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Schauder decompositions in non-separable Banach spaces

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It is shown that Schauder decompositions exist in non-separable weakly compactly generated spaces and in certain non-separable conjugate spaces. Some results are obtained concerning shrinking and boundedly complete Schauder decompositions in non-separable spaces.

Introduction

A sequence (P_n) of (continuous) projections in a Banach space X is called a *Schauder decomposition* of X if

(i) no P_n is the identity I in X,

(ii) $P_n P_m = P_m P_n = P_n \quad (n \le m)$,

(iii) $\|P_n x - x\| \to 0 \quad \forall x \in X$.

In this note we shall be concerned mainly with the existence of Schauder decompositions in non-separable spaces. Section 1 deals with weakly-compactly generated spaces, while in Section 2 we demonstrate the existence of Schauder decompositions for certain conjugate spaces.

In Section 3, we are concerned with specific types of decompositions (the "shrinking" and "boundedly complete" decompositions of Sanders [8] and Ruckle [7]), and some examples are given.

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1. Weakly compactly generated spaces

A Banach space X is called *weakly compactly generated* if there exists a weakly compact set K in X such that X is the closed subspace generated by K. Our main result of this section is the following:

THEOREM 1.1. *Let X be a non-separable weakly compactly generated space. Then X has a Schauder decomposition.*

The proof of Theorem 1.1 is based on the following result of Amir and Lindenstrauss [1].

LEMMA 1.2. *Let X be a linear space with two norms $\|\cdot\|$, $|||\cdot|||$, such that the unit ball in $(X, |||\cdot|||)$ is $\|\cdot\|$-weakly compact. Let μ be the first ordinal of cardinality the density character of $(X, \|\cdot\|)$ and let ω be the first countable ordinal. If $\{x_\alpha : \alpha < \mu\}$ is a dense subset of X then there exists a family $\{P_\alpha : \omega \le \alpha < \mu\}$ of projections in X such that*

1. $\|P_\alpha\| = |||P_\alpha||| = 1$,
2. $x_\alpha \in P_{\alpha+1}X$,
3. *the density character of* $(P_\alpha X, \|\cdot\|)$ *is less than or equal to* $\operatorname{card}(\alpha)$,
4. $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta \quad \omega \le \beta \le \alpha < \mu$.

Proof of Theorem 1.1. Let K be a weakly compact set which generates X. Then, [4, p. 434], the closed convex hull U of $K \cup (-K)$ is also weakly compact. If Y is the linear span of U, we can define a new norm on Y by

$$|||x||| = \operatorname{Inf}\{\lambda > 0 : x \in \lambda U\} .$$

It is easily seen that $|||\cdot|||$ satisfies the conditions of Lemma 1.2 in Y, so that the family $\{P_\alpha : \omega \le \alpha < \mu\}$ of projections with Properties 1-4 exists in Y. Since $\bigcup_{\alpha<\mu} P_\alpha X$ is dense in X, Property 3 implies that we can select an increasing sequence (α_i) of ordinals, $\omega \le \alpha_i < \mu$, such that the P_{α_i} are distinct. Let T_{α_i} denote the restriction of P_{α_i} to

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U . Since U is $\|\cdot\|$-weakly compact, Tychonoff's Theorem shows that U^U is compact in the product topology and so there exists a mapping $T : U \to U$ and a subnet $(T_\lambda : \Lambda)$ of $\left(T_{\alpha_i}\right)$ such that $T_\lambda x \to Tx$ weakly for every x in U . We may extend T to obtain an operator P on Y defined by

$$Px = \|\|x\|\| \, T(x/\|\|x\|\|) \; .$$

It is easily verified that P is a linear projection and that $P_\lambda x \to Px$ weakly. Since the norm $\|\cdot\|$ is weakly lower semicontinuous we have $\|P\| = 1$. Since Y is dense in X we may extend P_{α_n} and P uniquely to obtain projections $\hat{P}_{\alpha_n}$ and $\hat{P}$ on X . Let $f \in X^* = Y^*$ and $\varepsilon > 0$. If $x \in X$ and $y \in Y$ is such that $\|x-y\| < \varepsilon$, then

$$\begin{aligned} \left|f\left(\hat{P}_\lambda x-\hat{P}x\right)\right| &\leq \left|f\left(\hat{P}_\lambda x-P_\lambda y\right)\right| + \left|f\left(P_\lambda y-Py\right)\right| + \left|f\left(\hat{P}x-Py\right)\right| \\ &< \|f\|.3\varepsilon \quad \text{for sufficiently large } \lambda \; . \end{aligned}$$

Thus $\hat{P}_\lambda x \to \hat{P}x$ weakly for all x in X . Since $\cup\hat{P}_\lambda X = \cup\hat{P}_{\alpha_i} X$ and norm-closed subspaces are weakly closed, we see that $\hat{P}X \subset \overline{\cup\hat{P}_{\alpha_i} X}$.

Conversely, if $x \in \overline{\cup\hat{P}_{\alpha_i} X}$, we may select $y \in \cup\hat{P}_{\alpha_i} X$ such that $\|x-y\| < \varepsilon$. Then if $y \in \hat{P}_{\alpha_j} X$, and $i > j$

$$\begin{aligned} \left\|\hat{P}_{\alpha_i} x-x\right\| &\leq \left\|\hat{P}_{\alpha_i} x-\hat{P}_{\alpha_i} y\right\| + \left\|\hat{P}_{\alpha_i} y-y\right\| + \|y-x\| \\ &< 2\varepsilon \; . \end{aligned}$$

Hence $\hat{P}x = x$, $\hat{P}X = \overline{\cup\hat{P}_{\alpha_i} X}$ and $\left\|\hat{P}_{\alpha_i} x-\hat{P}x\right\| \to 0$ for every x in X . If we now take

$$Q_n x = \hat{P}_{\alpha_n} \hat{P}x + (I-\hat{P})x \; ,$$

then (Q_n) is a Schauder decomposition of X .

COROLLARY 1.3. *If X is a non-separable reflexive Banach space, then X has a Schauder decomposition.*

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We observe that if (Q_n) is the Schauder decomposition constructed in Theorem 1.1, then Q_nX is also weakly compactly generated. In fact Q_nK is a weakly compact subset of Q_nX which generates Q_nX. We have the following partial converse of Theorem 1.1.

THEOREM 1.4. *If X has a Schauder decomposition (P_n) such that P_nX is weakly compactly generated for each n, then X is weakly compactly generated.*

Proof. Let K_n be a weakly compact subset of P_nX which generates P_nX. Then K_n is weakly compact in X and is norm bounded. If $\|x\| \leq M_n$ $\forall x \in K_n$, let $B_n = K_n/nM_n$, and $K = \cup B_n \cup \{0\}$. It is clear that K generates X. To show that K is weakly compact, let (x_n) be a sequence in K. If $x_n = 0$ for infinitely many n or if for some m, $x_n \in B_m$ for infinitely many n, then (x_n) has a weakly convergent subsequence and we are finished. Otherwise there exists a subsequence $\left(x_{n_k}\right)$ of (x_n) and a sequence (m_k) of integers such that $m_k \to \infty$ and $x_{n_k} \in B_{m_k}$. Then $\left\|x_{n_k}\right\| \leq 1/m_k \to 0$ so that $\left(x_{n_k}\right)$ is weakly convergent. This completes the proof.

COROLLARY 1.5. *If X has a Schauder decomposition (P_n) such that each P_nX is reflexive, then X is weakly compactly generated.*

2. The conjugate of a smooth space

A Banach space X is called *smooth* if for every $x \in X$, there exists a unique functional f_x in X^* such that $\|f_x\| = \|x\|$ and $f_x(x) = \|f_x\|\|x\|$. It is known [3, p. 300] that if $x_n \to x$ in the norm topology and X is smooth, then $f_{x_n} \to f_x$ in the weak star topology. Following Tacon [9, p. 416], we say that a smooth space X has property A if, whenever $x_n \to x$ in the norm topology, we have $f_{x_n} \to f_x$, weakly. As

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in [9], if Y is a subspace of X, we denote by $D_{X^*}(Y)$ the set of all functionals in X^* which attain their norms on the unit sphere of Y.

Tacon has established the following result [9, p. 421]:

LEMMA 2.1. *Let X be a smooth space with property A. Let μ be the first ordinal of cardinality the density character of X and let ω be the first countable ordinal. For every α, $\omega \le \alpha < \mu$, there is a subspace X_α of X of density character less than or equal to the cardinality of α, together with a linear operator $T_\alpha : X_\alpha^* \to X^*$, such that P_α (defined by $P_\alpha f = T_\alpha f_\alpha$ where f_α is the restriction of f to X_α) is a bounded linear projection on X^* satisfying:*

1. $\|P_\alpha\| = 1$,

2. $P_\alpha X^* = \overline{D_{X^*}(X_\alpha)}$, *and is isometric to* X_α^*,

3. $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$, $\omega \le \beta \le \alpha < \mu$,

4. $\bigcup_{\alpha<\mu} P_\alpha X^*$ *is dense in* X^*,

5. $P_\alpha^* \hat{x} = \hat{x}$, $(x \in X_\alpha)$.

We observe that if X is non-separable then for every $\alpha < \mu$, the density character of X_α is less than that of X so that by 2 and [9, Lemma 6, p. 420], no P_α is the identity.

Lemma 2.1 will be used to establish the following:

THEOREM 2.2. *Let X be a non-separable smooth space with property A. Then X^* has a Schauder decomposition.*

Proof. By Lemma 2.1 and the above observation, we can select an increasing sequence (α_n) of ordinals such that the projections P_{α_n} of Lemma 2.1 are distinct. Let $Y = \overline{\cup X_{\alpha_n}}$ and for each n, define

$T'_{\alpha_n} : Y^* \to X^*$ by

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$$T'_{\alpha_n} g = T_{\alpha_n} g_{\alpha_n} \quad (g \in Y^*) ,$$

where g_{α_n} is the restriction of g to X_{α_n}. The unit ball of X^* is w^*-compact so that, following the method of Theorem 1.1, we may select a linear operator $T' : Y^* \to X^*$ and a subnet $(T'_\lambda : \Lambda)$ of $\left(T'_{\alpha_n}\right)$ such that for every $g \in Y^*$, $T'_\lambda g \to T'g \quad (w^*)$. For every $f \in X^*$, define $Pf \in X^*$ by $Pf = T'f_Y$, where f_Y is the restriction of f to Y. It is not difficult to show that P is a projection of norm 1 and that $P_{\alpha_n} f \to Pf \quad (w^*)$ for every $f \in X^*$.

If $x \in X_{\alpha_n}$, then we have, for any $f \in X^*$,

$$Pf(x) = \lim P_\lambda f(x) = \lim f\left(P^*_\lambda \hat{x}\right) = f\left(P^*_{\alpha_n} \hat{x}\right) = f(x) .$$

Consequently, $P^*\hat{x} = \hat{x}$ for every $x \in Y$. Let $f \in D_{X^*}(Y)$ and let $x \in Y$ be such that $\|x\| = 1$ and $f(x) = \|f\|$. Then $\|Pf\| \le \|f\|$ and we have $Pf(x) = P^*\hat{x}(f) = f(x)$. Since X is smooth, this implies that $Pf = f$. Conversely, let $Pf = f$. We know [2] that the set of functionals which attain their norms on the unit sphere of Y is dense in Y^*. Thus we may find sequences (g_n) in Y^* and (x_n) in Y such that $\|x_n\| = 1$, $g_n(x_n) = \|g_n\|$ and $\|g_n - f_Y\| \to 0$, where f_Y is the restriction of f to Y. Since X is smooth, g_n has a unique extension f_n to X such that $\|g_n\| = \|f_n\|$. Then

$$\|f_n - f\| = \|Pf_n - Pf\| = \|T'g_n - T'f_Y\| \to 0 .$$

Hence $f \in \overline{D_{X^*}(Y)}$ and consequently $PX^* = \overline{D_{X^*}(Y)}$.

Clearly $\overline{\cup D_{X^*}(X_{\alpha_n})} \subset \overline{D_{X^*}(Y)}$. If $f \in D_{X^*}(Y)$ and $x \in Y$ is such that $\|f\| = \|x\|$ and $f(x) = \|f\|\|x\|$, then there is a sequence (x_n) in $\cup X_{\alpha_n}$ such that $\|x_n - x\| \to 0$. Then $f_{x_n} \to f$ weakly. Thus f belongs to the weak closure and hence to the norm closure of $\overline{UD_{X^*}(X_{\alpha_n})}$. It follows that $\overline{\overline{UD_{X^*}(X_{\alpha_n})}} = \overline{D_{X^*}(Y)}$, that is we have $PX^* = \overline{UP_{\alpha_n}X^*}$. It is left to the reader to show that the sequence (Q_n) is a Schauder decomposition of X^* , where

$$Q_nf = P_{\alpha_n}Pf + (I-P)f .$$

A Schauder decomposition (P_n) is called *shrinking* if for every $f \in X^*$, we have $\|f\|_n \to 0$ where $\|f\|_n = \sup\{|f(x)|;\ P_nx = 0,\ \|x\| = 1\}$.

THEOREM 2.3. *Let* X *be a smooth space with property* A . *If* (P_n) *is a Schauder decomposition of* X *such that* $\|P_n\| = 1$ *for every* n , *then* (P_n) *is shrinking and* (P_n^*) *is a Schauder decomposition of* X^* .

Proof. Let $f \in X^*$ be such that f attains its norm on the unit sphere of X , and select x such that $\|x\| = \|f\|$ and $f(x) = \|f\|\|x\|$.

Let f_n be the unique linear functional in X^* such that $\|f_n\| = \|P_nx\|$ and $f_n(P_nx) = \|f_n\|\|P_nx\|$. Since X has property A , we see that $f_n \to f$ weakly. Now $\|P_n^*f_n\| \le \|f_n\|$ and also $P_n^*f_n(P_nx) = f_n(P_nx) = \|f_n\|\|P_nx\|$. Since X is smooth, $P_n^*f_n = f_n$ so that $f_n \in P_n^*X^*$. It follows that $f \in \overline{UP_n^*X^*}$, and so by the Bishop-Phelps Theorem [2] $\overline{UP_n^*X^*} = X^*$. If $f \in X^*$, then for some n , we may select $g \in P_n^*X^*$ such that $\|f-g\| < \varepsilon$. Then, for $m \ge n$,

$$\begin{aligned}\|P_m^*f-f\| &\le \|P_m^*f-P_m^*g\| + \|P_m^*g-g\| + \|f-g\| \\ &= \|P_m^*f-P_m^*g\| + \|f-g\| < 2\varepsilon .\end{aligned}$$

Hence $\|P_m^*f-f\| \to 0$ and (P_m^*) is a Schauder decomposition of X^* .

If $f \in X^*$, select x_n such that $\|x_n\| = 1$, $P_nx_n = 0$ and $f(x_n) > \|f\|_n - 1/n$. Then

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$$\|f\|_n < f(x_n) + 1/n = (f-P_n^*f)x_n + 1/n$$
$$\leq \|f-P_n^*f\| + 1/n \to 0 ,$$

so that (P_n) is shrinking.

3. Some further results

A Schauder decomposition (P_n) of X is called *boundedly complete* if, for every bounded sequence (x_n) in X satisfying $P_m x_n = x_m$, $(m \leq n)$, there exists $x \in X$ such that $\|x_n - x\| \to 0$. Ruckle [7, p. 553] and Sanders [8, p. 205] have shown that if (P_n) is a Schauder decomposition of X, then X is reflexive if and only if (P_n) is both shrinking and boundedly complete and each $P_n X$ is reflexive. In view of Corollary 1.3, we can improve on this result for non-separable spaces as follows:

THEOREM 3.1. *Let X be non-separable. Then X is reflexive if and only if X has a Schauder decomposition (P_n) satisfying:*

(i) (P_n) *is shrinking;*

(ii) (P_n) *is boundedly complete;*

(iii) each $P_n X$ *is reflexive.*

Each of the conditions *(i)-(iii)* is essential in Theorem 3.1 as will be shown by examples following Lemma 3.2. In fact there are separable non-reflexive spaces with Schauder decompositions satisfying *(i)* and *(ii)*.

LEMMA 3.2. *Let X be any Banach space, Y a complemented subspace of X and P a projection of X onto Y. Let (Q_n) be a Schauder decomposition of Y and for $x \in X$, define*

$$P_n x = Q_n P x + (I-P)x .$$

Then (P_n) *is a Schauder decomposition of* X *and*

(i) (P_n) *is shrinking if and only if* (Q_n) *is,*

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(ii) (P_n) *is boundedly complete if and only if* (Q_n) *is.*

Proof. It is easily seen that (P_n) is a Schauder decomposition of X . Suppose that (P_n) is shrinking and let $f \in Y^*$. By the Hahn-Banach Theorem there exists $g \in X^*$ such that $\|f\| = \|g\|$ and $f(y) = g(y)$ $(y \in Y)$. We note that $P_n x = 0$ if and only $x \in Y$ and $Q_n x = 0$. Thus

$$\sup\{|f(y)| : \|y\| = 1, Q_n y = 0\} = \sup\{|g(x)|; \|x\| = 1, P_n x = 0\} \to 0 \ (n \to \infty) ,$$

since (P_n) is shrinking. Conversely, if (Q_n) is shrinking let $f \in X^*$ and let g be the restriction of f to Y . Then as before,

$$\sup\{|f(x)| : \|x\| = 1, P_n x = 0\} = \sup\{|g(y)| : \|y\| = 1, Q_n y = 0\} \to 0 ,$$

since (Q_n) is shrinking. It follows that (P_n) is shrinking.

Next, suppose that (P_n) is boundedly complete. Let (y_n) be a bounded sequence in Y such that $Q_m y_n = y_m$ $(m \le n)$. Then for $m \le n$,

$$P_m y_n = Q_m P y_n + (I-P) y_n = Q_m y_n = y_m .$$

Since (P_n) is boundedly complete, there exists $y \in X$ such that $\|y_n - y\| \to 0$. Since $y \in Y$, this means that (Q_n) is boundedly complete. Finally, assume that (Q_n) is boundedly complete. Let (x_n) be a bounded sequence in X such that $P_m x_n = x_m$ $(m \le n)$. Then

$$\begin{aligned} Q_m P x_n &= P_m x_n - (I-P) x_n \\ &= P x_m + (I-P)\left(x_m - x_n\right) . \end{aligned}$$

Thus $Q_m P x_n = P x_m$ and $(I-P)\left(x_m - x_n\right) = 0$. (Q_m) is boundedly complete, so that for some $y \in Y$ we have $\|P x_n - y\| \to 0$. Also for all n , $(I-P) x_n = (I-P) x_1$. Thus

$$x_n = (I-P) x_n + P x_n \to (I-P) x_1 + y ,$$

and so (P_n) is boundedly complete.

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EXAMPLE 3.3. Let Y be a non-separable reflexive Banach space and let X be the direct sum $l_1 \oplus Y$. Let P be the projection satisfying $PX = l_1$, $(I-P)X = Y$. Define a Schauder decomposition (Q_n) of l_1 by

$$Q_n x = (x_1\ x_2\ \ldots\ x_n\ 0\ 0\ \ldots) \quad \text{where} \quad x = (x_n) \in l_1 .$$

(Q_n) is easily seen to be boundedly complete so that (P_n) defined as in Lemma 3.2 is a Schauder decomposition of the non-reflexive non-separable space X satisfying conditions *(ii)* and *(iii)* of Theorem 3.1.

EXAMPLE 3.4. Let Y be as in Example 3.3 and let $X = c_0 \oplus Y$. The 'natural' Schauder decomposition (Q_n) of c_0 defined by

$$Q_n x = (x_1\ x_2\ \ldots\ x_n\ 0\ 0\ \ldots) , \quad x = (x_n) \in c_0$$

is shrinking so that (P_n) defined by $P_n = Q_n P + (I-P)$ is a Schauder decomposition of the non-reflexive space X satisfying conditions *(i)* and *(iii)* of Theorem 3.1.

EXAMPLE 3.5. Let $X = l_2 \oplus m$ and let P satisfy $PX = l_2$, $(I-P)X = m$. For $x = (x_n) \in l_2$ let $Q_n x = (x_1\ \ldots\ x_n\ 0\ 0\ 0\ \ldots)$. (Q_n) is a Schauder decomposition of l_2 and so is both shrinking and boundedly complete. Consequently (P_n), where $P_n = Q_n P + I - P$, is a shrinking and boundedly complete Schauder decomposition of X such that no $P_n X$ is reflexive.

In fact X has no Schauder decomposition P_n such that each $P_n X$ is reflexive. Otherwise by Theorem 1.4, $l_2 \oplus m$ would be weakly compactly generated and hence [3, p. 38] isomorphic to a smooth space. But we know [4, p. 114] that m is not isomorphic to a smooth space.

THEOREM 3.6. *Let X be a Banach space. Then the following are equivalent:*

(i) *X has a complemented non-separable reflexive subspace;*

(ii) *X has a Schauder decomposition (P_n) such that (P_n) is shrinking and boundedly complete and for some n, $(I-P_n)X$ is reflexive and non-separable.*

Proof. The proof follows easily from Lemma 3.2.

Ruckle [7, p. 552] has shown that if X has a boundedly complete Schauder decomposition (P_n) such that each P_nX is reflexive, then X is isomorphic to a conjugate space. We conclude with the following example:

EXAMPLE 3.7. Let $X = L(0, 1)$. Then [6, p. 215] X is not isomorphic to a conjugate space. However X has a boundedly complete Schauder decomposition. Define P_nx by

$$P_nx(t) = \begin{cases} x(t) & 0 < t \leq 1-1/n , \\ 0 & 1 > t > 1-1/n . \end{cases}$$

We leave it to the reader to check that (P_n) is a boundedly complete Schauder decomposition of $L(0, 1)$

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