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# THE ASSOCIATIVE PART OF A CONVERGENCE DOMAIN IS INVARIANT

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Of special interest in summability theory are those conservative matrices possessing the "mean-value property". If $c_A=\{x: Ax\in c\}$ denotes the convergence domain of a conservative matrix $A$, then $A$ has the *mean-value property* in case, for each $x$ in $c_A$, there exists $M=M(A, x)>0$ such that

$$(1)\qquad \left|\sum_{k=1}^{r} a_{nk}x_k\right|\le M,\quad \forall n, r = 1, 2, \ldots.$$

This property has been considered by many writers and has been shown, among other things, to be equivalent to the requirement that the matrix be *associative*, i.e., for each $x$ in $c_A$,

$$(2)\qquad (tA)x \equiv \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} t_n a_{nk}\right)x_k$$

converges for each sequence $\{t_n\}$ in $\ell_1$. The two properties were generalized by Wilansky in [1], where he considers the subspaces $B$ and $L$ of $c_A$ having the corresponding properties. Thus

$$(3)\qquad \begin{aligned} B &= \{x\in c_A: \left|\sum_{k=1}^{r} a_{nk}x_k\right| \le M(A, x)\quad \forall n, r = 1, 2, \ldots\};\\ L &= \{x\in c_A: (tA)x \quad \text{exists}\quad \forall t\in \ell_1\}.\end{aligned}$$

It is shown in [1] that $B=L$ and the question is raised (question III, p. 348) as to whether or not $L$ is *invariant*, i.e., if $D$ is a matrix for which $c_A=c_D$, is $L_A=L_D$? Several conditions (pp. 338–339) are given for which this is so, i.e., for which $L$ can be expressed in a form depending only on the *FK* space $c_A$ and not the matrix $A$.

The purpose of this note is to point out that $L$ is always invariant and, consequently, to answer question III of [1] affirmatively.

LEMMA. *For any matrix $A$,*

$$(4)\qquad B = \left\{x\in c_A: \left\{\sum_{k=1}^{m} x_k\delta^k\right\}_{m=1}^{\infty} \text{ is a bounded set in } c_A\right\}.$$

**Proof.** The seminorms $p$ generating the *FK* topology of $c_A$ are of three types:

(i) $P_n(x)=|x_n|;\quad n=1, 2, \ldots$

(ii) $h_n(x)=\sup_r \left|\sum_{k=1}^{r} a_{nk}x_k\right|; n=1, 2, \ldots$

(iii) $q(x)=\sup_n |(Ax)_n|$



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If $x$ is in $B$ and we let $x^{(m)}=\sum_{k=1}^{m} x_k\delta^k$, then we need only show that $\{p(x^{(m)})\}_{m=1}^{\infty}$ is a bounded set for each of the generating seminorms. If, for example, $p=h_n$ for some $n$, then

$$p(x^{(m)}) = h_n(x^{(m)}) = \sup_r \left|\sum_{k=1}^{r} a_{nk}\, x_k^{(m)}\right|$$

$$= \sup_{1\le r\le m}\left|\sum_{k=1}^{r} a_{nk}x_k\right| \le \sup_r\left|\sum_{k=1}^{r} a_{nk}x_k\right| \le M, \quad \forall m = 1,2,\ldots$$

Similarly $p(x^{(m)})\le M$ if $p$ is of type (i) or (iii).

Conversely, if $\{x^{(m)}\}_{m=1}^{\infty}$ is bounded, then $q(x^{(m)})\le M$, $m=1,2,\ldots$ for some $M>0$. However,

$$q(x^{(m)}) = \sup_n\left|\sum_{k=1}^{\infty} a_{nk}x_k^{(m)}\right|$$

$$= \sup_n\left|\sum_{k=1}^{m} a_{nk}x_k\right|$$

Thus, for each $m$, $n=1,2,\ldots$, we have

$$\left|\sum_{k=1}^{m} a_{nk}x_k\right| \le M,$$

and $x$ is in $B$.

It follows that $L$ is invariant, since a convergence domain has precisely one *FK* topology.

**Added in Proof.** The above problem has been solved independently by Grahame Bennett, *Distinguished subsets and summability invariants*, (to appear).

## Reference

1. A. Wilansky, *Distinguished subsets and summability invariants*, J. Analyse Math. **12** (1964), 327–350.

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