $\widehat{\Phi}$ on the σ-algebra of Borel subsets of $[0, 1]$ (see [1], Theorem I.5.2). Moreover, $\widehat{\Phi}$ satisfies the inequality $\|\widehat{\Phi}(B)\| \le M|B|$ for all $B \in \mathcal{B}[0, 1]$ and may be extended to $\mathcal{L}$. According to our assumption, there exists a Pettis integrable function $f : [0, 1] \to X$ such that

$$\langle x^*, \widehat{\Phi}(E)\rangle = \int_E x^* f \, d\lambda \quad \text{for every } E \in \mathcal{L} \text{ and every } x^* \in X^*.$$

In particular,

$$\langle x^*, h(t)\rangle = \int_0^t x^* f \, d\lambda \quad \text{for every } t \in [0, 1] \text{ and every } x^* \in X^*.$$

According to the Lebesgue differentiation theorem, for every $x^* \in X^*$, we have $(x^* h)' = x^* f$ (almost everywhere). Thus, h is pseudo-differentiable. □

DEFINITION 4.3. A function $f : [0, 1] \to X$ is said to be *Lipschitz at the point* $t \in [0, 1]$ if there exist two positive constants C and η such that

$$\|f(t + h) - f(t)\| \le C\,|h|,$$

for all $h \in \mathbb{R}$, with $|h| < \eta$.

LEMMA 4.4. *Assume that X has the* WRNP *and let $f : [0, 1] \to X$ be an arbitrary function. Denote by G the set of all points $t \in [0, 1]$ at which f is Lipschitz. Then f is pseudo-differentiable on G.*

PROOF. For each $n \in \mathbb{N}$, let G_n denote the set of all $t \in G$ such that

$$\|f(t + h) - f(t)\| \le n\,|h|, \qquad \text{whenever } |h| < \frac{1}{n}.$$

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Clearly $G = \cup G_n$ and it is easy to see that each G_n is a closed set. Without loss of generality we may assume that in each set G_n there are no isolated points.

Let f_n be the extension of $f|_{G_n}$ to $[0, 1]$ such that f_n is linear on each contiguous interval of G_n. It is easy to prove that f_n is a Lipschitz function on $[0, 1]$. By Lemma 4.2, f_n is pseudo-differentiable with pseudo-derivative g_n. For $t \in G$ we define $g(t) = g_n(t)$, where n is the first natural such that $t \in G_n$, and we define $g(t) = 0$ if $t \in [0, 1] \setminus G$.

We now fix $x^* \in X^*$. For each $n \in \mathbb{N}$ let $\Gamma_n^{x^*} \subset [0, 1]$ be the set of all points $t \in [0, 1]$ such that $(x^* f_n)'(t) = x^* g_n(t)$. Then $|[0, 1] \setminus \Gamma_n^{x^*}| = 0$.

Denote by $\tilde{G}_n$ the set of all points $t \in G_n$ at which the distance function $\mathrm{dist}(t, G_n)$ is differentiable. Since $\mathrm{dist}(t, G_n)$ is Lipschitz, it follows that $|G_n \setminus \tilde{G}_n| = 0$. Hence $|G_n \setminus (\Gamma_n^{x^*} \cap \tilde{G}_n)| = 0$.

Define $N^{x^*} = \bigcup_n (G_n \setminus (\Gamma_n^{x^*} \cap \tilde{G}_n))$. Then $|N^{x^*}| = 0$. To complete the proof it is enough to show that $(x^* f)'(t) = x^* g(t)$ for all $t \in G \setminus N^{x^*}$.

For $t \in G \setminus N^{x^*}$ let n be the first natural such that $t \in \Gamma_n^{x^*} \cap \tilde{G}_n$.

Take an arbitrary $\varepsilon > 0$. Since $(x^* f_n)'(t) = x^* g(t)$ and $\mathrm{dist}(t, G_n)(t) = 0$, there exists $\delta_\varepsilon \in (0, \frac{1}{n})$ such that

$$\left| \frac{x^* f_n(t+h) - x^* f_n(t)}{h} - x^* g(t) \right| < \frac{\varepsilon}{2} \tag{4.1}$$

and

$$\mathrm{dist}(t+h, G_n) < \frac{\varepsilon |h|}{2(n + \|g(t)\|)},$$

for all $0 < |h| < \delta_\varepsilon$.

Then, for any fixed $0 < |h| < \delta_\varepsilon$, we can find $\bar{t} \in G_n$ such that $|t - \bar{t}| < h$ and

$$|t + h - \bar{t}| < \frac{\varepsilon |h|}{2(n + \|g(t)\|)}. \tag{4.2}$$

Now $f_n = f$ on G_n. Therefore, by (4.1) and (4.2),

$$\begin{aligned} &|x^* f(t+h) - x^* f(t) - x^* g(t)\, h| \\ &\quad \le |x^* f(\bar{t}) - x^* f(t) - x^* g(t)(\bar{t} - t)| + |x^* f(\bar{t}) - x^* f(t+h)| \\ &\qquad + |x^* g(t)|\, |t + h - \bar{t}| \\ &\quad \le \frac{\varepsilon}{2} |\bar{t} - t| + n\|x^*\|\, |t + h - \bar{t}| + |x^* g(t)|\, |t + h - \bar{t}| < \varepsilon |h|. \end{aligned}$$

This completes the proof. □

The following characterization of the WRNP is the main result of this paper.

THEOREM 4.5. *Let X be a Banach space. Then the following conditions are equivalent:*

(i) *X has the weak Radon–Nikodým property;*

(ii) *if $\Phi : \mathcal{I} \to X$ is BV_* on $[0, 1]$, then Φ is pseudo-differentiable on $[0, 1]$;*

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(iii) *if* $\Phi : \mathcal{I} \to X$ *is* BVG_* *on* $[0, 1]$, *then* Φ *is pseudo-differentiable on* $[0, 1]$;
(iv) *if* V_Φ *is* σ*-finite, then* Φ *is pseudo-differentiable on* $[0, 1]$;
(v) *if* $V_\Phi \ll \lambda$, *then* Φ *is pseudo-differentiable on* $[0, 1]$;
(vi) *if* $V_\Phi \ll \lambda$, *then* Φ *is pseudo-differentiable on* $[0, 1]$, $\Phi'_p \in \mathrm{HKP}([0, 1], X)$, *and*

$$\Phi(I) = (\mathrm{HKP}) \int_I \Phi'_p \, d\lambda \quad \text{for every } I \in \mathcal{I};$$

(vii) *if* $V_\Phi \ll \lambda$, *then there exists* $f \in \mathrm{HKP}([0, 1], X)$ *such that*

$$\Phi(I) = (\mathrm{HKP}) \int_I f \, d\lambda \quad \text{for every } I \in \mathcal{I}.$$

PROOF. (i) $\Rightarrow$ (ii) Define $f(t) = \Phi([0, t])$ for $t \in [0, 1]$, and set

$$G_n := \{t \in [0, 1] : \|f(u) - f(t)\| < n|u - t|, \ \forall\, u \in [0, 1] \text{ such that } |u - t| < 1/n\},$$

and $E = [0, 1] \setminus \bigcup_n G_n$. By Lemma 4.4 it is enough to prove that $|E| = 0$.

Assume, by way of contradiction, that $|E|_e > 0$. By definition of E, for each $t \in E$ and each $n \in \mathbb{N}$ there exists $u \in [0, 1]$ such that $|u - t| < 1/n$ and $\|f(u) - f(t)\| > n|u - t|$. Given $M > 0$, take $n \in \mathbb{N}$ such that $n|E|_e > 2M$ and let

$$\mathcal{F} := \{[t, u] : t \in E, \ |u - t| < 1/n, \ \|f(t) - f(u)\| > n|u - t|\}.$$

It is easy to check that $\mathcal{F}$ is a Vitali covering of E. Then there exist a finite number of disjoint intervals $\{[t_i, u_i]\}$ such that $[t_i, u_i] \in \mathcal{F}$ and $|E|_e < 2\sum_i |u_i - t_i|$. Consequently,

$$\sum_i \|\Phi[t_i, u_i]\| = \sum_i \|f(u_i) - f(t_i)\| > n \sum_i |u_i - t_i| > n\frac{|E|_e}{2} > M.$$

By the arbitrariness of M, this implies that Φ is not BV_*, which contradicts the hypothesis.

(ii) $\Rightarrow$ (iii) Let $[0, 1] = \bigcup_n E_n$ be a decomposition of $[0, 1]$ such that Φ is BV_* on each E_n. Then define $f(t) = \Phi([0, t])$ for $t \in [0, 1]$, and set

$$G_n := \{t \in [0, 1] : \|f(u) - f(t)\| < n|u - t|, \ \forall u \in [0, 1] \text{ such that } |u - t| < 1/n\},$$

and $E = [0, 1] \setminus \bigcup_n G_n$. By Lemma 4.4 it is enough to prove that $|E| = 0$.

Assume, by way of contradiction, that $|E|_e > 0$. Then there exists $k \in \mathbb{N}$ such that $|E_k \cap E|_e > 0$. In the same way as above, one can prove that Φ is not BV_* on E_k, which contradicts the hypothesis.

(iii) $\Rightarrow$ (iv) If V_Φ is σ-finite, then Φ is BVG_* (by Proposition 3.4) and so the result is a consequence of (iii).

(iv) $\Rightarrow$ (v) Assume that $V_\Phi \ll \lambda$. According to Proposition 3.3, V_Φ is σ-finite. Then condition (iv) implies the required pseudo-differentiability of Φ.

(v) $\Rightarrow$ (vi) Assume that $V_\Phi \ll \lambda$ and let Φ'_p be the pseudo-derivative of Φ. We will prove that Φ'_p is HKP-integrable with HKP-primitive Φ. Now we fix $x^* \in X^*$. Let N^{x^*} be the set of all $t \in [0, 1]$ such that $(x^*\Phi)'(t)$ does not exist. By hypothesis, $|N^{x^*}| = 0$ and $V_\Phi(N^{x^*}) = 0$. Therefore also $V_{x^*\Phi}(N^{x^*}) = 0$. Now define $f^{x^*} : [0, 1] \to \mathbb{R}$ as follows:

$$f^{x^*}(t) = \begin{cases} x^*\Phi'_p(t) & \text{if } (x^*\Phi)'(t) \text{ exists} \\ 0 & \text{otherwise.} \end{cases}$$

Fix $\varepsilon > 0$. If $t \in [0, 1] \setminus N^{x^*}$, define $\delta(t)$ such that

$$|x^*\Phi(I) - x^*\Phi'_p(t)\,|I|\,| < \varepsilon\,|I|, \tag{4.3}$$

for each interval $I \in \mathcal{I}$ having t as one of its end-points and such that $|I| < \delta(t)$. If $t \in N^{x^*}$, then taking into account the equality $V_{x^*\Phi}(N^{x^*}) = 0$, define $\delta(t)$ such that

$$\sum_{j=1}^{s} |x^*\Phi(J_j)| < \varepsilon, \tag{4.4}$$

for all δ-fine partitions $\{(J_j, t_j) : j = 1, \ldots, s\}$ anchored in N^{x^*}.

Now let $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ be a δ-fine partition of $[0, 1]$. We may assume that the tags t_i of the partition are end-points of the corresponding intervals I_i. Therefore, by (4.3) and (4.4),

$$\left|\sum_{i=1}^{p} f^{x^*}(t_i)|I_i| - x^*\Phi(I_i)\right| \le \sum_{t_i \in N^{x^*}} |x^*\Phi(I_i)| + \sum_{t_i \in [0,1]\setminus N^{x^*}} |f^{x^*}(t_i)|I_i| - x^*\Phi(I_i)| < 2\varepsilon.$$

Therefore f^{x^*} is HK-integrable and Φ'_p is HKP-integrable with HKP-primitive Φ. That gives (vi). Of course (vi) implies (vii).

(vii) $\Rightarrow$ (i) Assume that each additive function $\Phi : \mathcal{I} \to X$ such that $V_\Phi \ll \lambda$ is a HKP-primitive and let $\nu : \mathcal{L} \to X$ be a λ-continuous measure of finite variation. Define $\Phi : \mathcal{I} \to X$ by $\Phi(I) := \nu(I)$. It follows from Lemma 4.1 that $V_\Phi \ll \lambda$. Hence there is a Henstock–Kurtzweil–Pettis integrable function $f : [0, 1] \to X$ such that

$$\Phi(I) = (\text{HKP}) \int_I f \, d\lambda \quad \text{for every } I \in \mathcal{I}.$$

Consequently, for $x^* \in X^*$,

$$x^*\nu(I) = x^*\Phi(I) = (\text{HK}) \int_I x^* f \, d\lambda \quad \text{for every } I \in \mathcal{I}. \tag{4.5}$$

Thus, $V_{x^*\Phi} \ll \lambda$.

Moreover, since ν is countably additive and of finite variation, for every $x^* \in X^*$ the measure $x^*\nu$ is bounded and of finite variation. Therefore $V_{x^*\Phi}([0, 1]) \le |x^*\nu|([0, 1]) < \infty$. Then, by [7, Proposition 5], $x^* f \in L_1[0, 1]$ for all $x^* \in X^*$.

Let us fix $x^* \in B(X^*)$ and let $\alpha(\mathcal{I})$ be the algebra generated by all the intervals $(a, b] \subset [0, 1]$. Then it follows from (4.5) that

$$\int_A x^* f \, d\lambda = x^* \nu(A) \quad \text{for every } A \in \alpha(\mathcal{I}).$$

But both sides of the above equality are real set functions countably additive on the algebra $\alpha(\mathcal{I})$ and so they can be uniquely extended to a measure on the family $\mathcal{B}[0, 1]$ of all Borel subsets of $[0, 1]$. This means that

$$\int_E x^* f \, d\lambda = x^* \nu(E) \quad \text{for every } E \in \mathcal{B}[0, 1]. \tag{4.6}$$

Since both sides of (4.6) have unique extensions to $\mathcal{L}$, the above equality holds for all $E \in \mathcal{L}$. This proves that ν has a Pettis integrable Radon–Nikodým density f.

Thus, X has the WRNP. □

5. Proof of Proposition 3.4

We begin with the 'if' part. It is enough to prove that if Φ is BV_* on a set E, then V_Φ is σ-finite on E. So, let $M > 0$ be such that $\sum_{i=1}^n \omega(\Phi(J_i)) < M$, for each collection $\{J_1, \ldots, J_n\}$ of nonoverlapping intervals in $\mathcal{I}$ with end-points in E. The assertion is obvious if the set E is countable. If E is uncountable, then we may always find $F \subset E$ such that $E \setminus F$ is at most countable and the points inf F and sup F are not isolated in F. Fix such an F. For $k \in \mathbb{N}$ with $\inf(F) + 1/k < \sup(F) - 1/k$ we define $F_k = F \cap (\inf(F) + 1/k, \sup(F) - 1/k)$. We will prove that $V_\Phi(F_k) \le 2M$, for all k such that $\inf(F) + 1/k < \sup(F) - 1/k$. This will complete the proof, since $F = \bigcup_k F_k$.

Fix a gauge δ on F_k and take a δ-fine partition $\{(I_i, t_i), 1 \le i \le p\}$ anchored in F_k. We can assume that $t_1 < t_2 < \cdots < t_p$. Now take $t_0, t_{p+1} \in F$ with $t_0 < t_1 < t_p < t_{p+1}$, and define $J_1 = (t_0, t_1)$, $J_2 = (t_1, t_2), \ldots, J_{p+1} = (t_p, t_{p+1})$. Then

$$\sum_{i=1}^{p} \|\Phi(I_i)\| \le 2 \sum_{i=1}^{p+1} \omega(\Phi(J_i)) < 2M.$$

Hence $V_\Phi(F_k) \le V(\Phi, \delta, F_k) \le 2M$.

Turning now to the 'only if' part, let $A \subset [0, 1]$ be such that $V_\Phi(A) < \infty$. That is, there exist $M > 0$ and a gauge δ such that for all δ-fine partitions $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ anchored in A,

$$\sum_{i=1}^{p} \|\Phi(I_i)\| \le M. \tag{5.1}$$

Then, for $k \in \mathbb{N}$ define $A_k = \{t \in A : \delta(t) > 1/k\}$. Since $A = \bigcup_k A_k$ and $A_k = \bigcup_{s=0}^{k-1} (A_k \cap [s/k, (s+1)/k])$, it is enough to prove that the function Φ is BV_* on $A_k \cap [s/k, (s+1)/k]$, for all $k = 1, 2, \ldots,$ and for all $s = 0, \ldots, k-1$. Fix s and k and set $B_{ks} = A_k \cap [s/k, (s+1)/k]$. Now take any finite family of nonoverlapping

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intervals $\{(\alpha_1, \beta_1), \ldots, (\alpha_p, \beta_p)\}$ with end-points in A_k, and let $\alpha_j < u_j < v_j < \beta_j$, for all j. Then the families $\{(\alpha_j, \beta_j), \alpha_j\}$, $\{(\alpha_j, u_j), \alpha_j\}$, and $\{(v_j, \beta_j), \beta_j\}$ are δ-fine partitions anchored in B_{ks}. Hence, according to (5.1),

$$\sum_{j=1}^{p} \|\Phi(\beta_j) - \Phi(\alpha_j)\| \le M, \quad \sum_{j=1}^{p} \|\Phi(u_j) - \Phi(\alpha_j)\| \le M,$$

$$\sum_{j=1}^{p} \|\Phi(\beta_j) - \Phi(v_j)\| \le M.$$

Thus,

$$\sum_{j=1}^{p} \|\Phi(v_j) - \Phi(u_j)\| \le 3M.$$

Consequently, $\sum_{j=1}^{p} \omega(\Phi(\alpha_j, \beta_j)) \le 3M$, and Φ is BV_* on B_{ks}. □

References

[1] J. Diestel and J. J. Uhl, *Vector Measures*, Mathematical Surveys, 15 (American Mathematical Society, Providence, RI, 1977).

[2] L. Di Piazza, 'Variational measures in the theory of the integration in $\mathbb{R}^m$', *Czechoslovak Math. J.* **51** (2001), 95–110.

[3] L. Di Piazza and K. Musiał, 'Characterizations of Kurzweil–Henstock–Pettis integrable functions', *Studia Math.* **176** (2006), 159–176.

[4] K. Musiał, 'A characterization of the weak Radon–Nikodym property in terms of the Lebesgue measure', in: *Proceedings of the Conference Topology and Measure III, Part 1, 2 (Vitte/Hiddensee, 1980)* (Wissensch. Beitr., Ernst-Moritz-Arndt Univ., Greifswald, 1982), pp. 163–168.

[5] ———, 'Topics in the theory of Pettis integration', *Rend. Instit. Mat. Univ. Trieste* **23** (1991), 177–262.

[6] ———, 'Pettis integral', in: *Handbook of Measure Theory I* (ed. E. Pap) (Elsevier, Amsterdam, 2002), pp. 531–586.

[7] W. F. Pfeffer, 'The Lebesgue and Denjoy–Perron integrals from a descriptive point of view', *Ric. Mat.* **48** (1999), 211–223.

[8] M. Talagrand, 'Pettis integral and measure theory', *Mem. Amer. Math. Soc.* **51**(307) (1984).

[9] B. S. Thomson, 'Derivatives of interval functions', *Mem. Amer. Math. Soc.* **452** (1991).

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