ON DILATION EQUATIONS AND THE HÖLDER CONTINUITY OF THE DE RHAM FUNCTIONS

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(Received 15 January, 1993)

Abstract. We use a simple approximation method to prove the Hölder continuity of the generalized de Rham functions.

1. Consider the following dilatation equation

$$f(x) = f(3x) + (\tfrac{1}{2} - \alpha)(f(3x+1) + f(3x-1)) + (\tfrac{1}{2} + \alpha)(f(3x+2) + f(3x-2)), \tag{1}$$

where $|\alpha| < 1/2$. Suppose that f is an integrable solution of (1); then f must satisfy

$$\hat{f}(\xi) = p_\alpha(\xi/3)\hat{f}(\xi/3), \tag{2}$$

where $\hat{f}(\xi) = \int e^{ix\xi} f(x)\, dx$ is the Fourier transform of f, and

$$\begin{aligned} p_\alpha(\xi) &= [1 + (\tfrac{1}{2} - \alpha)(e^{i\xi} + e^{-i\xi}) + (\tfrac{1}{2} + \alpha)(e^{2i\xi} + e^{-2i\xi})]/3 \\ &= e^{-2i\xi}\left(\frac{1 + e^{i\xi} + e^{2i\xi}}{3}\right)[(\tfrac{1}{2} + \alpha) - 2\alpha e^{i\xi} + (\tfrac{1}{2} + \alpha)e^{2i\xi}], \end{aligned} \tag{3}$$

which immediately leads to

$$\hat{f}(\xi) = \hat{f}(0) \prod_{n=1}^{\infty} p_\alpha\left(\frac{\xi}{3^n}\right). \tag{4}$$

It turns out that (1) does have integrable solutions. (By (4), such a solution is unique up to multiplication by a constant.) When $\alpha = 1/6$, the function which is integrable, satisfies (1) and $\hat{f}(0) = 1$, is called the *de Rham function*. For other values of α, such functions are called the *generalized de Rham functions.*

In [**1**] and [**2**], Daubechies and Lagarias studied the existence and properties of solutions of general dilation equations. One of their results, obtained by using the time-domain method, shows that the generalized de Rham functions are supported on $[-1, 1]$, and are Hölder continuous, with Hölder exponent

$$\gamma_\alpha = \min[-\ln|\tfrac{1}{2} + \alpha|, -\ln|2\alpha|]/\ln 3.$$

It is stated in [**1**] that it is not known how to prove the continuity of the de Rham function directly from (3). The purpose of this note is to provide a proof of their result using precisely (3). For results on more general dilation equations, we refer the reader to [**1**] and [**2**].

2. Let $\alpha \in (-\frac{1}{2}, \frac{1}{2})$. The result mentioned above is stated as the following theorem.

THEOREM. *There exists an integrable function f which satisfies* (1) *and* $\hat{f}(0) = 1$. *The function f is Hölder continuous with exponent* γ_α; *i.e. there is an* $A_\alpha > 0$ *such that*

$$|f(x) - f(y)| \le A_\alpha |x - y|^{\gamma_\alpha}, \tag{5}$$

for $x, y \in \mathbf{R}$.

We now present our proof. Let

$$q_\alpha(\xi) = (\tfrac{1}{2} + \alpha) - 2\alpha e^{i\xi} + (\tfrac{1}{2} + \alpha)e^{2i\xi}.$$

By (3) and the equation

$$\prod_{n=1}^{\infty} e^{-2i\xi/3^n}\left(\frac{1 + e^{i\xi/3^n} + e^{2i\xi/3^n}}{3}\right) = \frac{1 - e^{-i\xi}}{i\xi},$$

which is the Fourier transform of $\chi_{(-1,0]}$, we are led to the following sequence of functions $\{f_n\}_{n=0}^{\infty}$, defined recursively by

$$f_0(x) = \chi_{(-1,0]}(x), \tag{6}$$

$$f_n(x) = (\tfrac{1}{2} + \alpha)f_{n-1}(x) - 2\alpha f_{n-1}\left(x - \frac{1}{3^n}\right) + (\tfrac{1}{2} + \alpha)f_{n-1}\left(x - \frac{2}{3^n}\right). \tag{7}$$

Let $k \in \mathbf{N}$, $j \in \mathbf{Z}$. Clearly, the function f_k assumes a constant value on every interval of the form $((j-1)/3^k, j/3^k]$, and $\operatorname{supp}(f_k) \subset [-1, 1]$. Let $v_{k,j}$ be the value of f_k on $((j-1)/3^k, j/3^k]$. We find that

$$v_{k+1,3j-2} = (\tfrac{1}{2} + \alpha)v_{k,j} + (\tfrac{1}{2} - \alpha)v_{k,j-1}, \tag{8}$$

$$v_{k+1,3k-1} = (\tfrac{1}{2} - \alpha)v_{k,j} + (\tfrac{1}{2} + \alpha)v_{k,j-1}, \tag{9}$$

$$v_{k+1,3j} = v_{k,j}. \tag{10}$$

Therefore

$$\sup_x |f_{k+1}(x) - f_k(x)| \le \sup_j |v_{k,j} - v_{k,j-1}|.$$

On the other hand, we have

$$v_{k+1,3j} - v_{k+1,3j-1} = (\tfrac{1}{2} + \alpha)(v_{k,j} - v_{k,j-1}), \tag{11}$$

$$v_{k+1,3j-1} - v_{k+1,3j-2} = (-2\alpha)(v_{k,j} - v_{k,j-1}), \tag{12}$$

$$v_{k+1,3j-2} - v_{k+1,3j-3} = (\tfrac{1}{2} + \alpha)(v_{k,j} - v_{k,j-1}). \tag{13}$$

Let $\beta = \max\{|\frac{1}{2} + \alpha|, |2\alpha|\}$. Using (11)–(13), we get

$$\sup_j |v_{k,j} - v_{k,j-1}| \le \beta^k.$$

Since $0 \le \beta < 1$, the sequence $\{f_n\}$ converges pointwise to a function f; i.e. $f = \lim f_n$. Clearly $\operatorname{supp}(f) \subset [-1, 1]$. We now show that f satisfies (1) and (5).

Take x, y which satisfy

$$3^{-N-1} \le |x - y| < 3^{-N},$$

for some integer $N>0$. Then

$$\begin{aligned}|f(x)-f(y)| &\le |f(x)-f_N(x)|+|f_N(x)-f_N(y)|+|f_N(y)-f(y)| \\ &\le \sum_{k=N}^{\infty}|f_{k+1}(x)-f_k(x)|+\sum_{k=N}^{\infty}|f_{k+1}(y)-f_k(y)|+\sup_j |v_{N,j}-v_{N,j-1}| \\ &\le 2\sum_{k=N}^{\infty}\beta^k+\beta^N \le A_\alpha \beta^N \le A_\alpha |x-y|^{\gamma_\alpha}.\end{aligned}$$

To show that f satisfies (1), we observe that $0 \le \lim\|f-f_n\|_1 \le 2\lim\|f-f_n\|_\infty = 0$,

$$\hat{f}_n(\xi)=q_\alpha(\xi/3^n)\hat{f}_{n-1}(\xi)=\dots=\frac{1-e^{-i\xi}}{i\xi}\prod_{k=1}^{n}q_\alpha(\xi/3^k),$$

and

$$q_\alpha(\xi/3^{n+1})\hat{f}_n(\xi)=p_\alpha(\xi/3)\hat{f}_n(\xi/3).$$

Let $n\to\infty$; we find that

$$\hat{f}(\xi)=p_\alpha(\xi/3)\hat{f}(\xi/3).$$

This shows that f is indeed a continuous, integrable solution of (1). Finally we show that (5) becomes false if γ_α is replaced by any larger exponent.

From (10) it is clear that $f_m(x)=f_k(x)$, if $x=j\,.\,3^{-k}$, and $m\ge k$. Hence, for such x, $f(x)=f_k(x)$. Let

$$x_k=-3^{-k}\left(\sum_{j=0}^{k-1}3^j\right),\qquad y_k=-3^{-k}\left(\sum_{j=0}^{k-1}3^j+1\right),\qquad z_k=-1+3^{-k}.$$

Then $|x_k-y_k|=|z_k-(-1)|=3^{-k}$. By (11)–(13), we get

$$\begin{aligned}|f(x_k)-f(y_k)| &= |f_k(x_k)-f_k(y_k)| \\ &= |2\alpha|\,|f_{k-1}(x_{k-1})-f_{k-1}(y_{k-1})|=\dots=|2\alpha|^k,\end{aligned}$$

and $|f(z_k)-f(-1)|=|1/2+\alpha|^k$, which show that (5) cannot hold if γ_α is replaced by any larger number.

REFERENCES

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