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PARITY BIAS IN FUNDAMENTAL UNITS OF REAL QUADRATIC FIELDS

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Abstract

We compute primes $p \equiv 5 \bmod 8$ up to 10^{11} for which the Pellian equation $x^2 - py^2 = -4$ has no solutions in odd integers; these are the members of sequence A130229 in the Online Encyclopedia of Integer Sequences. We find that the number of such primes $p \leqslant x$ is well approximated by

$$\frac{1}{12}\pi(x) - 0.037 \int_2^x \frac{dt}{t^{1/6}\log t},$$

where $\pi(x)$ is the usual prime counting function. The second term shows a surprising bias away from membership of this sequence.

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1. Introduction

For a prime $p \equiv 5 \bmod 8$, consider the real quadratic field $K = \mathbb{Q}(\sqrt{p})$, with ring of integers $\mathcal{O}_K = \mathbb{Z}[\frac{1}{2}(1+\sqrt{p})]$ and fundamental unit $\varepsilon_p = \frac{1}{2}(x_0 + y_0\sqrt{p}) > 1$. Then, (x_0, y_0) is a fundamental solution to the Pellian equation

$$x^2 - py^2 = -4. \tag{1.1}$$

The prime 2 is inert in $K/\mathbb{Q}$, and $\varepsilon_p \equiv 1 \bmod 2\mathcal{O}_K$ if and only if (1.1) has no odd integer solutions. Primes $p \equiv 5 \bmod 8$ satisfying the above equivalent conditions define sequence A130229 in [5]. They also appear in [1, 2, 7].

Since $\varepsilon_p \bmod 2\mathcal{O}_K$ can take any of three nonzero values in $\mathcal{O}_K/2\mathcal{O}_K \cong \mathbb{F}_4$, it is reasonable to expect roughly one third of all primes $p \equiv 5 \bmod 8$ to be members of this sequence.

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Define

$$\chi(p) := \begin{cases} 1 & \text{if } p \equiv 5 \bmod 8 \text{ and } \varepsilon_p \equiv 1 \bmod 2\mathcal{O}_K, \\ -\frac{1}{2} & \text{if } p \equiv 5 \bmod 8 \text{ and } \varepsilon_p \not\equiv 1 \bmod 2\mathcal{O}_K, \\ 0 & \text{if } p \not\equiv 5 \bmod 8, \end{cases}$$

and define the modified counting function

$$\theta_\chi(x) := \sum_{p \leqslant x} \chi(p) \log p.$$

Then, the above heuristic leads us to expect $\theta_\chi(x) = o(x)$ as $x \to \infty$.

In this note, we report on computations of $\theta_\chi(x)$ for $x \leqslant 10^{11}$, which show a surprising bias away from the $\varepsilon_p \equiv 1 \bmod 2\mathcal{O}_K$ case, hinted at in related computations reported in [1, Section 4]. We thus pose a conjecture.

Conjecture 1.1. There exists a constant $c \approx -0.066$ for which

$$\theta_\chi(x) \sim cx^{5/6}$$

as $x \to \infty$.

2. Results

We computed ε_p using the continued fraction method in [3, Section 3.3], with the modification that B_i and G_i are only computed modulo 2, since we only need to know the parity of ε_p. This significantly reduces the memory requirements of the calculation.

We implemented the algorithm to run on a GPU using the Python `Numba` library [4]. The final computation for all $p < 10^{11}$ took approximately 17 hours on an entry-level gaming laptop with an Nvidia RTX 3050 GPU. The source code and data are available at https://github.com/florianbreuer/A130229.

Table 1 lists some values for the naive counting function

$$\pi_1(x) = \sum_{p \leqslant x,\, \chi(p)=1} 1.$$

However, it is advantageous to study the 'smoothed' counting function $\theta_\chi(x) = \sum_{p \leqslant x} \chi(p) \log p$. Figure 1 plots $-\theta_\chi(x)$ for $x \leqslant 10^{11}$ on logarithmic axes. The plot approximates a straight line with slope $5/6$. The least squares best fit of the form $f(x) = cx^{5/6}$ is found to have $c \approx -0.06626$, computed using the `find_fit` method in SageMath v9.3 [6]. The error term $\theta_\chi(x) - cx^{5/6}$ is shown in Figure 2. This provides evidence for Conjecture 1.1. Moreover, it appears likely that the error is of the order $O(x^{1/2+\varepsilon})$.

From this, we may also deduce a good approximation for $\pi_1(x)$. Define

$$\pi_{-1/2}(x) = \sum_{p \leqslant x,\, \chi(p)=-1/2} 1 \quad \text{and} \quad \pi_\chi(x) = \sum_{p \leqslant x} \chi(p) = \pi_1(x) - \frac{1}{2}\pi_{-1/2}(x).$$

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TABLE 1. Some values of the counting function $\pi_1(x)$ for sequence A130229.

x	$\pi_1(x)$	Approximation	x	$\pi_1(x)$	Approximation
10^2	1	1	2×10^{10}	72770931	72761719
10^3	15	11	3×10^{10}	107298975	107293481
10^4	98	90	4×10^{10}	141363308	141357259
10^5	741	735	5×10^{10}	175085540	175080418
10^6	6200	6187	6×10^{10}	208542967	208537579
10^7	53382	53348	7×10^{10}	241775700	241776120
10^8	468223	468144	8×10^{10}	274823028	274829667
10^9	4164936	4165422	9×10^{10}	307723656	307723171
10^{10}	37490293	37483463	10^{11}	340472393	340476359

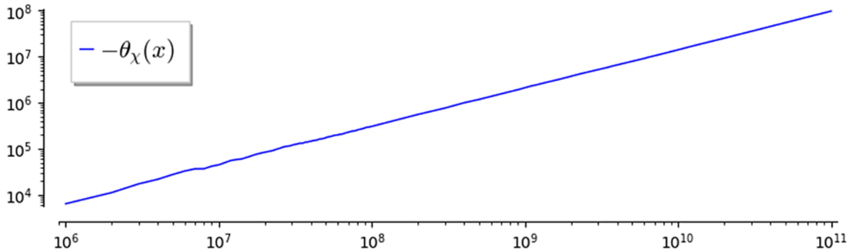

FIGURE 1. Log-log plot of $-\theta_\chi(x)$ for $x \leqslant 10^{11}$.

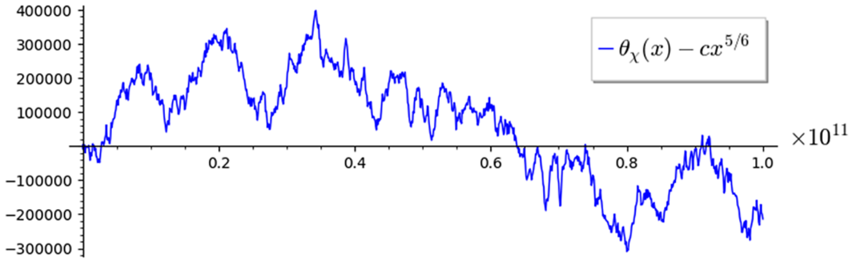

FIGURE 2. Plot of the error term $\theta_\chi(x) - cx^{5/6}$ for $c \approx -0.06626$.

Then, $\theta_\chi(x) \sim cx^{5/6} \approx c \cdot \frac{5}{6}\int_2^x t^{-1/6}\,dt$ suggests

$$\pi_\chi(x) \approx c \cdot \frac{5}{6}\int_2^x \frac{t^{-1/6}}{\log t}\,dt \sim c\,\frac{x^{5/6}}{\log x}.$$

Then, from $\pi_1(x) + \pi_{-1/2}(x) \approx \frac{1}{4}\pi(x)$, where $\pi(x)$ is the usual prime counting function, we arrive at

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$$\pi_1(x) \approx \frac{1}{12}\pi(x) + \frac{2}{3}\pi_\chi(x) \approx \frac{1}{12}\pi(x) + c \cdot \frac{5}{9}\int_2^x \frac{t^{-1/6}}{\log t}\,dt.$$

These approximations are compared with the computed values of $\pi_1(x)$ in Table 1.

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