$\mathfrak{F}$-INJECTORS OF LOCALLY SOLUBLE *FC*-GROUPS

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(Received 19 June, 1968)

1. Introduction. A group G is said to be an *FC-group* if each element of G has only a finite number of conjugates in G. We are concerned with the class $\mathfrak{G}$ of periodic locally soluble *FC*-groups. Clearly subgroups and factor groups of $\mathfrak{G}$-groups are also $\mathfrak{G}$-groups.

Every finite soluble group is a $\mathfrak{G}$-group, and we consider here the generalization of a concept from the theory of finite soluble groups.

In [**1**], B. Fischer, B. Hartley and W. Gaschütz defined a *Fitting class* to be a class $\mathfrak{E}$ of finite soluble groups satisfying the following conditions:

(i) If $G \in \mathfrak{E}$ and $N \lhd G$, then $N \in \mathfrak{E}$; and

(ii) if N_1 and N_2 are normal $\mathfrak{E}$-subgroups of G such that $G = N_1 N_2$, then G is also an $\mathfrak{E}$-group.

If $\mathfrak{X}$ is any class of groups, then an $\mathfrak{X}$-*injector* of the group G is defined to be an $\mathfrak{X}$-subgroup X of G such that, for each subnormal subgroup S of G, $X \cap S$ is a maximal $\mathfrak{X}$-subgroup of S.

The following result was proved in [**1**]:

THEOREM 1.1. *If $\mathfrak{E}$ is a Fitting class, then a finite soluble group G possesses $\mathfrak{E}$-injectors and any two such subgroups are conjugate in G.*

We shall extend this result to the class $\mathfrak{G}$.

We define a *Fitting class* of $\mathfrak{G}$-groups to be a subclass $\mathfrak{F}$ of $\mathfrak{G}$ satisfying the following conditions:

(i) If $G \in \mathfrak{F}$ and $N \lhd G$, then $N \in \mathfrak{F}$; and

(ii) if N_λ $(\lambda \in \Lambda)$ are normal $\mathfrak{F}$-subgroups of the $\mathfrak{G}$-group G such that $G = \mathrm{gp}\{N_\lambda : \lambda \in \Lambda\}$, then G is also an $\mathfrak{F}$-group.

With the usual notations for closure operations, these two conditions may be written

$$\text{(i) } \mathrm{S}_n\mathfrak{F} = \mathfrak{F} \qquad \text{and} \qquad \text{(ii) } \mathrm{N}\mathfrak{F} \cap \mathfrak{G} = \mathfrak{F}.$$

These conditions ensure that every $\mathfrak{G}$-group G has a unique maximal normal $\mathfrak{F}$-subgroup, called the $\mathfrak{F}$-*radical* of G.

An automorphism ϕ of a group G is said to be *locally inner* if, for each finite set of elements $g_1, \ldots, g_n \in G$, there is an element $x \in G$ (depending on the set $g_1, \ldots, g_n$) such that

$$g_i\phi = x^{-1}g_ix \qquad (i = 1, 2, \ldots, n).$$

If there is a locally inner automorphism of G mapping a subgroup H onto a subgroup K, then H and K are said to be *locally conjugate* in G. In a number of results for *FC*-groups which are generalizations of results for finite groups it will be seen that local conjugacy replaces conjugacy. This holds, for example, in the case of Sylow subgroups [**2**], Carter subgroups [**5**] and covering $\mathfrak{F}$-subgroups [6]. We shall have a further illustration in the present paper where we establish the following result:

If $\mathfrak{F}$ is a Fitting class of $\mathfrak{G}$-groups, then a $\mathfrak{G}$-group G possesses $\mathfrak{F}$-injectors and any two such subgroups are locally conjugate in G.

A. P. Dicman proved that every finite set of elements of a periodic FC-group G is contained in a finite normal subgroup of G (see [4], pp. 154–155). To construct the $\mathfrak{F}$-injectors of G from the $\mathfrak{F}$-injectors of the finite normal subgroups of G, we also use the theory of projection sets due to A. G. Kuroš ([4], pp. 167–169). We refer to the outline of this theory given in Section 3 of [6].

In Section 4 we consider what conditions are necessary for the $\mathfrak{F}$-injectors of G to be conjugate in G.

2. Fitting classes of $\mathfrak{G}$-groups. In this section we characterize the Fitting classes of $\mathfrak{G}$-groups in terms of the Fitting classes of finite soluble groups. The first result is an immediate consequence of the definitions.

THEOREM 2.1. *If $\mathfrak{F}$ is a Fitting class of $\mathfrak{G}$-groups, then the class of finite $\mathfrak{F}$-groups is a Fitting class of finite soluble groups.*

THEOREM 2.2. *If $\mathfrak{E}$ is a Fitting class of finite soluble groups, then $\mathrm{L}\mathfrak{E} \cap \mathfrak{G}$ is a Fitting class of $\mathfrak{G}$-groups.*

Proof. (i) Let H be a normal subgroup of the $\mathrm{L}\mathfrak{E} \cap \mathfrak{G}$-group G and let $h_1, \ldots, h_n$ be a finite set of elements of H. There is an $\mathfrak{E}$-subgroup E of G containing $h_1, \ldots, h_n$. $H \cap E$ is a normal subgroup of E and so is an $\mathfrak{E}$-subgroup of H containing $h_1, \ldots, h_n$. Thus $H \in \mathrm{L}\mathfrak{E} \cap \mathfrak{G}$ and $\mathrm{L}\mathfrak{E} \cap \mathfrak{G}$ is s_n-closed.

(ii) Let G be a $\mathfrak{G}$-group generated by the normal $\mathrm{L}\mathfrak{E}$-subgroups N_λ $(\lambda \in \Lambda)$ and let $g_1, \ldots, g_n$ be a finite set of elements of G. There is a finite normal subgroup N of G containing $g_1, \ldots, g_n$, and we may choose $\lambda_1, \ldots, \lambda_r \in \Lambda$ such that $N \leqq N_{\lambda_1} N_{\lambda_2} \ldots N_{\lambda_r} = L$, say. It is now sufficient to prove that $\mathrm{N}_0 \mathrm{L}\mathfrak{E} \cap \mathfrak{G} = \mathrm{L}\mathfrak{E} \cap \mathfrak{G}$, for then $L \in \mathrm{L}\mathfrak{E} \cap \mathfrak{G}$ and so, by (i), N is an $\mathfrak{E}$-subgroup of G containing $g_1, \ldots, g_n$.

Accordingly let H and K be normal $\mathrm{L}\mathfrak{E}$-subgroups of the $\mathfrak{G}$-group G such that $HK = G$. Let $g_1, \ldots, g_n$ be a finite set of elements of G and, for each $i = 1, 2, \ldots, n$, write $g_i = h_i k_i$, where $h_i \in H$ and $k_i \in K$. There are normal $\mathfrak{E}$-subgroups E_1 and E_2 of G contained in H and K and containing $h_1, \ldots, h_n$ and $k_1, \ldots, k_n$ respectively. It follows that

$$g_1, \ldots, g_n \in E_1 E_2 \in \mathrm{N}_0 \mathfrak{E} = \mathfrak{E}.$$

Thus $G \in \mathrm{L}\mathfrak{E} \cap \mathfrak{G}$, as required.

It follows from these two results that the Fitting classes of $\mathfrak{G}$-groups are precisely the classes $\mathrm{L}\mathfrak{E} \cap \mathfrak{G}$, where $\mathfrak{E}$ is a Fitting class of finite soluble groups.

COROLLARY 2.3. *If $\mathfrak{F}$ is a Fitting class of $\mathfrak{G}$-groups, then $\mathrm{L}\mathfrak{F} \cap \mathfrak{G} = \mathfrak{F}$.*

3. $\mathfrak{C}$-injectors of $\mathfrak{G}$-groups.

LEMMA 3.1. *Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{G}$-groups. If V is an $\mathfrak{F}$-injector of the $\mathfrak{G}$-group G and $H \lhd G$, then $V \cap H$ is an $\mathfrak{F}$-injector of H.*

Proof. $V \cap H$ is a normal subgroup of V and so is an $\mathfrak{F}$-subgroup.

If S is a subnormal subgroup of H, then S is also subnormal in G, and so $V \cap S$ is a maximal $\mathfrak{F}$-subgroup of S.

THEOREM 3.2. *Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{G}$-groups. Then the $\mathfrak{G}$-group G possesses $\mathfrak{F}$-injectors.*

Proof. Let $\{N_\lambda: \lambda \in \Lambda\}$ be the set of all finite normal subgroups of G. Since the class of finite $\mathfrak{F}$-groups is a Fitting class of finite soluble groups (Theorem 2.1), each N_λ possesses $\mathfrak{F}$-injectors (Theorem 1.1). For each $\lambda \in \Lambda$, let $\mathscr{A}_\lambda$ denote the set of $\mathfrak{F}$-injectors of N_λ; then each $\mathscr{A}_\lambda$ $(\lambda \in \Lambda)$ is finite and non-empty. The sets $\mathscr{A}_\lambda$ $(\lambda \in \Lambda)$ may be partially ordered by defining $\mathscr{A}_\lambda \prec \mathscr{A}_\mu$ if and only if $N_\lambda \leqq N_\mu$.

If $\mathscr{A}_\lambda \prec \mathscr{A}_\mu$ and V_μ is an $\mathfrak{F}$-injector of N_μ, then, by Lemma 3.1, $V_\mu \cap N_\lambda$ is an $\mathfrak{F}$-injector of N_λ. Thus we may define a projection $\pi_{\mu\lambda}$ from $\mathscr{A}_\mu$ into $\mathscr{A}_\lambda$ by

$$V_\mu \pi_{\mu\lambda} = V_\mu \cap N_\lambda.$$

Clearly $\pi_{\lambda\lambda}$ is the identity mapping on $\mathscr{A}_\lambda$, and if $\mathscr{A}_\lambda \prec \mathscr{A}_\mu \prec \mathscr{A}_\nu$, then

$$V_\nu \pi_{\nu\mu} \pi_{\mu\lambda} = V_\nu \cap N_\mu \cap N_\lambda = V_\nu \cap N_\lambda = V_\nu \pi_{\nu\lambda},$$

and so $\pi_{\nu\mu}\pi_{\mu\lambda} = \pi_{\nu\lambda}$.

It follows that there is a complete projection set, i.e. a set $\mathscr{P} = \{V_\lambda; \lambda \in \Lambda\}$ such that, whenever $N_\lambda \leqq N_\mu$, $V_\mu \cap N_\lambda = V_\lambda$.

We define $V = \bigcup_{\lambda \in \Lambda} V_\lambda$ and show that V is an $\mathfrak{F}$-injector of G. If $x_1, \ldots, x_n$ is a finite set of elements of V, then there is a finite normal subgroup N_λ of G containing $x_1, \ldots, x_n$. But it is clear from the properties of the complete projection set $\mathscr{P}$ that $V \cap N_\lambda = V_\lambda$, and so $x_1, \ldots, x_n$ are contained in the $\mathfrak{F}$-subgroup V_λ of V. Thus V is an $\mathrm{L}\mathfrak{F} \cap \mathfrak{G}$-group; hence, by Corollary 2.3, V is an $\mathfrak{F}$-group.

Now let S be a subnormal subgroup of G and let W be an $\mathfrak{F}$-subgroup of S containing $V \cap S$. For each $\lambda \in \Lambda$, $W \cap N_\lambda$ is an $\mathfrak{F}$-subgroup of $S \cap N_\lambda$ containing $V \cap S \cap N_\lambda = V_\lambda \cap S$. Since $S \cap N_\lambda$ is subnormal in N_λ, $V_\lambda \cap S$ is a maximal $\mathfrak{F}$-subgroup of $S \cap N_\lambda$ and so $W \cap N_\lambda = V_\lambda \cap S$. Therefore

$$W = \bigcup_{\lambda \in \Lambda} (W \cap N_\lambda) = \bigcup_{\lambda \in \Lambda} (V_\lambda \cap S) = V \cap S;$$

hence $V \cap S$ is a maximal $\mathfrak{F}$-subgroup of S.

THEOREM 3.3. *Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{G}$-groups. Then any two $\mathfrak{F}$-injectors of the $\mathfrak{G}$-group G are locally conjugate in G.*

Proof. Let V_1 and V_2 be two $\mathfrak{F}$-injectors of G and again let $\{N_\lambda: \lambda \in \Lambda\}$ be the set of all finite normal subgroups of G. $V_1 \cap N_\lambda$ and $V_2 \cap N_\lambda$ are $\mathfrak{F}$-injectors of N_λ (Lemma 3.1) and so are conjugate in N_λ (Theorem 1.1). For each $\lambda \in \Lambda$, let $\mathscr{A}_\lambda$ be the set of automorphisms of

N_λ which are induced by inner automorphisms of G and which map $V_1 \cap N_\lambda$ onto $V_2 \cap N_\lambda$. Then each $\mathscr{A}_\lambda$ $(\lambda \in \Lambda)$ is finite and non-empty.

If $N_\lambda \leqq N_\mu$ and $\phi_\mu \in \mathscr{A}_\mu$, then the automorphism induced in N_λ by ϕ_μ is a member of $\mathscr{A}_\lambda$. It follows from a result due to S. E. Stonehewer ([5], Lemma 2.2) that G has a locally inner automorphism ϕ which induces in each N_λ an automorphism in the set $\mathscr{A}_\lambda$. In particular,

$$(V_1 \cap N_\lambda)\phi = V_2 \cap N_\lambda$$

and so

$$V_1\phi = \bigcup_{\lambda \in \Lambda} (V_1 \cap N_\lambda)\phi = \bigcup_{\lambda \in \Lambda} (V_2 \cap N_\lambda) = V_2;$$

i.e. V_1 and V_2 are locally conjugate in G.

It was proved in [1] that, if V is an $\mathfrak{C}$-injector of G and H is a subgroup of G containing V, then V is also an $\mathfrak{C}$-injector of H. This result is easily extended to the class $\mathfrak{G}$.

THEOREM 3.4. *Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{G}$-groups and let V be an $\mathfrak{F}$-injector of the $\mathfrak{G}$-group G. If $V \leqq H \leqq G$, then V is an $\mathfrak{F}$-injector of H.*

Proof. Let $\{N_\lambda : \lambda \in \Lambda\}$ be the set of all finite normal subgroups of G. By Lemma 3.1, $V \cap N_\lambda$ is an $\mathfrak{F}$-injector of N_λ. Since the class of finite $\mathfrak{F}$-groups is a Fitting class of finite soluble groups and $V \cap N_\lambda \leqq H \cap N_\lambda \leqq N_\lambda$, $V \cap N_\lambda$ is an $\mathfrak{F}$-injector of $H \cap N_\lambda$. It now follows as in the proof of Theorem 3.2 that $V = \bigcup_{\lambda \in \Lambda} (V \cap N_\lambda)$ is an $\mathfrak{F}$-injector of $H = \bigcup_{\lambda \in \Lambda} (H \cap N_\lambda)$.

A subgroup A of a group G is said to be *pronormal* in G if, for each $x \in G$, A and A^x are conjugate in $\mathrm{gp}\{A, A^x\}$.

THEOREM 3.5. *Let $\mathfrak{F}$ be a Fitting class of G-groups. If V is an $\mathfrak{F}$-injector of the $\mathfrak{G}$-group G, then V is pronormal in G.*

Proof. If $x \in G$, then V^x is an $\mathfrak{F}$-injector of G and so, by Theorem 3.4, V and V^x are $\mathfrak{F}$-injectors of $H = \mathrm{gp}\{V, V^x\}$. Therefore there is a locally inner automorphism ϕ of H such that $V\phi = V^x$ (Theorem 3.3). But $H \leqq \mathrm{gp}\{V, x\}$, and so V has finite index in H. It follows that $V\phi = V^h$ for some $h \in H$, so that V and V^x are conjugate in H.

4. Conjugacy of $\mathfrak{F}$-injectors. If A is a subgroup of the group G, then the *local conjugacy class* containing A is the set of all subgroups of G which are locally conjugate to A, and is denoted by $\mathrm{Lcl}(A)$. Similarly the conjugacy class containing A is denoted by $\mathrm{Cl}(A)$.

We showed in [7] that, with suitable conditions on A, $\mathrm{Lcl}(A) = \mathrm{Cl}(A)$ if and only if $\mathrm{Cl}(A)$ is finite. The main restriction on A was that, for any set $\{H_\lambda : \lambda \in \Lambda\}$ of normal subgroups of G, $\bigcap_{\lambda \in \Lambda} (AH_\lambda) = A(\bigcap_{\lambda \in \Lambda} H_\lambda)$. This condition is not satisfied by the $\mathfrak{F}$-injectors, and so we prove a result similar to that in [7] with different conditions on the subgroup A. The proof of this result is essentially the same as that given by M. I. Kargapolov [3] for the case in which A is a Sylow p-subgroup, but, as Kargapolov's result is not available in translation, it seems worth giving the details in full.

THEOREM 4.1. *Let A be a subgroup of the periodic FC-group G satisfying the following conditions*:

(i) *A is pronormal in G; and*

(ii) *for each finite normal subgroup N of G and for each $x \in G$, $A \cap N$ and $A^x \cap N$ are conjugate in N.*

Then $\mathrm{Lcl}(A) = \mathrm{Cl}(A)$ *if and only if* $\mathrm{Cl}(A)$ *is finite.*

Proof. It has already been shown in [7] that, for any subgroup A with $\mathrm{Cl}(A)$ finite, $\mathrm{Lcl}(A) = \mathrm{Cl}(A)$. It remains to prove that if $\mathrm{Cl}(A)$ is infinite, then $\mathrm{Lcl}(A) \neq \mathrm{Cl}(A)$.

We shall define inductively the following ascending chains of subgroups:

$$1 = N_0 < N_1 < \ldots < N_i < \ldots, \tag{1}$$

where each N_i is a finite normal subgroup of G,

$$1 = \bar{A}_0 < \bar{A}_1 < \ldots < \bar{A}_i < \ldots, \tag{2}$$

$$1 = A_0^* < A_1^* < \ldots < A_i^* < \ldots, \tag{3}$$

where, for each integer $i > 0$, $\bar{A}_i$ and A_i^* are subgroups of G which are conjugate to $N_i \cap A$ and which satisfy the condition

$$|\bar{A}_i : \bar{A}_i \cap A_i^*| = n_i,$$

where

$$1 = n_0 < n_1 < \ldots < n_i < \ldots. \tag{4}$$

By Lemma 2.2 of [5], $\bigcup_{i=0}^{\infty} \bar{A}_i$ and $\bigcup_{i=0}^{\infty} A_i^*$ are locally conjugate to $\bigcup_{i=0}^{\infty} (N_i \cap A)$ and so are contained in subgroups $\bar{A}$ and A^*, respectively, which are locally conjugate to A.

$N_i \cap A^*$ is conjugate to $N_i \cap A$ and contains A_i^* Therefore $N_i \cap A^* = A_i^*$ and so

$$\begin{aligned} \bar{A}_i \cap (\bar{A} \cap A^*) &= \bar{A}_i \cap A^* \\ &= \bar{A}_i \cap N_i \cap A^* \\ &= \bar{A}_i \cap A_i^*. \end{aligned}$$

Therefore

$$|\bar{A} : \bar{A} \cap A^*| \geqq |\bar{A}_i : \bar{A}_i \cap A_i^*| = n_i, \qquad \text{for all } i.$$

Thus $|\bar{A} : \bar{A} \cap A^*|$ is infinite and so $\bar{A}$ and A^* cannot both be conjugate to A ([7], Lemma 2.3). Therefore $\mathrm{Lcl}(A) \neq \mathrm{Cl}(A)$.

To construct the chains (1), (2) and (3), we assume that we have constructed the first k terms of each chain and show that subgroups N_{k+1}, $\bar{A}_{k+1}$ and A_{k+1}^* may be defined such that N_{k+1} is a finite normal subgroup of G containing N_k, $\bar{A}_{k+1} > \bar{A}_k$, $A_{k+1}^* > A_k^*$, $\bar{A}_{k+1}$ and A_{k+1}^* are conjugate to $A \cap N_{k+1}$ and $|\bar{A}_{k+1} : \bar{A}_{k+1} \cap A_{k+1}^*| > n_k$.

Let $\{N_\lambda : \lambda \in \Lambda\}$ be the set of all finite normal subgroups of G containing N_k. If $\bar{A}$ and A^* are subgroups locally conjugate to A and containing $\bar{A}_k$ and A_k^* respectively, then, as above, $N_k \cap \bar{A} = \bar{A}_k$ and $N_k \cap A^* = A_k^*$

Thus, for each $\lambda \in \Lambda$,

$$\bar{A}_k \cap (\bar{A} \cap N_\lambda \cap A^*) = \bar{A}_k \cap A_k^*.$$

Therefore

$$|\bar{A} \cap N_\lambda : \bar{A} \cap N_\lambda \cap A^*| \geqq |\bar{A}_k : \bar{A}_k \cap A_k^*| = n_k,$$

for all $\lambda \in \Lambda$ and for all $\bar{A}, A^*$ satisfying

$$\bar{A} \geqq \bar{A}_k, \quad A^* \geqq A_k^*, \quad \bar{A} \text{ and } A^* \text{ are members of } \mathrm{Lcl}(A). \tag{5}$$

Suppose, if possible, that

$$|\bar{A} \cap N_\lambda : \bar{A} \cap N_\lambda \cap A^*| = n_k, \tag{6}$$

for all $\lambda \in \Lambda$ and for all $\bar{A}$ and A^* satisfying (5).

Now

$$\mathrm{gp}\{\bar{A}_k, \bar{A} \cap N_\lambda \cap A^*\} \leqq \bar{A} \cap N_\lambda$$

and so

$$|\mathrm{gp}\{\bar{A}_k, \bar{A} \cap N_\lambda \cap A^*\} : \bar{A} \cap N_\lambda \cap A^*| \leqq n_k.$$

But clearly

$$|\mathrm{gp}\{\bar{A}_k, \bar{A} \cap N_\lambda \cap A^*\} : \bar{A} \cap N_\lambda \cap A^*| \geqq |\bar{A}_k : \bar{A}_k \cap A_k^*| = n_k.$$

Therefore

$$|\mathrm{gp}\{\bar{A}_k, \bar{A} \cap N_k \cap A^*\} : \bar{A} \cap N_\lambda \cap A^*| = |\bar{A} \cap N_\lambda : \bar{A} \cap N_\lambda \cap A^*|$$

and so

$$\mathrm{gp}\{\bar{A}_k, \bar{A} \cap N_\lambda \cap A^*\} = \bar{A} \cap N_\lambda, \tag{7}$$

for all $\lambda \in \Lambda$ and for all $\bar{A}, A^*$ satisfying (5).

By Lemma 2.4 of [7],

$$|A^* \cap N_\lambda : \bar{A} \cap N_\lambda \cap A^*| = |\bar{A} \cap N_\lambda : \bar{A} \cap N_\lambda \cap A^*| = n_k$$

and so we may prove, corresponding to (7), that

$$\mathrm{gp}\{A_k^*, \bar{A} \cap N_\lambda \cap A^*\} = A^* \cap N_\lambda, \tag{8}$$

for all $\lambda \in \Lambda$ and for all $\bar{A}, A^*$ satisfying (5).

It now follows from (7) and (8) that

$$\begin{aligned} N_k \bar{A} \cap N_\lambda &= N_k(\bar{A} \cap N_\lambda) \\ &= N_k(\bar{A} \cap N_\lambda \cap A^*) \\ &= N_k(A^* \cap N_\lambda) \\ &= N_k A^* \cap N_\lambda, \end{aligned}$$

for all $\lambda \in \Lambda$ and for all $\bar{A}, A^*$ satisfying (5).

Thus

$$N_k \bar{A} = \bigcup_{\lambda \in \Lambda} (N_k \bar{A} \cap N_\lambda) = \bigcup_{\lambda \in \Lambda} (N_k A^* \cap N_\lambda) = N_k A^*, \tag{9}$$

for all $\bar{A}$ and A^* satisfying (5).

Let $x \in G$; then, by (ii), $N_k \cap \bar{A}^x$ is conjugate to $N_k \cap \bar{A}$ in N_k, and so there is an element $n \in N_k$ such that

$$\bar{A}^{xn} \geqq N_k \cap \bar{A} = A_k.$$

It follows from (9) that

$$N_k \bar{A}^{xn} = N_k \bar{A},$$

i.e. that

$$N_k \bar{A}^x = N_k \bar{A}.$$

Thus $N_k A \lhd G$.

For each $x \in G$, $A^x \leqq N_k A$ and hence, by (i), A^x is conjugate to A in $N_k A$. But $|N_k A : A|$ is finite, so that A has only a finite number of conjugates in $N_k A$. Thus A has only a finite number of conjugates in G, contrary to our initial hypothesis.

We have shown that our assumption (6) leads to a contradiction, and so there must be a normal subgroup N_λ $(\lambda \in \Lambda)$ and two subgroups $\bar{A}$ and A^* satisfying (5) such that

$$|\bar{A} \cap N_\lambda : \bar{A} \cap N_\lambda \cap A^*| > n_k.$$

If we define

$$N_{k+1} = N_\lambda, \quad \bar{A}_{k+1} = \bar{A} \cap N_\lambda, \quad A^*_{k+1} = A^* \cap N_\lambda, \quad n_{k+1} = |\bar{A}_{k+1} : \bar{A}_{k+1} \cap A^*_{k+1}|,$$

then we have defined the $(k+1)$th terms of the series (1), (2) and (3). This completes the proof of Theorem 4.1.

COROLLARY 4.2. *Let $\mathfrak{F}$ be a Fitting class of $\mathfrak{G}$-groups. Then the $\mathfrak{F}$-injectors of the $\mathfrak{G}$-group G are conjugate in G if and only if there is only a finite number of them.*

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https://doi.org/10.1017/S0017089500000689 Published online by Cambridge University Press