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TOPOLOGIES EXTENDING VALUATIONS

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Let K be a field complete for a proper valuation (absolute value) v. It is classic that a finite-dimensional K-vector space E admits a unique Hausdorff topology making it a topological K-vector space, and that that topology is the "cartesian product topology" in the sense that for any basis $c_1, \ldots, c_n$ of E, $(\lambda_1, \ldots, \lambda_n) \mapsto \sum_{i=1}^{n} \lambda_i c_i$ is a topological isomorphism from K^n to E [**1**, Chap. I, § 2, no. 3; **2**, Chap. VI, § 5, no. 2]. It follows readily that any multilinear mapping from E^m to a Hausdorff topological K-vector space is continuous. In particular, any multiplication on E making it a K-algebra is continuous in both variables. If for some such multiplication E is a field extension of K, then by valuation theory the unique Hausdorff topology of E is given by a valuation (absolute value) extending v.

Our purpose here is to determine what happens if E is a simple algebraic extension of K but K is no longer assumed to be complete. More precisely, we shall determine all the ring topologies on a simple algebraic extension of K that induce on K the topology defined by v (a *ring topology* is one making addition and multiplication continuous in both variables).

If v is a valuation or an absolute value on a field K, we denote by $\mathcal{T}_v$ the topology on K defined by v, and by $K_v^\wedge$ the completion of K for $\mathcal{T}_v$. If each of v and w is either a valuation or an absolute value on K, we shall say v and w are *independent* if $\mathcal{T}_v \neq \mathcal{T}_w$. The Approximation Theorem, usually stated either for valuations or for absolute values, actually holds for a mixture [**3**, Theorem 3.4, p. 292]: If for each $k \in [1, q]$, v_k is either a proper absolute value or a proper valuation on K, and if v_i and v_j are independent whenever $i \neq j$, then the diagonal mapping $x \mapsto (x, x, \ldots, x)$ from K, equipped with $\sup_{1 \leqq k \leqq q} \mathcal{T}_{vk}$, into $\prod_{k=1}^{q} K_{vk}^\wedge$ is a topological isomorphism onto a dense subfield, and consequently the completion of K for $\sup_{1 \leqq k \leqq q} \mathcal{T}_{vk}$ can be identified with $\prod_{k=1}^{q} K_{vk}^\wedge$.

If L is a finite-dimensional field extension of K and if v is a valuation (absolute value) on K, a sequence $v_1, \ldots, v_m$ of valuations (absolute values) on L is a *complete family of independent valuations* (*absolute values*) *on L extending v* if each v_i is an extension of v, if v_i and v_j are independent whenever $i \neq j$, and if for any valuation (absolute value) w on L extending v there exists $i \in [1, m]$ such that $\mathcal{T}_w = \mathcal{T}_{vi}$.

THEOREM 1. *Let K be a field and L a simple algebraic extension of K of degree n. Let $c \in L$ be such that $L = K[c]$, and let f be the minimal polynomial of c. Let v*

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be a proper valuation (absolute value) on K, and let $D(f)$ be the set of all non-constant monic divisors of f in $K_v^\wedge[X]$. There is a bijection $g \mapsto \mathcal{T}_g$ from $D(f)$ onto the set of all ring topologies on L inducing $\mathcal{T}_v$ on K such that for all $g, h \in D(f)$, $g|h$ if and only if $\mathcal{T}_g \subseteq \mathcal{T}_h$. Moreover, for each $g \in D(f)$, the completion $L_g^\wedge$ of L for $\mathcal{T}_g$ is a finite-dimensional $K_v^\wedge$-algebra generated by 1 and c, and the minimal polynomial over $K_v^\wedge$ of c in $L_g^\wedge$ is g.

Proof. For each $g \in D(f)$, let A_g be the $K_v^\wedge$-algebra $K_v^\wedge[X]/(g)$, and let $c_g = X + (g) \in A_g$. Clearly $A_g = K_v^\wedge[c_g]$, and the minimal polynomial over $K_v^\wedge$ of c_g is g. Since $g|f$ in $K_v^\wedge[X]$, $f(c_g) = 0$; but f is a prime polynomial over K; hence f is the minimal polynomial of c_g over K. Thus there is a unique K-isomorphism u_g from L onto $K[c_g]$ satisfying $u_g(c) = c_g$. We equip A_g with its unique Hausdorff topology making it a $K_v^\wedge$-topological algebra, and we define $\mathcal{T}_g$ to be the (ring) topology on L making u_g a topological K-isomorphism from L onto $K[c_g]$, equipped with the topology it inherits from A_g. Clearly $\mathcal{T}_g$ induces $\mathcal{T}_v$ on K. Also, $K[c_g]$ is dense in A_g, so there is a unique topological isomorphism $u_g^\wedge$ from $L_g^\wedge$, the completion of L for $\mathcal{T}_g$, onto A_g extending u_g; since u_g is a topological K-isomorphism, clearly u_g is a $K_v^\wedge$-isomorphism. Since $u_g^\wedge(c) = c_g$, the minimal polynomial over $K_v^\wedge$ of c in $L_g^\wedge$ is g.

Suppose that $\mathcal{T}_g \subseteq \mathcal{T}_h$ where $g, h \in D(f)$. Then the identity map from L, equipped with $\mathcal{T}_h$, to L, equipped with $\mathcal{T}_g$, is a continuous K-isomorphism and hence has an extension to a continuous $K_v^\wedge$-homomorphism w from $L_h^\wedge$ into $L_g^\wedge$. Thus k, defined by $k = u_g^\wedge \circ w \circ u_h^{\wedge -1}$, is a continuous $K_v^\wedge$-homomorphism from A_h into A_g taking c_h into c_g. Consequently, as $h(c_h) = 0$, $0 = k(h(c_h)) = h(k(c_h)) = h(c_g)$, so the minimal polynomial g of c_g divides h. In particular, if $\mathcal{T}_g = \mathcal{T}_h$, then $g = h$.

Conversely, suppose that $g|h$. Then the canonical epimorphism from $A_h = K_v^\wedge[X]/(h)$ onto $A_g = K_v^\wedge[X]/(g)$ is $K_v^\wedge$-linear and hence continuous and takes c_h into c_g; its restriction q to the subfield $K[c_h]$ of A_h is therefore a continuous isomorphism onto $K[c_g]$ satisfying $q(c_h) = c_g$. Hence $u_g^{-1} \circ q \circ u_h$ is the identity map of L and is continuous from L, equipped with $\mathcal{T}_h$, to L, equipped with $\mathcal{T}_g$. Thus $\mathcal{T}_g \subseteq \mathcal{T}_h$.

To complete the proof, it therefore suffices to show that if $\mathcal{T}$ is a ring topology on L inducing $\mathcal{T}_v$ on K, then $\mathcal{T} = \mathcal{T}_g$ for some $g \in D(f)$. As $\mathcal{T}$ induces $\mathcal{T}_v$ on K, we may consider $L^\wedge$, the completion of L for $\mathcal{T}$, as a topological $K_v^\wedge$-algebra. As $\deg f = n$, $K_v^\wedge + K_v^\wedge c + \ldots + K_v^\wedge c^{n-1}$ is a closed subspace of $L^\wedge$ containing L and hence is all of $L^\wedge$. Thus $L_v^\wedge = K_v^\wedge[c]$. The minimal polynomial g of c in $L^\wedge$ divides f in $K_v^\wedge[X]$ and hence belongs to $D(f)$. Thus there is a unique $K_v^\wedge$-linear isomorphism from $L^\wedge$ onto A_g taking c into c_g, and that isomorphism is a topological isomorphism since both $L^\wedge$ and A_g are finite-dimensional; its restriction to L is clearly u_g, so $\mathcal{T} = \mathcal{T}_g$.

COROLLARY 1. *Let $p_1, \ldots, p_n$ be the prime factors of f in $K_v^\wedge[X]$, and let $f = p_1^{t_1} \ldots p_n^{t_n}$. For each $i \in [1, n]$, $\mathcal{T}_{p_i}$ is given by a valuation (absolute value)*

v_i on L extending v. The valuations (absolute values) $v_1, \ldots, v_m$ form a complete family of independent valuations (absolute values) on L extending v, and

$$\sum_{i=1}^{m} t_i[L_{vi}^{\wedge}: K_v^{\wedge}] = [L : K].$$

Proof. Clearly each A_{p_i} is a field, and by valuation theory its unique Hausdorff topology making it a $K_v^{\wedge}$-topological vector space is given by a valuation (absolute value) v_i extending v. If $g \in D(f)$ and $g \neq p_i$ for all $i \in [1, m]$, then the completion of L for $\mathscr{T}_g$, being isomorphic to A_g, is not a field, so $\mathscr{T}_g$ is not given by a valuation (absolute value). Therefore $v_1, \ldots, v_m$ is a complete family of independent valuations (absolute values) on L extending v, and

$$[L : K] = \deg f = \sum_{i=1}^{m} t_i (\deg p_i) = \sum_{i=1}^{m} t_i[L_{vi}^{\wedge}: K_v^{\wedge}].$$

COROLLARY 2. *For each $i \in [1, m]$, $\mathscr{T}_{vi} = \mathscr{T}_{pi} \subset \mathscr{T}_{p_i^2} \subset \ldots \subset \mathscr{T}_{p_i^{t_i}}$, and the topologies $\mathscr{T}_{p_i^k}$, where $1 \leqq k \leqq t_i$, are precisely the ring topologies on L inducing $\mathscr{T}_v$ on K that are stronger than $\mathscr{T}_{vi}$ but not stronger than $\mathscr{T}_{vj}$ for any $j \neq i$. Furthermore, if $g \in D(f)$ and if $g = p_1^{s_1} \ldots p_m^{s_m}$, then $\mathscr{T}_g = \sup_{1 \leqq i \leqq m} \mathscr{T}_{p_i^{s_i}}$, where $\mathscr{T}_1 = \mathscr{T}_{p_i^0}$ is the topology whose only open sets are L and $\emptyset$.*

Proof. The statement follows at once from Theorem 1, since $g \mapsto \mathscr{T}_g$ is an isomorphism from $D(f)$, equipped with the ordering $|$, to the set of ring topologies on L inducing $\mathscr{T}$ on K, equipped with the ordering $\subseteq$.

THEOREM 2. *Let v be a proper valuation (absolute value) on a field K, let L be a finite-dimensional separable extension of K, and let $v_1, \ldots, v_m$ be a complete family of independent valuations (absolute values) on L extending v. There are precisely $2^m - 1$ ring topologies on L inducing $\mathscr{T}_v$ on K, namely, the topologies $\sup_{k \in M} \mathscr{T}_{vk}$ for all nonempty subsets M of $[1, m]$. Also*

$$[L : K] = \sum_{i=1}^{m} [L_{vi}^{\wedge}: K_v^{\wedge}].$$

Proof. By the theorem of the primitive element, L is a simple extension of K. In the terminology of Theorem 1, f is a separable prime polynomial over K and hence is separable over $K_v^{\wedge}$, so f is a product of distinct prime polynomials of $K_v^{\wedge}[X]$. The assertions therefore follow from Theorem 1 and its corollaries.

THEOREM 3. *Let v be a proper valuation (absolute value) on a field K, and let L be a simple algebraic extension of K. Of all the ring topologies on L inducing $\mathscr{T}_v$ on K there is a strongest. Moreover, for any ring topology $\mathscr{T}$ on L inducing $\mathscr{T}_v$ on K, the following statements are equivalent:*

1°. *There is a basic $c_1, \ldots, c_n$ of the K-vector space L such that u: $(\lambda_1, \ldots, \lambda_n) \mapsto \sum_{i=1}^{n} \lambda_i c_i$ is a topological isomorphism from K^n to L.*

2°. *$[L^{\wedge} : K_v^{\wedge}] = [L : K]$, where $L^{\wedge}$ is the completion of L for $\mathscr{T}$.*

3°. *$\mathscr{T}$ is the strongest ring topology on L inducing $\mathscr{T}_v$ on K.*

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4°. *For any basis* $c_1, \ldots, c_n$ *of the* K*-vector space* L, u: $(\lambda_1, \ldots, \lambda_n) \mapsto \sum_{i=1}^n \lambda_i c_i$ *is a topological isomorphism from* K^n *to* L.

Proof. We shall use the terminology of Theorem 1 and Corollary 1. Clearly $\mathcal{T}_f$ is the strongest ring topology on L inducing $\mathcal{T}_v$ on K. Assume 1°. Now $u^\wedge$: $(\lambda_1, \ldots, \lambda_n) \mapsto \sum_{i=1}^n \lambda_i c_i$ is a multilinear and hence continuous function from the $\hat{K_v}$-vector space $(\hat{K_v})^n$ into the $\hat{K_v}$-vector space $L^\wedge$. It is the unique continuous extension of the topological isomorphism u and hence is itself a topological isomorphism. Therefore $c_1, \ldots, c_n$ is a basis of the $\hat{K_v}$-vector space $L^\wedge$, and 2° holds.

Let $g \in D(f)$ be such that $\mathcal{T} = \mathcal{T}_g$, and let $g = p_1^{s_1} \ldots p_m^{s_m}$. Now $[L^\wedge : \hat{K_v}] = \deg g = \sum_{i=1}^m s_i(\deg g_i)$, and $[L : K] = \deg f = \sum_{i=1}^m t_i(\deg g_i)$. Hence 2° holds if and only if $s_i = t_i$ for all $i \in [1, m]$, that is, if and only if $g = f$, or equivalently, if and only if 3° holds.

Assume finally that 2° holds, and let $c_1, \ldots, c_n$ be any basis of the K-vector space L. Since $\hat{K_v}c_1 + \ldots + \hat{K_v}c_n$ is a closed dense subspace of the $\hat{K_v}$-vector space $L^\wedge$, $c_1, \ldots, c_n$ is a set of generators of the $\hat{K_v}$-vector space $L^\wedge$. By 2°, $c_1, \ldots, c_n$ is a basis of the $\hat{K_v}$-vector space $L^\wedge$. Thus 4° holds.

Example. Let v be a proper valuation on a field K of prime characteristic p, let $L = K[c]$ where c is radical over K, and let $f = X^{p^n} - a$ be the minimal polynomial of c over K. Let m be the largest integer such that $\hat{K_v}$ contains a p^mth root b of a. Then $f = (X^{p^{n-m}} - b)^{p^m} \in \hat{K_v}[X]$, and $X^{p^{n-m}} - b$ is irreducible over $\hat{K_v}$. By Corollary 2, there are p^m ring topologies on L inducing $\mathcal{T}_v$ on K, and they are totally ordered by inclusion. The weakest is the topology defined by the unique valuation w on L extending v, and that is the only topology whose completion is a field. The strongest of these topologies is the only one for which $(\lambda_1, \ldots, \lambda_q) \mapsto \sum_{i=1}^q \lambda_i c_i$ is a topological isomorphism from K^q to L for some (or any) basis $c_1, \ldots, c_q$ of the K-vector space L $(q = p^n)$. The completion of L for each topology is a local algebra over $\hat{K_v}$ whose maximal ideal is nilpotent and whose residue field is the completion of L for w; the dimension of the maximal ideal (or its index of nilpotency) completely determines the topology.

These theorems may be extended to the case where K is topologized by the supremum of finitely many valuation or absolute value topologies by virtue of the following theorem.

THEOREM 4. *Let* K *be a topological field whose topology* $\mathcal{T}_0 = \sup_{1 \leq k \leq q} \mathcal{T}_{v_k}$, *where for each* $k \in [1, q]$, v_k *is either a proper valuation or a proper absolute value on* K, *and* v_i *and* v_j *are independent whenever* $i \neq j$. *Let* L *be an extension field of* K, *and let* $\mathcal{T}$ *be a ring topology on* L *inducing* $\mathcal{T}_0$ *on* K. *There exists a sequence* $\mathcal{T}_1, \ldots, \mathcal{T}_q$ *of ring topologies on* L *such that* $\mathcal{T}_\kappa$ *induces* $\mathcal{T}_{v_k}$ *on* K *for each* $k \in [1, q]$ *and* $\mathcal{T} = \sup_{1 \leq k \leq q} \mathcal{T}_k$.

Proof. Let $K^\wedge$ be the completion of K for $\mathcal{T}_0$, $L^\wedge$ the completion of L for $\mathcal{T}$, $\mathcal{T}^\wedge$ the topology of $L^\wedge$. By the Approximation Theorem there is an orthogonal sequence $(e_k)_{1 \leqq k \leqq q}$ of idempotents whose sum is 1 such that each $K^\wedge e_k$ is a (complete) field and the topology of its dense subfield Ke_k is the image $\mathscr{T}_{v_k}e_k$ of $\mathscr{T}_{v_k}$ under the isomorphism $x \mapsto xe_k$ from K to Ke_k. For each $k \in [1, q]$ let $\mathscr{T}_k$ be the topology on L such that its image $\mathscr{T}_k e_k$ under the isomorphism $x \mapsto xe_k$ from L to Le_k is the topology induced on Le_k by $\mathscr{T}^\wedge$. Then $\mathscr{T}_k$ induces $\mathscr{T}_{v_k}$ on K. Let $L_0 = Le_1 + \ldots + Le_q$, a subring of $L^\wedge$ that contains L. Since the projection $x \mapsto xe_k$ from L_0 to Le_k is continuous for each $k \in [1, q]$, L_0 is the topological direct sum of $Le_1, \ldots, Le_q$. Thus the sets $U_1e_1 + \ldots + U_qe_q$ form a fundamental system of neighborhoods of zero in L_0, where for each $k \in [1, q]$, U_k runs through all neighborhoods of zero for $\mathscr{T}_k$. But $L \cap (U_1e_1 + \ldots + U_qe_q) = U_1 \cap \ldots \cap U_q$. Thus $\mathscr{T} = \sup_{1 \leqq k \leqq q} \mathscr{T}_k$.

In the remaining two theorems, K is a topological field whose topology $\mathscr{T}_0$ is as described in the statement of Theorem 4.

THEOREM 5. *Let L be a finite-dimensional separable extension of K, and for each $k \in [1, q]$ let $v_{k,1}, \ldots, v_{k,m(k)}$ be a complete family of independent valuations (absolute values) on L extending v_k. There are precisely $\prod_{k=1}^{q} (2^{m(k)} - 1)$ ring topologies on L extending $\mathscr{T}_0$, namely, the topologies $\sup_{1 \leqq k \leqq q} (\sup_{i \in M_k} \mathscr{T}_{v_{k,i}})$ as M_k runs through all nonempty finite subsets of $[1, m(k)]$ for each $k \in [1, q]$.*

Proof. The assertions follow from Theorems 2 and 4, together with the observation that if

$$\sup_{1 \leqq k \leqq q} \left(\sup_{i \in M_k} \mathscr{T}_{v_{k,i}} \right) = \sup_{1 \leqq k \leqq q} \left(\sup_{i \in N_k} \mathscr{T}_{v_{k,i}} \right),$$

then the completions

$$\prod_{k=1}^{q} \left(\prod_{i \in M_k} L^\wedge_{v_{k,1}} \right) \quad \text{and} \quad \prod_{k=1}^{q} \left(\prod_{i \in N_k} L^\wedge_{v_{k,i}} \right)$$

of L for those two topologies are topologically isomorphic, from which it follows readily that $M_k = N_k$ for all $k \in [1, q]$.

THEOREM 6. *Let L be a simple algebraic extension of K. Of all the ring topologies on L inducing $\mathscr{T}_0$ on K, there is a strongest. Moreover, for any ring topology $\mathscr{T}$ on L inducing $\mathscr{T}_0$ on K, statements 1°, 3°, and 4° of Theorem 3 are equivalent.*

Proof. The first assertion follows from Theorems 3 and 4. By Theorem 4, let $\mathscr{T} = \sup_{1 \leqq k \leqq q} \mathscr{T}_k$, where each $\mathscr{T}_k$ induces $\mathscr{T}_{v_k}$ on K. Let $L^\wedge$ be the completion of L for $\mathscr{T}$, $K^\wedge$ the closure of K in $L^\wedge$. By the Approximation Theorem, there is an orthogonal sequence $(e_k)_{1 \leqq k \leqq q}$ of idempotents in $K^\wedge$ whose sum is 1 such that each $K^\wedge e_k$ is the completion of Ke_k for the topology $\mathscr{T}_{v_k}e_k$. Assume 1°. As the projection $x \mapsto xe_k$ is a continuous, open mapping from $L^\wedge$ onto $L^\wedge e_k$ and also from $K^\wedge$ onto $K^\wedge e_k$, the mapping $(\lambda_1 e_k, \ldots, \lambda_n e_k) \mapsto \sum_{i=1}^{n} \lambda_i c_i e_k$ is also a topological isomorphism from $(K^\wedge e_k)^n$ onto $L^\wedge e_k$ and hence its restriction to $(Ke_k)^n$ is a topological isomorphism onto Le_k. Since the topology of Ke_k is given

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by a valuation (absolute value), $\mathscr{T}_k e_k$ is the strongest ring topology on Le_k inducing $\mathscr{T}_{v_k} e_k$ on Ke_k by Theorem 3. Hence each $\mathscr{T}_k$ is the strongest ring topology on L inducing $\mathscr{T}_{v_k}$ on K, so 3° holds by Theorem 4. If 3° holds, then each $\mathscr{T}_k$ is the strongest ring topology on L inducing $\mathscr{T}_{v_k}$ on K, so by Theorem 3, u is a topological isomorphism when K is equipped with $\mathscr{T}_{v_k}$ and L with $\mathscr{T}_k$; but then u is also a topological isomorphism when K is equipped with $\mathscr{T}_0 = \sup_{1 \leqq k \leqq q} \mathscr{T}_{v_k}$ and L with $\mathscr{T} = \sup_{1 \leqq k \leqq q} \mathscr{T}_k$.

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