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COMPUTABLE TOPOLOGICAL GROUPS

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Abstract. We investigate what it means for a (Hausdorff, second-countable) topological group to be computable. We compare several potential definitions based on classical notions in the literature. We relate these notions with the well-established definitions of effective presentability for discrete and profinite groups, and compare our results with similar results in computable topology.

§1. Introduction.

1.1. A brief overview. Our paper contributes to a fast developing branch of effective mathematics which combines methods of computable algebra [1, 8, 9] with tools of computable analysis [4, 39, 47] to study computable presentations of topological spaces and topological groups (e.g., [12, 25, 28, 30, 38]). Whenever a theory emerges, one of the first tasks is to compare the most basic definitions arising and see which ones are equivalent. In the context of algorithmic group theory, one of the most well-known examples of such a separation result is the celebrated work of Novikov [34] and Boone [3] who proved that not every finitely presented group has decidable Word Problem. Such investigations often lead to a deeper understanding of the various notions of presentability that are being studied in that area. For instance, in his search for a more elegant proof of the Novikov–Boone theorem, Higman [16] discovered that "recursively presented" groups are exactly the subgroups of finitely presented groups, thus characterising one notion of presentability for groups in terms of another.

In the present paper we prove several *positive* characterization-type results that are fundamental to the emerging theory of computable topological groups. As the main result of the paper, we prove that in the locally compact and the abelian cases, a seemingly weak notion of effective topological presentability is in fact equivalent to the apparently much stronger notion of right-c.e. Polish presentability (up to topological group-isomorphism). The second main result of the paper further improves the result, but under the extra assumption of effective local compactness. All of these terms will be defined in the paper. Finally, we also support these positive results with counter-examples that separate several notions of computable presentability up to topological group isomorphism. These counter-examples and

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their proofs relate our notions with the aforementioned recursive groups studied by Higman [16], computable groups as defined by Maltsev [26] and Rabin [40], and with recursive profinite groups investigated by Metakides and Nerode [32]. Indeed, we will see that in many important cases some of these definitions turn out to be equivalent.

Before we formally state our results, we give a bit more background and briefly discuss the related literature.

1.2. Notions of computable presentability. Maltsev [26], Rabin [40], Higman [16], Metakides and Nerode [32], and others (e.g., [12, 25, 28, 30, 38]) suggested various notions of computability for various classes of groups. The most well-established notions of computable presentability of groups are restricted to discrete and profinite groups, as we discuss below.

We have already mentioned the notion of a "recursive presentation" [16] which is standard in combinatorial group theory. Since the term "recursive presentation" can mean many different and non-equivalent notions (as we shall see shortly), we instead call such presentations *computably enumerable* (c.e.), or Σ^0_1. These presentations are groups of the form F_ω/H, where F_ω is the standard reduced-word presentation of the free group upon ω-many generators and H is its computably enumerable normal subgroup. The equality relation modulo H, also known as the "Word Problem," does not have to be computable even in a finitely presented group [3, 34]. Perhaps motivated by this early fundamental result, Mal'cev [26] and Rabin [40] suggested a stronger notion of computable presentability for a group. In the notation above, a group is *computably presented* if it is isomorphic to F_ω/H, where H is a computable normal subgroup of F_ω. In other words, a group is computable if it is c.e. presented and the Word Problem is decidable in the presentation. Equivalently, a countably infinite group is computable if its domain is a computable set and the group operations are represented by computable functions upon elements of the set. This notion can be extended to an arbitrary discrete algebraic structure in the obvious way. The notion is well-established and is central to the technically deep theory of computable algebraic structures (see [1, 9]).

From the perspective of computable presentability, the second most well-understood class is the class of profinite groups. Metakides and Nerode [32] and then La Roche [41, 42] and Smith [43, 44] studied "recursively presented" and "co-r.e.-presented" profinite groups. To define computability in this class we just need to say that some inverse system of finite groups representing the group is effective in a certain sense; we omit the definitions here. Among other results, they proved the effective versions of Galois correspondence that relate recursive and co-r.e. presentations with computable and computably enumerable field extensions, respectively. Another duality established in the cited papers is a Stone-type duality between recursive groups and decidable classes in 2^ω, and between co-r.e. presented groups and Π^0_1 classes in 2^ω. In both cases, the classes are also equipped with group operations. The recent paper [27] proves that the recursive profinite abelian groups are exactly the Pontryagin duals of computable discrete torsion abelian groups. There are many ways to apply these dualities to establish that co-r.e. presentability is strictly weaker than recursive presentability for profinite groups.

One of the characteristic features of the results discussed above is that *the notions of computability in the profinite and the discrete case are interconnected and related*

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via dualities of various kinds. Are these two subjects just pieces of a bigger puzzle? More specifically, can we develop a general theory of computable topological groups that is not restricted to the profinite and discrete case? In the present paper, we are mainly interested in Polish groups but some of the technical results proven here also work for Hausdorff second-countable groups. Thus, for the remaining of the paper we will adopt the convention:

All our groups and spaces are Hausdorff and second countable.

Indeed, we are mainly interested in locally compact groups, and it well-known that every Hausdorff and second countable locally compact space is Polish.

Following the analogy with the discrete and profinite cases, a computable topological group should mean a *computable space* together with *computable group operations*. The situation becomes much more complex when we consider groups which are neither profinite nor discrete. Unfortunately, even in the nice case of a Polish(able) space, it is not even clear what "computable space" should mean exactly. Computable topology is notorious for its zoo of different notions of computability for a topological space (and a topological group). In contrast with effective algebra [1, 9] where all standard notions of computable presentability in common classes had been separated more than half a century ago (e.g., Novikov [34], Boone [3], Feiner [10], Khisamiev [20], and Odintsov and Selivanov [35]), some of the basic notions of computable presentability in topology have been separated recently [2, 15, 17, 25, 29]. Arranged from strong to weak, some common notions of effective presentations for a Polish space are as follows:

$$\text{computably compact} \longrightarrow \text{computable Polish} \longrightarrow \text{right-c.e Polish} \longrightarrow \text{computable topological.}$$

All these notions will be formally defined in the preliminaries and also briefly discussed below; we only mention here that the first one clearly works only for compact spaces, and that all implications are known to be *strict up to homeomorphism*, as established in [2, 6, 17, 25, 29]. This means that, in each case, there is a space that is effectively presentable in the weaker sense but is not homeomorphic to any space effectively presentable in the stronger sense.[1]

As discussed above, a "computable topological group" should mean "computable topological space" + "computable group operations," where computability of operations is defined as usual in terms of approximations, i.e., through the use of effective operators. Thus each of the four notions above leads to a potentially different definition of a computable topological group. Up to topological group isomorphism, are these four notions equivalent? How are these notions related to the well-established approach in the discrete case? What about the profinite case? We will answer these questions shortly. But first, we clarify and compare the notions above with respect to topological *spaces*.

[1] For the last implication see [29], for right-c.e. vs. computable Polish we cite [2], and for the first implication see [6, 17, 25]. Also, for the closely related notion of a left-c.e. Polish space which will not be used in this paper, we cite [29].

There are several variations of the definition of a computable topological space that can be found in, e.g., [19, 45]. We will use the following, perhaps the *weakest possible*, approach. A *computable topological presentation* is given by a countable base of topology $(B_i)_{i\in\omega}$ consisting of non-empty basic open sets together with the c.e. set W that allows one to list intersections in the following weak sense:

$$B_i \cap B_j = \bigcup\{B_k : (i, j, k) \in W\}.$$

The standard examples of computable topological spaces include right-c.e. Polish spaces that will be defined shortly. However note that the definition of a computable topological space is *point-free*. In fact, one can simply define a computable topological presentation as a c.e. set of triples (i, j, k) and do not contain any information about the points in the space they represent. Thus it is entirely possible that a single computable topological presentation can represent non-homeomorphic spaces; for instance the unit circle and the unit open interval share the same fixed computable topological presentation. In fact this strange feature can be exploited to show that there is a Polish space that is computable topological but is not homeomorphic to any right-c.e. Polish space [29]. Indeed, it follows from the simple proof in the companion paper [29] that in general, *for a computable topological locally compact (Polish) space there is no bound on the complexity of its Polish presentation, up to homeomorphism.* Nonetheless, the notion of a computable topological space is rather popular in the literature, and authors often consider additional assumptions since the basic notion is very weak.

The much stronger classical notion of a *computable Polish space* can be traced back to Ceitin [5] and Moschovakis [33]. We say that a Polish space is computable Polish if there is a countable dense subset $(x_i)_{i\in\omega}$ and a complete metric d compatible with the topology such that $d(x_i, x_j)$ can be uniformly computed to precision 2^{-n}. If we only require that the real $d(x_i, x_j)$ can be effectively approximated from above by enumerating its right cut then we get the notion of a right-c.e. Polish space. These are also known as upper-semicomputable Polish spaces. The reason why the right-c.e. case is so important in the literature is because it is *the* standard example of a computable topological space. Also, it is known that Stone duality associates effectively compact (to be defined) right-c.e. Stone spaces with c.e. presented Boolean algebras [2]. In particular, it follows from results in [15, 17] and the aforementioned classical result of Feiner [10] that there is a right-c.e. Polish space not homeomorphic to a computable Polish space; see [2] for a detailed explanation.

Finally, we say that a space is computably (or effectively) compact if it is computable Polish and additionally, we can effectively list all finite basic open covers of the space. It has been proven in [6, 17, 25] that there are compact spaces that are computable Polish but not homeomorphic to any computably compact space. Interestingly, the proof in [25] builds a connected compact *group* with this property. It follows from the proof in [25] that there is a connected compact abelian *group* that has a computable Polish presentation (as a group, i.e., in which the operations are also computable), but so that its underlying *space* is not homeomorphic to any computably compact space. Thus, at least one implication is known to be strict for groups. The notion of computable compactness is restricted to compact spaces and therefore will not be too important to us. Instead we will consider the natural generalisation to locally compact spaces.

1.3. Results. We are now ready to discuss our results. Recall that in [29] we illustrate that there is a computable topological locally compact Polish *space* that is not homeomorphic to any arithmetical (or even analytical, and beyond) Polish space, let alone a right-c.e. Polish space. In stark contrast, the principal result of the present paper shows that, up to topological group-isomorphism, these two notions of presentability are equivalent for groups:

THEOREM 1.1. *For a Polish group G that is either abelian or locally compact, the following are equivalent*:

(1) *G has a computable topological presentation*.
(2) *G has a right-c.e. Polish presentation*.

Furthermore, in (2) *the metric can be taken to be left-invariant* (*or right-invariant*).

The implication (2) $\rightarrow$ (1) is obvious, but (1) $\rightarrow$ (2) is non-trivial and surprising. This unexpectedness of our result is due to the fact that, as per our discussion in the previous subsection, a computable topological presentation is *point-free* and therefore extremely undiscriminating. Nevertheless for groups, the concept of a point-free computable presentation is enough to recover a right-c.e. Polish presentation in these two important cases.

It should not be surprising that one of the crucial steps in the proof is a new effective version of the classical Birkhoff–Kakutani metrization theorem; this is Theorem 3.2. The proof of the effective version of the Birkhoff–Kakutani theorem requires much care since we use the weak point-free definition of a computable topological presentation. Even more care is needed to reconstruct the dense sequence from the point-free effective topology; this is Theorem 4.1. We shall also explain why in the locally compact and in the abelian cases the metric produced in Theorem 3.2 is complete; this is not obvious at all, but several classical results from topological group theory will come to our aid. The proof of Theorem 1.1 is spread through the paper; for the abelian case see Corollary 4.7, and for the locally compact case see Corollary 5.3.

In Corollary 6.3 we also show that Theorem 1.1 is *sharp* in the sense that, in general, we cannot produce a computable metric. Our counter-example is a discrete abelian group that admits a right-c.e. Polish copy but is not topologically isomorphic to any computable Polish group. In Section 6 we also illustrate that, in the discrete case, computable Polish presentability is equivalent to computable presentability in the sense of Mal'cev [26] and Rabin [40]. We also mentioned earlier that, for compact groups, computable Polish does not imply computable compact up to topological isomorphism. Combined with Theorem 1.1, these results imply that for abelian and for locally compact Polish groups and up to topological group isomorphism, the diagram looks like:

$$\begin{array}{c}
\text{computably compact} \\
\downarrow \not\uparrow \\
\text{computable Polish} \\
\downarrow \not\uparrow \\
\text{right-c.e Polish} \\
\downarrow \uparrow \\
\text{computable topological.}
\end{array}$$

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Clearly, for the top implication the counter-examples are compact; such examples can be found among connected compact groups [25] (as discussed above) and also among profinite groups [6].

We now discuss the second main result of the present article. One of the two cases in Theorem 1.1 is when the group is locally compact. Struble [46] showed that a Hausdorff, second countable locally compact topological group admits a compatible left-invariant *proper* metric; recall that a metric is proper if every closed bounded set is compact. The natural question is: Can we improve Theorem 1.1 and prove an effective version of Struble's result in the locally compact case?

In the literature, most *effective* arguments that involve the use of compactness assume that the space satisfies a version of *effective* compactness. One such notion we have already mentioned above. Remarkably, many definitions of effective compactness in the literature turn out to be equivalent; see [6, 18, 37]. We will formally define and discuss the notion of effective compactness that we use in the preliminaries section. Roughly speaking, a set is effectively compact if we can effectively list all of its covers by finitely many basic open balls.

To have access to *local* compactness, we need to generalize this notion to locally compact spaces. There are several definitions in the literature [37, 49, 50]. We shall not attempt to compare these definitions up to homeomorphism; we suspect that they are non-equivalent for groups. The notion suggested in [37] seems most suitable for our purpose. Roughly speaking, it is a direct effectivisation of the classical notion of local compactness that says that every point is contained in a compact neighbourhood. We will define it formally in the preliminaries.

We are ready to state the second main result of the paper:

THEOREM 1.2. *Every effectively locally compact computable topological group admits a proper right-c.e. Polish presentation in which the metric is left-invariant and proper.*

Furthermore, the metric produced in the theorem is itself effectively locally compact and indeed *effectively proper* in the sense that will be clarified in the preliminaries. We also point out that in both Theorem 1.1 and Theorem 1.2, the right-c.e. Polish presentations that we build are *computably homeomorphic* (indeed, *effectively compatible*) to the given computable topological presentation of the group. These standard notions will be clarified later. Theorem 1.2 will be derived as a corollary of a rather general technical result Theorem 5.4 combined with Theorem 1.1; see Corollary 5.6.

We do not know whether the assumption of effective local compactness can be dropped from Theorem 1.2. However, similarly to Theorem 1.1, we do know that the result cannot be improved to give a computable proper metric. This is essentially due to the fact that the aforementioned Corollary 6.3 actually gives a *discrete* example. Since having a discrete example is perhaps not particularly exciting, in Proposition 6.4 we produce an example of a profinite group that is right-c.e. Polish and effectively compact, but is not homeomorphic to any effectively compact computable Polish group. Note that in the compact case every metric is automatically proper.

The plan is as follows. Section 2 is a preliminaries section. In Section 3 we prove the effective version of the classical Birkhoff–Kakutani theorem. Section 4 contains a technical result that allows to reconstruct an effective dense sequence of points in the group; in this section we also prove the abelian case of the first main result. Section 5 is concerned with the locally compact case; it contains the proofs of the locally compact clause of the first main result and the proof of the second main result. Section 6 contains results concerning discrete and profinite groups, where the notions of effective presentability for such groups from the literature are characterized in terms of computable Polish and right-c.e. presentations. Section 7 contains a further discussion of various notions of effective presentability for groups in the literature, most notably for totally disconnected non-compact groups, and how these notions are related to the notions studied in the paper. Finally, Section 8 is a brief and informal conclusion.

§2. Preliminaries.

2.1. Computable topological spaces. As we already stated in the introduction, *we assume that all our topological spaces and groups are Hausdorff and second-countable.* The definition below is central to the paper.

Definition 2.1 (See, e.g., Definition 2.1 of [24] and Definition 4 of [48]). A *computable topological space* is given by a computable, countable basis of its topology for which the intersection of any two basic open sets ("basic balls") can be uniformly computably listed. More formally, it is a tuple (X, τ, β, ν) such that:

- β is a base of τ consisting of non-empty sets,
- $\nu : \omega \to \beta$ is a computable surjective map, and
- there exists a c.e. set W such that for any $i, j \in \omega$,

$$\nu(i) \cap \nu(j) = \bigcup\{\nu(k) : (i, j, k) \in W\}.$$

We say that a topological space has a *computable topological presentation* if it is homeomorphic to a computable topological space (with is of course called a computable topological presentation of the space).

Remark 2.2. We establish the convention at this point. In Definition 2.1, we have opted for a basis consisting of non-empty sets. However, some authors allow basis sets to be empty while requiring the collection of non-empty basic open sets to be computably enumerable (c.e.). It is worth noting that assuming the space is not empty, these two definitions are indeed equivalent.

Our approach readily ensures the relation $\nu(i) \neq \emptyset$ is vacuously computably enumerable. On the other hand, if we assume $\nu(i) \neq \emptyset$ to be c.e. but permit empty sets to be listed, we can simply limit ν to the non-empty sets. (If needed, we can apply padding to one of the non-empty basic sets to ensure that the domain of ν equals ω.) However, we find it more appropriate to adopt the above definition as it leads to a more concise description of spaces, given that we never consider the empty space. Additionally, it's worth noting that without the assumption of non-emptiness for basic sets (equivalently, $\nu(i) \neq \emptyset$ being c.e.), *any* second countable topological space becomes computable, which hardly gives an insightful notion of effective presentability.

Let (X, τ, β, ν) be a computable topological space. (As mentioned above, all spaces considered in this paper are non-empty.) For $i \in \omega$, by B_i we denote the open set $\nu(i)$. As usual, we identify basic open sets B_i and their ν-indices. In order to simplify our notation even further, we will never actually use the notation (X, τ, β, ν) and will just say that τ is a computable topological presentation of X.

We note that computable topological spaces are closed under taking finite direct products; the computable topology is given by the product topology (equiv., box topology). We will not need infinite direct products of spaces in the paper.

Perhaps, the most natural examples of computable topological Polish spaces are right-c.e. spaces; see, e.g., Theorem 2.3 of [24]; we also cite [2, 6] for a detailed proof. For instance, every computably metrized Polish space is a computable topological space. We discuss these notions in the next subsection.

Definition 2.3. We call

$$N^x = \{i : x \in B_i\}$$

the neighbourhood base of x (in a computable topological space X). A name for a point x is any $p \in \omega^\omega$ such that $rng(p) = N^x$, i.e., p enumerates N^x.

We can also use basic open balls to produce names of open sets, as follows.

Definition 2.4. A *name* of an open set U in a computable topological space X is the enumeration of a set $W \subseteq \mathbb{N}$ such that $U = \bigcup_{i \in W} B_i$, where B_i stands for the i-th basic open set in the basis of X.

If an open set U has a computable name, then we say that U is *effectively open* (sometimes known as c.e. open). If C is closed then a closed name for C is an open name of its complement. We say that C is effectively closed if its complement is effectively open. We also say that a closed set C is effectively overt if we can effectively list all basic open sets that intersect C. A compact set is computable compact if it is both effectively overt and effectively closed.

The notions defined below are also standard.

Definition 2.5. Let f be a map between two computable topological spaces.

(1) We say that f is effectively continuous if there is a Turing functional such that given any name for an open set U, outputs a name for $f^{-1}(U)$.
(2) We say that f is effectively open if there is a Turing functional such that given any name for an open set V, outputs a name for $f(V)$.

If f is a homeomorphism, then it is effectively open if and only if f^{-1} is effectively continuous. We thus say that a homeomorphism between two computable topological spaces is effective (or computable) if it's is both effectively open and effectively continuous.

One special case of a computable f is the following:

Definition 2.6. If X is a computable topological space then a metric d compatible with the topology of X is computable if there is a Turing functional Φ such that given any $y, z \in X$ and any name p, q of y and z respectively, $\Phi(p \oplus q)$ produces an enumeration of both the left and the right cuts of the real $d(y, z)$.

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Equivalently, we could require that the metric is a computable map $X^2 \to \mathbb{R}$, where $\mathbb{R}$ is equipped with the usual computable topology generated by rational intervals.

DEFINITION 2.7. In the notation of the previous definition, a metric d is said to be right-c.e. (upper-semicomputable) if $\Phi(p \oplus q)$ enumerates the right cut of $d(y, z)$. (A left-c.e. metric is defined similarly.)

Notice that we do not require the metric to be complete, and we do not require the existence of a computable dense sequence in the space. Of course, completeness and effective separability are highly desirable properties, especially because we are mainly interested in Polish(able) spaces. Many natural computable metrics in standard Polish spaces are complete and also effectively separable. We will discuss the complete effectively separable case in Section 2.2.

2.1.1. Computable topological groups. We are ready to formally define the notion of a computable topological group.

DEFINITION 2.8. A computable topological group is a triple $(G, \cdot, ^{-1})$, where G is a computable topological space and the group operations $\cdot : G \times G \to G$ and $^{-1} : G \to G$ are effectively continuous.

We also say that a topological group has a *computable topological presentation* if it is homeomorphic to a computable topological group. (We call the latter a computable topological presentation of the group.) In order to simplify our notation, we will usually simply say "G is computable topological" rather than "$(G, \cdot, ^{-1})$ is a computable topological group."

FACT 2.9. *Multiplication and inverse operators are both effectively open in a computable topological group.*

PROOF. Given some name for an effectively open set U, in order to enumerate the name for U^{-1}, simply enumerate the preimage of $^{-1}$ on U. This gives a name for U^{-1} since $\left(U^{-1}\right)^{-1} = U$. Thus $^{-1}$ is effectively open.

Now given names for open U, V, we want to produce computably a name for $U \cdot V$. The map $(x, y) \to x^{-1}y$ is computable. Enumerate all basic open B s.t. there is some basic open set A where $A \cap U \neq \emptyset$ and

$$A^{-1} \cdot B \subseteq V.$$

Now we claim that the union of all such B is equal to $U \cdot V$. First of all, if B is enumerated by the procedure above, let $a \in A \cap U$. For each $b \in B$ we have $b = a \cdot a^{-1}b \in U \cdot A^{-1} \cdot B \subseteq U \cdot V$, and so $B \subseteq U \cdot V$. Conversely let $a \in U$ and $b \in V$. Since $a^{-1} \cdot ab = b \in V$, let A, B be basic open sets containing a and ab respectively such that $A^{-1} \cdot B \subseteq V$. Then B will be enumerated by the procedure above, and where $ab \in B$. ⊣

FACT 2.10. *Every computable topological group admits a c.e. local base for the identity element. That is, there is a uniformly c.e. sequence of basic open sets $(\mathcal{U}_n)_{n\in\omega}$ such that $\mathcal{U}_{n+1} \subseteq \mathcal{U}_n$ for each n, and $\bigcap_{n\in\omega} \mathcal{U}_n = \{e\}$.*

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PROOF. Let $\{B_n\}_{n\in\omega}$ be the effective basis for G, where each B_n is nonempty. We extract a local base for e_G in the following way. Define

- $\mathcal{U}_0 = G$,
- $\mathcal{U}_{n+1} = \mathcal{U}_n \cap \left(B_n \cdot B_n^{-1}\right)$ for $n \geq 0$.

Note that each $\mathcal{U}_n$ is effectively open, uniformly in n. First we check that $\{\mathcal{U}_n\}_{n\in\omega}$ is a local base for e_G. Consider the continuous function $f(x, y) = xy^{-1}$. Then for any open set U containing e_G, there must be some basic open ball B_k s.t. $f(B_k, B_k) \subseteq U$, since $x \cdot x^{-1} = e_G$ for any $x \in G$. Thus $\{B_n \cdot B_n^{-1}\}_{n\in\omega}$ gives a local base for e_G. Since each $B_n \neq \emptyset$, hence $e_G \in B_n \cdot B_n^{-1}$ for every n and so it follows that $\{\mathcal{U}_n\}_{n\in\omega}$ is also a local base for e_G.

Next we check that $\mathcal{U}_n = \mathcal{U}_n^{-1}$ for all n. Base case $n = 0$ is trivially true, then assume $\mathcal{U}_n$ is true for some n. Let $x \in \mathcal{U}_{n+1} = \mathcal{U}_n \cap \left(B_n \cdot B_n^{-1}\right)$. By inductive hypothesis, we know that $x^{-1} \in \mathcal{U}_n$. Furthermore, since $\left(B_n \cdot B_n^{-1}\right)^{-1} = \left\{\left(ab^{-1}\right)^{-1} \mid a, b \in B_n\right\} = \left\{ba^{-1} \mid a, b \in B_n\right\} = B_n \cdot B_n^{-1}$, then $x^{-1} \in B_n \cdot B_n^{-1}$, and thus $x^{-1} \in \mathcal{U}_{n+1}$.

Since G is Hausdorff, for any $x \neq e_G$ there is an open set U such that $e_G \in U$ and $x \notin U$. There is some n such that $\mathcal{U}_n \subseteq U$ which means that $x \notin \mathcal{U}_n$. So this means that $\bigcap_{n\in\omega} \mathcal{U}_n = \{e_G\}$. ⊣

2.1.2. Relativization. We can relativise all definitions in this section. For instance, we can talk about topological presentations that are not necessarily computable. For instance, given a space we could fix $\tau = \{B_i : i \in \omega\}$ (that can be identified with ω) and a set W_τ such that

$$B_i \cap B_j = \bigcup\{B_k : (i, j, k) \in W_\tau\}$$

and call it a topological presentation of the space. We stretch our terminology slightly and denote the representation by τ rather than W_τ. Here we do not require W_τ to be c.e.

DEFINITION 2.11. Topological presentations (which are not necessarily computable) τ_0 and τ_1 of X are *effectively compatible* on X if the identity function on X is a homeomorphism that is computable w.r.t. τ_0 and τ_1. That is, each basic open set in $\mathcal{B}_0$ is effectively open with respect to $\mathcal{B}_1$ and vice versa, uniformly in the indices for the basic open sets.

2.2. Effectively metrized spaces and groups. The notions below are standard.

DEFINITION 2.12. Fix a Polish(able) space M.

(1) M is *computable Polish* or *computably (completely) metrized* if there is a compatible, complete metric d and a countable sequence of *special points* (x_i) dense in M such that, on input i, j, n, we can compute a rational number r such that $|r - d(x_i, x_j)| < 2^{-n}$.
(2) M is *right-c.e. Polish* or *upper-semicomputable Polish* if there is a compatible, complete metric d and a countable sequence of *special points* (x_i) dense in M such that, on input i, j, the right cut $\{r \in \mathbb{Q} : d(x_i, x_j) < r\}$ of $d(x_i, x_j)$ can be uniformly computably enumerated.

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Clearly, any computable Polish space is right-c.e. Polish, but there are examples of right-c.e. Polish spaces that are not even homeomorphic to any computable Polish spaces [2]. As we have already mentioned above, every right-c.e. Polish space can be viewed as a topological space. Indeed, in a right-c.e. Polish space define a *basic open ball* to be an open ball having a rational radius and centred in a special point. Let (B_i) be the effective list of all its basic open balls, perhaps with repetition. It is not too difficult to show using triangle inequality that (B_i) induces a computable topological structure on the space; e.g., [2, 6, 24]. In contrast, in the satellite paper [29] we show that there is a Polish(able) space that admits a computable topological presentation, but is not homeomorphic to any right-c.e. Polish space.

We remark here that in a computable Polish space, the metric d is also computable in the sense of Definition 2.6, and in a right-c.e. Polish space the metric is right-c.e. in the sense of Definition 2.7. Definitions 2.6 and 2.7 are more general and work with respect to any computable topological presentation.

REMARK 2.13. The computability of a function between computable (more generally, right-c.e.) Polish spaces admits several reformulations equivalent to Definition 2.5. For instance, we can require that a map is computable if it uniformly transforms fast converging sequences of special points to fast converging sequences of special points; see [6, 18] for a detailed exposition of computable metric space theory (which we omit here). In [28], such lemmas were formally verified for computable topological spaces with a property called "c.e. strong (formal) inclusion"; both computable Polish and right-c.e. Polish spaces have this property.

Recall that all our topological spaces and groups are Hausdorff and second-countable. The following notions are direct generalizations of Definition 2.8.

DEFINITION 2.14. (1) A computable Polish group is a computable Polish space together with computable group operations $\cdot$ and $^{-1}$.
(2) A right-c.e. Polish group is a right-c.e. Polish space together with computable group operations $\cdot$ and $^{-1}$.

REMARK 2.15. The identity element is computable in any right c.e. Polish group; non-uniformly fix any non-zero special x and consider $x \cdot x^{-1}$ (i.e., N^e introduced in Definition 2.3 is c.e.); one could also extract this observation from Fact 2.10. Thus, without loss of generality we can also require the identity element to be computable in both computable and right-c.e. Polish groups.

The notion of a computable Polish group is due to Melnikov and Montalbán [28]. We will see that in the discrete case it is equivalent to computable presentability of the group in the sense of Mal'cev [26] and Rabin [40]. We will also see that the notion of a right-c.e. Polish group can be viewed as a generalization of the standard notion of a c.e.-presented (Σ^0_1-presented, positive) group in effective algebra [9], and of a co-c.e. presented profinite group studied by LaRoche and Smith [44]. The latter notion, however, also additionally assumes a certain weak version of *effective compactness* which is another important notion that we discuss next.

We also note that the notions defined in this section can be relativized. For instance, we could consider Polish presentations in which we do not restrict the (algorithmic) complexity of the metric.

2.3. Effective compactness and effective local compactness. The definition below is standard in the literature:

DEFINITION 2.16. We say that a computable compact topological space X is *effectively compact* (as a space) if the set of all tuples $\langle i_1, \dots, i_k \rangle$ such that $X = B_{i_1} \cup B_{i_2} \cup \dots \cup B_{i_k}$ is computably enumerable.

In the context of a computable topological space, one could also talk about effective compactness of a subset of the space.

DEFINITION 2.17. We say that a compact subset K of computable topological space X is *effectively compact* (as a subset) if the set of all tuples $\langle i_1, \dots, i_k \rangle$ such that $K \subseteq B_{i_1} \cup B_{i_2} \cup \dots \cup B_{i_k}$ is computably enumerable.

This can be generalized in the straightforward way:

DEFINITION 2.18 (See [50]). Given a compact subset K of a computable topological space X, we define a compact name for K to be any $p \in \omega^\omega$ that enumerates the set of all finite covers of K by basic open sets of X.

More formally, a compact name for K is any enumeration of

$$N^K = \{\langle i_0, \dots, i_k \rangle : K \subseteq \bigcup_{j \leq k} B_{i_j}\},$$

where B_i is the i-th basic open set in X. Evidently K is effectively compact iff it has a computable compact name. (Note that we do not require that each B_{i_j} has to intersect the set.)

The following fact is well-known (for example, see [50, Lemma 2.3]):

FACT 2.19. *Given a name for a closed set A and a name for a compact set K, we can list a name for $A \cap K$.*

REMARK 2.20 (Comparing with other notions of compactness). We will get the stronger notion of a *computably compact* set if we additionally require in Definition 2.17 that $B_{i_j} \cap K \neq \emptyset$ for every cover enumerated by a name, i.e., that the set K is effectively overt. Of course, this would not be an issue in Definition 2.16 because each basic open B_{i_j} is non-empty and thus vacuously intersects the entire space. The difference is already seen in the context of 2^ω, where a compact set is effectively compact (as a subset of 2^ω) if and only if it is Π^0_1. Recall also that a non-empty Π^0_1 class does not have to contain any computable points. In other words, even in 2^ω we get that effective compactness of K implies only that the open complement of K can be effectively listed without necessarily being able to effectively list all basic open B such that $B \cap K \neq \emptyset$. In other words, an effectively compact subset of a computable Polish space does not have to be computably compact. Indeed, it has been shown in [2], there is a Π^0_1-class in 2^ω that is not even *homeomorphic* to any computable Polish space (thus, to any effectively compact computable Polish space). Nonetheless, there is a very nice correspondence between effectively compact *spaces* and computably compact *sets*, as essayed in [6, 18]. The reader should be aware that a subtle distinction exists between considering effective (in some sense) subsets of a fixed space (e.g., the Hilbert Cube) and presenting these sets without reference to any larger ambient space. However, explaining and comparing these two (often

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equivalent) approaches to computability falls beyond the scope of this paper. For a comprehensive explanation, we recommend referring to the surveys mentioned earlier [6, 18].

We shall adopt the following convention when it comes to compact sets:

> When we talk about effectively compact subsets of a computable topological space, we mean it in the sense of Definition 2.17.

2.3.1. Effective local compactness. Unlike the notion of an effectively compact space which is robust [6, 18], there are several (seemingly) non-equivalent notions of effective local compactness in the literature [37, 49, 50]. We shall not attempt to verify whether these notions of effective local compactness are equivalent up to homeomorphism since there is yet not enough evidence that these notions are equally important and useful. We adopt the following:

Definition 2.21 [37]. A second countable topological space X is effectively locally compact if there is a Turing operator which given a name for a point $x \in X$ and a name for an open set $U \ni x$, outputs a name for an open set V and a name for a compact set K such that $x \in V \subseteq K \subseteq U$.

As explained in the companion paper [29], in the context of computable Polish spaces, we can drop U in the definition above and assume K is a computably compact (closed) ball of an arbitrary small computable radius. However, groups in the present paper are rarely computable Polish. Nonetheless, later in the paper we will be dealing with computable topological groups that additionally have a dense sequence of computable points in them. The following lemma shall be useful:

Lemma 2.22 [37, Proposition 8]. *Suppose that a computable topological space X has a dense set of uniformly computable points. Then X is effectively locally compact if and only if there is a triple $(\{U_n\}_{n\in\omega}, \{K_m\}_{m\in\omega}, R)$, where*:

- *U_n is a computable sequence of (uniformly) effectively open sets.*
- *K_m is a computable sequence of (uniformly) effectively compact sets.*
- *$R \subseteq \mathbb{N} \times \mathbb{N}$ is a c.e. set such that $(n, m) \in R \Rightarrow U_n \subseteq K_m$.*
- *For any open set U, we have*

$$U = \bigcup_{\{m \mid K_m \subseteq U\}} \bigcup_{\{n \mid (n,m) \in R\}} U_n.$$

In other words, under the assumptions of the lemma, any open set can be essentially approximated by compact neighbourhoods from within, with a sufficient degree of effectiveness. The triple $(\{U_n\}_{n\in\omega}, \{K_m\}_{m\in\omega}, R)$ was called an ercs for X in [37]. Compare this to the related notion of a *computably locally compact Hausdorff space* in [50, Definition 3.2].

2.4. Effectively proper metrics. Recall that a metric d is proper if every closed bounded ball $\{y \mid d(x, y) \leq r\}$ is compact; equivalently, every closed bounded set is compact. Recall that a name of a closed set is a name of its open complement, and the name of a compact set is the list of all of its finite basic open covers. The obvious effectivisation of properness is the following:

DEFINITION 2.23. A right-c.e. Polish space $(\mathcal{M}, d)$ is effectively proper if there exists a Turing functional which, given a name of a closed set A and a basic open ball $B_d(\alpha, r) \supseteq A$, outputs a name for some compact set $K \supseteq A$.

The lemma below relates the notion with local effective compactness.

LEMMA 2.24. *Given a Polish space $(\mathcal{M}, d)$, we have $(i) \Leftrightarrow (ii) \Rightarrow (iii)$. If $(\mathcal{M}, d)$ is a computable Polish space then all three are equivalent*:

(i) *$(\mathcal{M}, d)$ is effectively proper.*
(ii) *Given a name for a closed set A and a basic open ball containing A, we can list a compact name for A.*
(iii) *Given a name (as a closed set) for the closed ball $B_d^{\leq}(\alpha, r) = \{x : d(\alpha, x) \leq r\}$ and the parameters α, r, we can compute a compact name for $B_d^{\leq}(\alpha, r)$.*

PROOF. (i) $\Rightarrow$ (ii): Apply Fact 2.19, which holds even if d is not computable. The implications (ii) $\Rightarrow$ (i),(iii) are trivial. If d is computable then assuming (iii) holds, for each basic open $B(\alpha, r)$ where α is a special point and $r \in \mathbb{Q}^+$, we have $B_d^{\leq}(\alpha, r)$ is effectively closed. (This is because a special point $x \notin B_d^{\leq}(\alpha, r)$ together with a q-ball $B_d^{<}(x, q)$ iff $d(\alpha, x) > r + q$, which is c.e. if the metric is computable, but could be not c.e. for a right-c.e. metric. In the case of a computable metric we do not have to assume $B_d^{\leq}(\alpha, r)$ comes together with a closed name, as it automatically can be reconstructed from its parameters.) So given a name for a closed set A and some α, r such that $A \subseteq B_d(\alpha, r)$ we can obtain a name for $B_d^{\leq}(\alpha, r)$ and, therefore, a compact name for $B_d^{\leq}(\alpha, r)$. By Fact 2.19, we obtain a compact name for A. $\dashv$

The first item in the above lemma is the effective version of the fact that every closed and bounded set is contained in a compact set. The second item corresponds to the fact that every closed and bounded set is compact, while the third item corresponds to the fact that every closed and bounded ball is compact. An effectively locally compact computable Polish space will satisfy a version of (iii) that says that, for every x, *there exists* a sufficiently small r and a special α such that $x \in B_d^{\leq}(\alpha, r)$ is effectively compact. In [29] we additionally show that, in such a space, $B_d^{\leq}(\alpha, r)$ can be picked computably closed as well. But of course, to claim that we can compute a compact name for $B_d^{\leq}(\alpha, r)$ for *any* r, α the metric has to be proper in the first place.

2.5. A unified generalization: represented spaces. The definition of a compact name Definition 2.18 is reminiscent of Definition 2.3 for points. It is also somewhat similar to Definition 2.4 for open sets. Also, the definitions of a computable topological, a computable Polish, and a right-c.e. Polish space have a similar flavour too. More specifically, in each case we can define names of points and define the notion of a computable map between presentations.

All these notions have some clear similarities and seem to be special instances of something more general. This intuition can be made formal using the theory of *represented spaces*. We do not need this degree of generality in the present paper. Indeed, one of the main goals of our paper is to illustrate that, in the case of Polish groups, we can safely restrict ourselves to the classical theory of effectively metrized spaces without any loss of generality. We cite [36] for a detailed exposition of the theory of represented topological spaces.

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§3. Effective Birkhoff–Kakutani theorem. The classical Birkhoff–Kakutani theorem is the following:

Theorem 3.1 (Birkhoff–Kakutani theorem). *Let G be a topological group. Then G is metrisable iff G is Hausdorff and first countable. Moreover, if G is metrisable, then G admits a compatible left-invariant metric.*

We consider the effective version of the Birkhoff–Kakutani theorem and restrict attention to computable topological groups. We obtain the following effective version of Theorem 3.1.

Theorem 3.2. *Let G be a computable topological group. Then G admits a right-c.e. compatible left-invariant metric.*

Proof. Let $\{B_n\}_{n\in\omega}$ be the effective basis for G, where each B_n is nonempty. Using Fact 2.10, fix a c.e. local base $(\mathcal{U}_n)_{n\in\omega}$ of the identity element:

$$\bigcap_{n\in\omega} \mathcal{U}_n = \{e_G\},$$

where the sets $\mathcal{U}_n$ are uniformly c.e. open.

Now we define $\{\mathcal{V}_n\}_{n\in\omega}$ satisfying the following properties for every $n \in \omega$:

(1) $\mathcal{V}_0 = G; \mathcal{V}_{n+1} \subseteq \mathcal{V}_n$.
(2) $\mathcal{V}_n = \mathcal{V}_n^{-1}$.
(3) $\mathcal{V}_{n+1}^3 \subseteq \mathcal{V}_n$.
(4) $\mathcal{V}_n \subseteq \mathcal{U}_n$.

Consider the function $f(x_0, y_0, x_1, y_1, x_2, y_2) = \Pi_{i=0}^2 x_i y_i^{-1}$. Assume that $\mathcal{V}_n$ has been defined. Search for basis elements B and $B_{i_0}, \dots, B_{i_5}$ satisfying the following:

- $\langle B_{i_j}\rangle_{j<6} \subseteq f^{-1}(\mathcal{V}_n)$, and
- $B \subseteq \mathcal{V}_n \cap \bigcap_{j=0}^5 B_{i_j}$.

Since G is a computable topological group, multiplication and inverse are effectively continuous operations, and hence, given an index for $\mathcal{V}_n$, we can search for an index for B. Take $\mathcal{V}_{n+1} = (B \cdot B^{-1}) \cap \mathcal{U}_{n+1}$, and note that $\mathcal{V}_{n+1}$ is also effectively open, uniformly in n. Note also that B must exist, since $e_G \in \mathcal{V}_n$ and $f(e_G, e_G, \dots, e_G) = e_G$.

The properties 2 and 4 are immediate. By choice of B, we see $f(B, B, \dots, B) \subseteq \mathcal{V}_n$, and thus $\mathcal{V}_{n+1}^3 = \{x \cdot y \cdot z \mid x, y, z \in (B \cdot B^{-1}) \cap \mathcal{U}_{n+1}\} \subseteq \{x \cdot y \cdot z \mid x, y, z \in (B \cdot B^{-1})\} = f(B, B, \dots, B) \subseteq \mathcal{V}_n$. Finally, since $e_G \in \mathcal{V}_{n+1}$, then for any $x \in \mathcal{V}_{n+1}$, $x \cdot e_G \cdot e_G \in \mathcal{V}_{n+1}^3$, which gives $\mathcal{V}_{n+1} \subseteq \mathcal{V}_{n+1}^3 \subseteq \mathcal{V}_n$.

Now we define the functions $\varrho, d : G^2 \to \mathbb{R}$ by:

- $\varrho(x, y) = \inf\{2^{-n} \mid x^{-1}y \in \mathcal{V}_n\}$.
- $d(x, y) = \inf\{\sum_{i=0}^{l} \varrho(g_i, g_{i+1}) \mid g_i \in G, g_0 = x, g_{l+1} = y, l \in \omega\}$.

Since $\mathcal{V}_0 = G$, ϱ and d are total functions.

To verify that d is a metric on G, we follow the classical proof (see [11]) almost exactly. Since for each n, $\mathcal{V}_n = \mathcal{V}_n^{-1}$, then we have that $\varrho(x, y) = \varrho(y, x)$ for any $x, y \in G$, and since $\varrho(gx, gy) = \inf\{2^{-n} \mid (gx)^{-1}(gy) \in \mathcal{V}_n\} = \inf\{2^{-n} \mid x^{-1}y \in \mathcal{V}_n\} = \varrho(x, y)$, then d must also be both symmetric and left-invariant. It is easy to see that $d(x, x) = 0$, and since $\varrho(g, h) \geq 0$ for any $g, h \in G$, then $d(x, y) \geq 0$ for any

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$x, y \in G$. From the definition of d, it is also easy to see that the triangle inequality holds. Thus it remains to check that $d(x, y) = 0$ only if $x = y$.

We prove the following by induction:

$$\sum_{i=0}^{l} \varrho(g_i, g_{i+1}) \geq \frac{1}{2}\varrho(g_0, g_{l+1})$$

for any $g_0, g_1, \ldots, g_{l+1} \in G$. First note that by property 3 of $\{\mathcal{V}_n\}_{n\in\omega}$, ϱ has the following property:

$$\forall \varepsilon > 0, \text{ if } \varrho(g_0, g_1), \varrho(g_1, g_2), \varrho(g_2, g_3) \leq \varepsilon, \text{ then } \varrho(g_0, g_3) \leq 2\varepsilon. \qquad (*)$$

Then when $l \leq 2$, the proposition follows directly from $(*)$. Suppose that the proposition holds for all $l' < l$ for some l. Let $S = \sum_{i=0}^{l} \varrho(g_i, g_{i+1})$, and m be the largest (possibly $m = 1$) s.t. $\sum_{i=0}^{m-1} \varrho(g_i, g_{i+1}) \leq \frac{1}{2}S$. By the inductive hypothesis, $\varrho(g_0, g_m) \leq 2\sum_{i=0}^{m-1} \varrho(g_i, g_{i+1}) \leq S$. Since $\sum_{i=0}^{m} \varrho(g_i, g_{i+1}) > \frac{1}{2}S$, then $\sum_{i=m+1}^{l} \varrho(g_i, g_{i+1}) \leq \frac{1}{2}S$, then by inductive hypothesis again, we have $\varrho(g_{m+1}, g_{l+1}) \leq S$. But clearly $\varrho(g_m, g_{m+1}) \leq S$, that is,

$$\varrho(g_0, g_m),\ \varrho(g_m, g_{m+1}), \varrho(g_{m+1}, g_{l+1}) \leq S,$$

and hence by $(*)$, $\varrho(g_0, g_{l+1}) \leq 2S$. Then if $d(x, y) = 0$, it must be that $\frac{1}{2}\varrho(x, y) = 0$ and this is only the case when $x^{-1}y = e_G$, i.e., $x = y$.

Now we check that d is compatible with the topology of G. Let U be open in G and $g \in U$. Then for some $n \in \mathbb{N}$, $g\mathcal{V}_n \subseteq U$. We check that $B_d(g, 2^{-n-1}) \subseteq U$. Let $h \in B_d(g, 2^{-n-1})$, then $d(h, g) < 2^{-n-1}$. By the claim above, $\varrho(g, h) \leq 2d(g, h) < 2^{-n}$, and by the definition of ϱ, $g^{-1}h \in \mathcal{V}_n$, thus $h \in g\mathcal{V}_n \subseteq U$. Conversely, let U be open in the topology given by d and let $g \in U$. For some $n \in \mathbb{N}$, we have that $B_d(g, 2^{-n}) \subseteq U$. We check that $g\mathcal{V}_{n+1} \subseteq U$. Let $h \in g\mathcal{V}_{n+1}$, then $\varrho(g, h) \leq 2^{-n-1}$, and by definition of d, $d(g, h) \leq \varrho(g, h) \leq 2^{-n-1} < 2^{-n}$, therefore $h \in B_d(g, 2^{-n}) \subseteq U$.

Now we check that the right cut of $d(x, y)$ can be enumerated given an enumeration of N^x and N^y. List out all finite tuples, and search for $(l+1)$-tuples $\langle p_m\rangle_{m\leq l}$, $\langle q_m\rangle_{m\leq l}$, and $\langle n_m\rangle_{m\leq l}$ such that the sequence $(B_{p_0}, B_{p_1}, \ldots, B_{p_l})$ and $(B_{q_0}, B_{q_1}, \ldots, B_{q_l})$ satisfy:

- B_{p_0} is enumerated in N^x,
- B_{q_l} is enumerated in N^y,
- $B_{p_m}^{-1} \cdot B_{q_m} \subseteq \mathcal{V}_{n_m}$ for each $m \leq l$, and
- $B_{q_m} \cap B_{p_{m+1}} \neq \emptyset$ for each $m < l$.

Note that the third condition is c.e. and implies that $(B_{p_m}, B_{q_m}) \subseteq \varrho^{-1}([0, 2^{-n_m}])$. If a suitable $\langle n_m\rangle_{m<l}$ is found, enumerate $\sum_{j=0}^{l} 2^{-n_j}$ into our approximation to the right cut of $d(x, y)$.

Now we verify that the procedure described above in fact does enumerate the right cut of $d(x, y)$. Let $q = \sum_{j=0}^{l} 2^{-n_j}$ be enumerated by the procedure at some stage. This means that some $\langle n_j\rangle_{j\leq l}$ and corresponding $(B_{i_0}, B_{i_1}, \ldots, B_{i_l})$ and $(B_{k_0}, B_{k_1}, \ldots, B_{k_l})$

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have been found where $x \in B_{i_0}$ and $y \in B_{k_l}$. Now for each $j < l$, let $g_j \in B_{k_j} \cap B_{i_{j+1}}$. Then we have:

- $x^{-1}g_0 \in \mathcal{V}_{n_0}$,
- $g_j^{-1}g_{j+1} \in \mathcal{V}_{n_{j+1}}$ for each $j < l-1$, and
- $g_{l-1}^{-1}y \in \mathcal{V}_{n_l}$.

Since $d(x,y) = \inf\{\sum_{i=0}^{l} \varrho(g_i, g_{i+1}) \mid g_i \in G, g_0 = x, g_{l+1} = y, l \in \omega\}$, so we have $d(x,y) \leq \sum_{i=0}^{l} \varrho(g_i, g_{i+1}) \leq \sum_{j=0}^{l} 2^{-n_j} = q$.

Now to check that the procedure enumerates some rational q s.t $q < d(x,y) + \varepsilon$ for each $\varepsilon > 0$. We can assume that $x \neq y$, since if $x = y$, it can be easily seen that the procedure described before enumerates the right cut of 0. Let $g_i \in G$ for $i \leq l+1$ where $g_0 = x, g_{l+1} = y$ be given s.t. $\sum_{i=0}^{l} \varrho(g_i, g_{i+1}) < d(x,y) + \varepsilon$, for some $\varepsilon > 0$. We may of course assume that $\varrho(g_i, g_{i+1}) > 0$ for each i, since $\varrho(g_i, g_{i+1}) = 0$ iff $g_i = g_{i+1}$. Hence $\varrho(g_i, g_{i+1}) = 2^{-n_i}$ for some n_i. At some stage $(B_{i_0}, B_{i_1}, \dots, B_{i_l})$ and $(B_{k_1}, B_{k_2}, \dots, B_{k_l})$ must be found satisfying the conditions above and we enumerate $\sum_{i=0}^{l} 2^{-n_i}$ into the right cut of $d(x,y)$. Thus the procedure enumerates rationals arbitrarily close to $d(x,y)$. ⊣

§4. Computing a dense sequence. In Theorem 3.2 we produced a compatible right-c.e. metric for any given Hausdorff computable topological group. However the effectivisation was point-free, in the sense of lacking a countable dense subset of points. Perhaps unexpectedly, if we *assume* the metric that we produce is actually complete, then we can show that the metric admits a dense computable sequence of points. In other words, in this case we obtain a right-c.e. Polish presentation of the group.

THEOREM 4.1. *Let G be a computable topological group where the metric d produced in Theorem 3.2 is complete. Then G has a right-c.e. Polish presentation.*

Most of the rest of the section is devoted to the proof of the theorem. We begin with several technical lemmas that establishes several useful properties of the metric produced in the proof of Theorem 3.2.

4.1. Two technical lemmas. Let $\mathcal{M} = (\{\alpha_i\}_{i\in\omega}, d)$ be a countable metric space and $\overline{\mathcal{M}}$ be its completion. Let τ_d be the topology on $\overline{\mathcal{M}}$ generated by the metric d with basis elements $B_d(\alpha_i, \varepsilon)$ where $i \in \omega$ and $\varepsilon \in \mathbb{Q}^+$. Note these balls also form a base of the restricted topology on $\mathcal{M}$. We say that a computable topological space G (we mainly care about topological groups) is *effectively compatible* with $\mathcal{M}$ if $\mathcal{M} \subseteq G \subseteq \overline{\mathcal{M}}$ and where τ and τ_d (restricted as the subspace topology) are effectively compatible on G (see Definition 2.11, note that here G is a subset of $\overline{M}$ and contains all the special points of $\overline{M}$).

In the definitions above, we do not require τ or τ_d to be computable topologies, even though our topologies will typically be computable; we discussed relativization in Section 2.1.2. Also, we do not restrict the complexity of d, even though we are interested in right-c.e. metrics which induce a computable topology. The technical reason why we need this extra degree of generality will be explained in Remark 4.3 shortly. Recall that all our groups are Hausdorff.

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LEMMA 4.2. *Let $(G, \tau, \mathcal{B})$ be a computable topological group and let d be the metric produced in Theorem 3.2. Suppose that G contains a dense set of points $\{\alpha_i\}_{i\in\omega}$ w.r.t. τ, and there is a computable function φ such that for every i, s we have $d(\alpha_i, g) \leq 2^{-s}$ for any $g \in B_{\varphi(i,s)} \in \mathcal{B}$. Then G is effectively compatible with $(\{\alpha_i\}_{i\in\omega}, d)$.*

REMARK 4.3. Even though d is a right-c.e. point-free metric defined on G, however, since $\{\alpha_i\}_{i\in\omega}$ need not be computable points w.r.t. τ, it is not immediately obvious why $d(\alpha_i, \alpha_j)$ is right-c.e. uniformly in i, j. This is in fact true, but we will not need it here (yet). We shall revisit this later.

PROOF OF LEMMA 4.2. Since $\{\alpha_i\}_{i\in\omega}$ is dense with respect to τ, it is also dense in G with respect to d since τ and τ_d are compatible as shown in Theorem 3.2. Thus $G \subseteq \overline{(\{\alpha_i\}_{i\in\omega}, d)}$.

Given $B_d(\alpha_i, r)$, where $r \in \mathbb{Q}^+$, we want to effectively (in i and r) produce a name for $U = B_d(\alpha_i, r)$ with respect to τ. For each s where $2^{-s} < r$, list all finite tuples and search for $\langle n_m\rangle_{m\leq l}, \langle p_m\rangle_{m\leq l}$ and $\langle q_m\rangle_{m\leq l}$ s.t. the sequences $(B_{p_0}, B_{p_1}, \dots, B_{p_l})$ and $(B_{q_0}, B_{q_1}, \dots, B_{q_l})$ satisfy:

- $B^{-1}_{p_m} \cdot B_{q_m} \subseteq \mathcal{V}_{n_m}$ for each $m \leq l$,
- $B_{q_m} \cap B_{p_{m+1}} \neq \emptyset$ for each $m < l$,
- $B_{p_0} \cap B_{\varphi(i,s)} \neq \emptyset$, and
- $2^{-s} + \sum_{m\leq l} 2^{-n_m} < r$.

If such sequences are found, enumerate B_{q_l} into the name for U.

We follow a similar argument in the proof of Theorem 3.2. It is clear that whenever a sequence satisfying the above properties is found, we have the property that for any $g \in B_{q_l}$, $d(g, \alpha_i) < r$. Thus $B_{q_l} \subseteq U$. Conversely, for any $g \in U = B_d(\alpha_i, r)$, there must be some sequence which witnesses that $d(\alpha_i, g) < r$, then at some stage we must find it and thus enumerate a set containing g into the name of U. Thus U is effectively open with respect to τ.

Now given a basic open set $B_j \in \tau$, we produce a c.e. name for B_j w.r.t. τ_d. Consider the function $f(x, y, z) = x \cdot (y \cdot z^{-1})$. Since f is effectively continuous w.r.t. τ, we can effectively enumerate $f^{-1}(B_j)$. Search for τ-basic open sets X, Y, Z s.t.:

- $Y \cap Z \neq \emptyset$,
- $X \cap B_{\varphi(i,s)} \neq \emptyset$ for some i, and $s > n + 2$ where n is found so that $B_n \subseteq Y \cap Z$, and
- $X \cdot Y \cdot Z^{-1} \subseteq B_j$.

For each X, Y, Z and i, s, n found satisfying the above, we enumerate the ball $B_d(\alpha_i, 2^{-n-1} - 2^{-s})$ into the name for B_j w.r.t. τ_d.

Suppose X, Y, Z, i, s, n are found by the procedure above. Since $B_n \subseteq Y \cap Z$, we have that $X \cdot (B_n \cdot B_n^{-1}) \subseteq B_j$, and hence for any $g \in X$, $g\mathcal{V}_n \subseteq B_j$, since $\mathcal{V}_n \subseteq B_n \cdot B_n^{-1}$. Since $X \cap B_{\varphi(i,s)} \neq \emptyset$, we can fix $g \in X \cap B_{\varphi(i,s)}$, and so $d(\alpha_i, g) \leq 2^{-s} < 2^{-n-2}$. As a result, for any $h \in B_d(\alpha_i, 2^{-n-1} - 2^{-s})$, $d(g, h) \leq d(g, \alpha_i) + d(\alpha_i, h) < 2^{-s} + 2^{-n-1} - 2^{-s} = 2^{-n-1}$, that is $h \in B_d(g, 2^{-n-1})$. However from the classical proof of compatibility (see the proof of Theorem 3.2), we know that since $g\mathcal{V}_n \subseteq B_j$, then $B_d(g, 2^{-n-1}) \subseteq B_j$. Therefore we conclude that $B_d(\alpha_i, 2^{-n-1} - 2^{-s}) \subseteq B_j$.

Now conversely fix some $g \in B_j$. There are τ-basic open sets X, Y, Z such that $X \cdot Y \cdot Z^{-1} \subseteq B_j$ such that $g \in X$ and $e \in Y \cap Z$. Now fix any n (such that $B_n \subseteq$

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$Y \cap Z)$. Since τ and τ_d are compatible, and $\{\alpha_i\}_{i\in\omega}$ is dense in G wrt τ_d, we can pick some i and $s > n+2$ such that $B_d(\alpha_i, 2^{-s+1}) \subseteq X$. (We may also assume that $d(\alpha_i, g) < 2^{-s}$.) But this means that $B_{\varphi(i,s)} \subseteq X$. The above procedure must therefore find six items X, Y, Z, i, s, n with the desired properties. Furthermore, since $d(\alpha_i, g) < 2^{-s} < 2^{-n-1} - 2^{-s}$, then $g \in B_d(\alpha_i, 2^{-n-1} - 2^{-s})$.

Hence the procedure above witnesses that B_j is effectively open wrt τ_d. Thus, τ and τ_d are effectively compatible. ⊣

LEMMA 4.4. *Let G be a computable topological group that contains a dense set of uniformly computable points. Then G is effectively compatible with a right-c.e. metric space. Furthermore, the compatible metric (on G) is also left-invariant.*

PROOF. Apply Theorem 3.2 to produce a compatible right-c.e. metric d for G. Let $\{\alpha_i\}_{i\in\omega} \subseteq G$ be the set of uniformly computable points with respect to the original topology τ on G. Now $d(\alpha_i, \alpha_j)$ is right-c.e. uniformly in i, j by following the procedure in Theorem 3.2 and feeding to the procedure the c.e. names for N^{α_i} and N^{α_j}. Now take $\mathcal{M} = (\{\alpha_i\}_{i\in\omega}, d)$.

It remains to check that (G, τ) is effectively compatible with $\mathcal{M}$. By Lemma 4.2, we need only check that there is a computable function $\varphi(i, s) = j$ s.t. $d(\alpha_i, g) \leq 2^{-s}$ for any $g \in B_j$. (Here B_j are the basic open sets of τ.) Given i, s, search for some basic open set B_j such that $B_j \subseteq X \cap Y$, $\alpha_i \in B_j$, and $X^{-1} \cdot Y \subseteq \mathcal{V}_s$. Some B_j must be found since $\alpha_i^{-1} \cdot \alpha_i = e_G \in \mathcal{V}_s$. Now define $\varphi(i, s) = j$. Since $B_j^{-1} \cdot B_j \subseteq \mathcal{V}_s$, it must be that $\forall g, h \in B_j$, $d(g, h) \leq \varrho(g, h) \leq 2^{-s}$, in particular, $\forall g \in B_j$, $d(\alpha_i, g) \leq 2^{-s}$. ⊣

By effective compatibility, the group operations remain computable with respect to the metric. Thus, to argue that the group from Theorem 4.1 has a right-c.e. Polish presentation, all we need to show is that there is a dense set of computable points.

REMARK 4.5. We can easily reconstruct a computable dense sequence in a computable topological group with c.e. formal (strong) inclusion. The latter is an axiomatic generalisation of formal inclusion in metric spaces that is defined as follows:

$$B(x, q) \subset_{form} B(y, r) \text{ iff } d(x, y) + q < r.$$

Note that it is a c.e. relation in a right-c.e. space. We omit the definition of *abstract* strong inclusion $\gg$ and refer the reader to [28], but we note that [28] contains an example of a non-metrisable computable topological space with c.e. abstract strong (formal) inclusion. We claim that Lemma 4.4 implies the following: *Every computable topological group with a c.e. strong inclusion is effectively compatible with a right-c.e. metric space.* To see why, extract a dense set of uniformly computable points by considering for each i_0 the first found effective sequence $B_{i_0} \gg B_{i_1} \gg \cdots$ where $\bigcap_k B_{i_k} = \{\alpha_{i_0}\}$. It follows from the definition of formal inclusion in [28] that $N^{\alpha_{i_0}}$ is c.e.; we omit the details.

4.2. Proof of Theorem 4.1.. In order to identify special points in G, we use some ideas in [13] by utilising the group operations. Fix G and d as in the proof of Theorem 3.2. The first lemma below is designed to implement the idea sketched in Remark 4.5, but in the absence of c.e. "formal inclusion."

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LEMMA 4.6. *Given a basic open set B_i in G, there exists a computable sequence of basic open sets $\{B_{i_s}\}_{s\in\omega}$ such that*:

(1) *$B_{i_0} = B_i$ and $B_{i_{s+1}} \subseteq B_{i_s}$ for every s.*
(2) *$dm\left(\overline{B_{i_s}}\right) := \sup\left\{d(x,y) \mid x,y \in \overline{B_{i_s}}\right\} \leq 2^{-s}$ for every s.*
(3) *$f^*\left(B_{i_s}, B_{i_s}\right) \subseteq \mathcal{V}_s$ for every s, where $f^*(x,y) = x^{-1}y$.*

Notice that we only claim that the sequence $\{B_{i_s}\}_{s\in\omega}$ is computable (uniformly in $i = i_0$). We make no claims about how difficult it might be to approximate $dm\left(\overline{B_{i_s}}\right)$.

PROOF. Let $B_{i_0} = B_i$. Since $\mathcal{V}_0 = G$, $d(x,y) \leq 1$ for every $x,y \in G$ and so properties 2 and 3 are trivially true for $s = 0$. Now suppose inductively that B_{i_s} satisfying the desired properties has been defined. Since f^* is both effectively open and effectively continuous, and since $e_G \in f^*\left(B_{i_s}, B_{i_s}\right) \cap \mathcal{V}_{s+1}$, we can search for some basic open set B s.t. $f^*(B,B) \subseteq f^*(B_{i_s}, B_{i_s}) \cap \mathcal{V}_{s+1}$ and $B \cap B_{i_s} \neq \emptyset$. Take $B_{i_{s+1}}$ to be any basic open set contained in $B \cap B_{i_s}$. This gives property 1. Note that $f^*\left(B_{i_{s+1}}, B_{i_{s+1}}\right) \subseteq \mathcal{V}_{s+1}$ which gives property 3.

It remains to check property 2. Let $x,y \in \overline{B_{i_s}}$ be given. Since the original topology on G is compatible with the topology on G induced by the metric d, we can find sequences $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ such that $x_n, y_n \in B_{i_s}$ and $d(x_n,x), d(y_n,y) \leq 2^{-n}$ for all n. Therefore, $\forall\varepsilon > 0$, $\exists m,n$ for which $d(x_n,x) < \frac{\varepsilon}{2}$ and $d(y_m,y) < \frac{\varepsilon}{2}$, then

$$\begin{aligned} d(x,y) &\leq d(x,x_n) + d(x_n,y_m) + d(y_m,y) \\ &\leq \frac{\varepsilon}{2} + 2^{-s} + \frac{\varepsilon}{2} \quad \left(\text{since } f^*\left(B_{i_s}, B_{i_s}\right) \subseteq \mathcal{V}_s\right) \\ &\leq 2^{-s} + \varepsilon. \end{aligned}$$

Since the above holds for any ε, then $d(x,y) \leq 2^{-s}$. ⊣

Since d is complete and compatible with the original topology, by property 3 of Lemma 4.6, there is a unique point $\alpha_i \in G$ s.t.

$$\alpha_i \in \bigcap_{s\in\omega} \overline{B_{i_s}}.$$

Here α_i corresponds to the sequence $\{B_{i_s}\}_{s\in\omega}$ with $B_{i_0} = B_i$. Repeating this for all i produces an infinite sequence $\alpha_0, \alpha_1, \ldots$ such that $\alpha_i \in \overline{B_i}$ for each i. When we want to distinguish between the different sequences $\{B_{i_s}\}_{s\in\omega}$ we will use the notation $\{B^k_{i_s}\}_{s\in\omega}$ if $B^k_{i_0} = B_k$.

We claim that the set $\{\alpha_i\}_{i\in\omega}$ produced in Lemma 4.6 is dense in the original topology of G.

To see why, fix a basic open set B_i and let $g \in B_i$. Since G is metrisable, G is also (classically) regular. Then $\exists F \ni g$ s.t. $F \subseteq B_i$ and F is a closed neighbourhood of g. Since the basic open balls form a basis for the topology of G, $\exists B_j \subseteq F$, and hence $\overline{B_j} \subseteq F \subseteq B_i$. Then we must have that $\alpha_j \in \overline{B_j} \subseteq B_i$.

We now finish the proof of the theorem. Since $\{\alpha_i\}_{i\in\omega}$ is dense in τ and using the procedure in Lemma 4.6, given any i,s, we are able to effectively identify a basic open set B_j for which $\forall g \in B_j$, $d(\alpha_i, g) \leq 2^{-s}$. By Lemma 4.2, we have that τ is effectively compatible with τ_d.

Finally notice that $d(\alpha_i, \alpha_j)$ is a right-c.e. real uniformly in i,j. To see this, note that an easy modification of the right c.e. approximation procedure of d

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in Theorem 3.2 by requiring that $B_{p_0} \cap B_{\varphi(i,s+1)} \neq \emptyset$ and $B_{q_l} \cap B_{\varphi(j,s+1)} \neq \emptyset$ and enumerating $2^{-s} + \sum_{m \leq l} 2^{-n_m}$ allows us to produce the right cut of $d(\alpha_i, \alpha_j)$. But since $\{\alpha_i\}_{i \in \omega}$ are uniformly computable points with respect to τ_d, and since τ and τ_d are effectively compatible, and also that d is right-c.e., we conclude that $\{\alpha_i\}_{i \in \omega}$ are also uniformly computable points with respect to τ.

4.3. Consequences of Theorem 4.1. A topological space will be called topologically complete if it admits a metric under which it is complete. Recall also that a metric is invariant if it is both left and right invariant. For instance, every left- or right-invariant metric in an abelian group is invariant. Klee [23] proved the following result. Suppose G is a group with invariant metric d. If the space (G, d) is topologically complete, then G is actually complete under d. We therefore obtain the following, rather satisfying:

COROLLARY 4.7. *Let G be a computable topological group that is Polish(able) abelian. Then G admits an effectively compatible right-c.e. Polish presentation.*

Of course, the metric is invariant in this case.

Since all compact metric spaces are complete, by Theorem 4.1 we see that any compact computable topological group must have an effectively compatible with a right-c.e. Polish presentation too. In fact, we will see that the same is true of locally compact groups; this fact will be established at the end of the next section.

§5. Locally compact groups and proper metrisation. Recall that a metric d is proper if every closed bounded ball $\{y \mid d(x, y) \leq r\}$ is compact; equivalently, every closed bounded set is compact. As we have already mentioned above, Struble showed the following:

THEOREM 5.1 (Struble [46]; see also [14]). *Let G be a topological group which is Hausdorff, second countable and locally compact. Then G admits a compatible left-invariant proper metric.*

We refer the reader to [46] for the classical proof. This section is devoted to proving that the following effective version of Theorem 5.1. (Recall that all our groups are Hausdorff and second countable.)

The first observation made about Struble's proof is the following:

LEMMA 5.2. *Let G be a computable topological group that is (classically) locally compact. Then the metric produced by Theorem 3.2 is complete.*

PROOF. Let δ be the left-invariant right-c.e. compatible metric produced in Theorem 3.2. Struble (see [46]) used δ to produce another metric d on G such that d is compatible with δ and d is a proper metric. Furthermore there is some N such that d and δ are equal whenever $d(x, y) < N$ or $\delta(x, y) < N$. Since every proper metric space is complete, this means that (G, δ) is complete. ⊣

This gives us the immediate corollary:

COROLLARY 5.3. *Let G be a computable topological group that is locally compact. Then G admits an effectively compatible right-c.e. Polish presentation.*

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PROOF. By Lemma 5.2 we know that (G, δ) is complete, where δ is the metric produced in Theorem 3.2. By Theorem 4.1, G is effectively compatible with a right-c.e. metric space. ⊣

Now we prove an effective version of Struble's result.

THEOREM 5.4. *Let (G, τ) be an effectively locally compact computable topological group. Then G is effectively compatible with an effectively proper right-c.e. metric space. Furthermore the metric is left-invariant.*

PROOF. By the proof of Theorem 4.1, and applying Lemma 5.2, we see that (G, τ) contains a dense set of uniformly computable (wrt τ) points $\{\alpha_i\}_{i\in\omega}$. By Lemma 4.4, G is effectively compatible with $(\{\alpha_i\}_{i\in\omega}, \delta)$ where δ is right-c.e. Furthermore δ is left-invariant.

By Lemma 2.22 we fix the triple $(\{B_n\}_{n\in\omega}, \{K_m\}_{m\in\omega}, R)$. Note that each B_n is τ-effectively open (uniformly in n) and collectively form a basis for (G, τ). We may also assume that for every n there is some m such that $(n, m) \in R$. By set product and power we mean the corresponding operation with respect to the group operation.

Note that the group identity e is (in any computable topological group) a computable point with respect to τ, therefore there is some τ-effectively compact set K and some τ-open set B such that $e \in B \subseteq K$. Since δ is compatible with τ we fix some $r \in \mathbb{Q}^+$ such that $B_\delta(e, r) \subseteq B \subseteq K$. By scaling δ we can assume that $r = 2$, and so we may assume that $B_\delta(e, 2) \subseteq K$. Note that we do not claim that $\overline{B_\delta(e, 2)}$ or $B_\delta^{\leq}(e, 2)$ is effectively compact, merely that some open ball around e is contained in an effectively compact set K.

We now define a collection $\{U_r\}_{r\in\mathbb{Q}^+}$ of τ-effectively open sets satisfying the following properties.

(1) For each $r \in \mathbb{Q}^+$, U_r is contained in some τ-effectively compact set.
(2) For each $r \in \mathbb{Q}^+$, $U_r = U_r^{-1}$.
(3) For each $r, s \in \mathbb{Q}^+$, $U_r \cdot U_s \subseteq U_{r+s}$.
(4) $\forall r < 2$, $U_r = B_\delta(e, r)$.
(5) $\bigcup_{r\in\mathbb{Q}^+} U_r = G$.
(6) For each $r \in \mathbb{Q}^+$, $e \in U_r$.

For $0 < r < 2$, define $U_r = B_\delta(e, r)$. To check that the properties hold, since $\delta(g, e) = \delta(g^{-1}, e)$ for any $g \in G$, U_r is closed under inverse for $r < 2$. For $r + s < 2$, let $x \in U_r$ and $y \in U_s$, then by triangle inequality and left-invariance of δ, $\delta(xy, e) \leq \delta(xy, x) + \delta(x, e) = \delta(y, e) + \delta(x, e)$, thus giving that $xy \in U_{r+s}$.

Now we define $U_2 = B_\delta(e, 2) \cup W_2$, where W_{2^n} is defined as follows. For each n, take $W_{2^n} = B_n \cup B_n^{-1}$. Then W_{2^n} is τ-effectively open (uniformly in n) and closed under inverse. If $B_n \subseteq K_m$, then by the effective continuity of $^{-1}$, $K_m^{-1} \supseteq B_n^{-1}$ is also effectively compact. We get that U_2 is contained in $K \cup K_m \cup K_m^{-1}$ which are effectively compact. It is then clear that we have $\{U_r\}_{r\leq 2}$ with the desired properties.

Suppose inductively that U_r for $r \leq 2^n$ have been defined s.t. each U_r is τ-effectively open and U_{2^n} is contained in some τ-effectively compact set. For each $2^n < r < 2^{n+1}$, list out all finite sequences of positive rationals $\langle t_i\rangle_{i\leq m}$ s.t. $t_i \leq 2^n$ for each i and $\sum_{i\leq m} t_i = r$. For each such sequence listed out, enumerate $\prod_{i\leq m} U_{t_i}$ into the open name of U_r. By inductive hypothesis, since each U_{t_i} is effectively open, and multiplication is effectively open, then U_r must also be effectively open (uniformly in the index r). Finally take $U_{2^{n+1}} = W_{2^{n+1}} \cup (U_{2^n} \cdot U_{2^n} \cdot U_{2^n} \cdot U_{2^n})$.

To see that property 3 holds, if $r + s < 2^{n+1}$, then the desired property follows easily from the definition of U_{r+s}. Suppose then that $r + s = 2^{n+1}$. If $r = s = 2^n$, then $U_r \cdot U_s = U_{2^n} \cdot U_{2^n} \subseteq U_{2^{n+1}}$ (note that $e \in U_{2^n}$). Therefore we may assume that $r > 2^n$ and $s < 2^n$. Then for any sequence $\langle t_i \rangle_{i \leq m}$ where $\sum_{i \leq m} t_i = r$, $\exists m_0, m_1$ s.t. $\sum_{i=0}^{m_0} t_i \leq 2^n$, $\sum_{i=m_0+1}^{m_1} t_i \leq 2^n$, and $\sum_{i=m_1+1}^{m} t_i \leq 2^n$. By inductive hypothesis, we have that $\prod_{i=0}^{m_0} U_{t_i}, \prod_{i=m_0+1}^{m_1} U_{t_i}, \prod_{i=m_1+1}^{m} U_{t_i} \subseteq U_{2^n}$. Thus this gives us that $U_r \subseteq (U_{2^n})^3$. Then note that $U_s \subseteq U_s \cdot U_{2^n-s} \subseteq U_{2^n}$ (again by the inductive hypothesis and the fact that $e \in U_{2^n-s}$), and thus $U_r \cdot U_s \subseteq (U_{2^n})^4 \subseteq U_{2^{n+1}}$.

To check that property 1 holds, note that if $r < 2^{n-1}$ then $U_r \subseteq U_r \cdot U_{2^{n+1}-r} \subseteq U_{2^{n+1}}$ by property 3 above, and so it is enough to check that $U_{2^{n+1}}$ is contained in an effectively compact set. By the inductive hypothesis, U_{2^n} is contained in some effectively compact set K^*, so $U_{2^{n+1}}$ is contained in $K_m \cup K_m^{-1} \cup (K^*)^4$, where m is s.t. $(n, m) \in R$. It is not hard to check that $(K^*)^4$ is effectively compact, and hence $U_{2^{n+1}}$ is contained in some effectively compact set.

From the definition of U_r, for any r where $2^n < r \leq 2^{n+1}$, $U_r = U_r^{-1}$ and so property 2 holds as well.

Finally, since $\{B_n\}_{n \in \omega}$ is a basis for (G, τ), we have $\bigcup_n W_{2^n} = G$ and hence $\bigcup_r U_r = G$.

Now we define the metric d on G by $d(x, y) = \inf\{r \mid x^{-1}y \in U_r\}$. To see that d is a metric, note that $d(x, y) = 0$ gives that $\forall 0 < r < 2, x^{-1}y \in U_r = B_\delta(e, r)$, meaning that $\delta(x^{-1}y, e) = 0$. Since δ is a metric, it has to be that $x = y$. By property 6, $d(x, x) = 0$. The symmetry of d and triangle inequality follow from property 2 and 3 of $\{U_r\}_{r \in \mathbb{Q}^+}$ respectively. d is obviously left-invariant. It remains to check that $(G, d, \{\alpha_i\}_{i \in \omega})$ is a right-c.e. metric space, G is effectively compatible with $(G, d, \{\alpha_i\}_{i \in \omega})$, and that d is effectively proper. First of all, we have:

Lemma 5.5. *For all $x, y \in G$, if $d(x, y) < 2$ or $\delta(x, y) < 2$ then $d(x, y) = \delta(x, y)$.*

Proof. If $d(x, y) < 2$ then $d(x, y) = \inf\{r < 2 \mid x^{-1}y \in U_r\} = \inf\{r < 2 \mid \delta(x, y) < r\} = \delta(x, y)$. If $\delta(x, y) < 2$ then $x^{-1}y \in U_r$ for some $r < 2$, which means that $d(x, y) < 2$ and so by the above, $d(x, y) = \delta(x, y)$. ⊣

Recall that the sequence $\{\alpha_i\}_{i \in \omega}$, apart from being used as special points for d and δ, are also uniformly computable points with respect to τ. Then together with the fact that U_r are τ-effectively open (uniformly in the index r), one can obviously give a right-c.e. approximation to $d(\alpha_i, \alpha_j)$, uniformly in i, j.

By Lemma 5.5, we have $\overline{(\{\alpha_i\}_{i \in \omega}, d)} = \overline{(\{\alpha_i\}_{i \in \omega}, \delta)} \supset G$, so it is sufficient to show that τ_d and τ_δ are effectively compatible on G. Let $B_d(\alpha_i, r)$ be given. Since d is right-c.e., $\ll_d$ is a c.e. relation, where $\ll_d$ is the usual formal inclusion relation for basic metric balls. Consider the τ_δ-effectively open set consisting of all $B_\delta(\alpha_j, q)$ such that $q < 2$ and $B_d(\alpha_j, q) \ll_d B_d(\alpha_i, r)$. This shows that $B_d(\alpha_i, r)$ is τ_δ-effectively open. To show that each $B_\delta(\alpha_i, r)$ is τ_d-effectively open is similar.

Finally to check that d is effectively proper, we note that by definition of d, $B_d(e, r) \subseteq U_r \subseteq U_{2^n}$ for some sufficiently large n. Given a closed set F and an open ball $B_d(\alpha_i, q) \supseteq F$, take $r = d(\alpha_i, e)[0] + q$, where $d(\alpha_i, e)[0]$ is the first

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rational enumerated by the right cut of $d(\alpha_i, e)$, then note that $F \subseteq B_d(\alpha_i, q) \subseteq B_d(e, r) \subseteq U_r$. For any $n > \log_2(r)$, $F \subseteq U_{2^n} \subseteq K$, where K is τ-effectively compact. But this means that K is also τ_d-effectively compact and a compact name can be found uniformly in n. ⊣

We see that the metrics δ and d are complete and effectively compatible assuming that the group is effectively locally compact, by Theorem 5.4. Thus, by Theorem 4.1 combined with Theorem 5.4, we have:

Corollary 5.6. *If G is an effectively locally compact computable topological group. Then G admits a right-c.e. Polish presentation in which the metric is (effectively) proper and left-invariant.*

In particular, effective local compactness (and the effective compatibility of δ and d in the notation above) implies the proper metric in the corollary above is also effectively locally compact. But of course, being effectively proper is nicer than just being effectively locally compact.

A natural question arises whether we can strengthen these corollaries further and additionally assume that the metric is computable in each of the corollaries above. In the next section we show that the answer is "no" in both cases. In fact, our counter-examples corresponding to Corollaries 5.3 and 5.6 are compact and discrete, respectively.

§6. Comparing and separating the notions. In Sections 4 and 5 we produced (left-invariant) right-c.e. Polish presentations of locally compact groups. We now aim to show that this is tight, i.e., we show that there are computable topological (Polish) groups which are not effectively compatible with any computable metric space. In this section we give two examples, one discrete and one profinite. In the process of proving the results we will establish several lemmas that are perhaps more valuable (or interesting) than the counter examples.

6.1. Discrete groups. Recall that a computable presentation of a discrete countable group is its isomorphic copy of the form F/H, where F is the standard decidable presentation of the free group upon omega generators, and H is its computable normal subgroup [26, 40]. If H is merely c.e., then we say that the group is "c.e.-presented." (These correspond to "recursive" groups with solvable and not necessarily solvable Word Problem, respectively.) We can pick representatives in each class and assume that the domain of a computable group is $\mathbb{N}$; then the group operations are computable (as functions on $\mathbb{N}$). In the c.e.-presented case, we have to also introduce a computably enumerable congruence on $\mathbb{N}$, but we can still keep the operations computable. The difference is that two elements can be at some stage declared equal. Note that this is very similar to the difference between computable and right-c.e. Polish presentations of a group. This intuition is made formal below.

Lemma 6.1. *A countable discrete group is computably presentable iff it admits a computable Polish presentation.*

Proof. Suppose that a group G is computably presentable, i.e., G is generated by $\{\alpha_i\}_{i\in\omega}$ on which the group operations and the equality relation are computable. We consider the standard discrete metric defined on the elements of the computable

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presentation of G, i.e., $d(\alpha_i, \alpha_j) = 0$ iff $\alpha_i = \alpha_j$ and $d(\alpha_i, \alpha_j) = 1$ otherwise. Since testing of equality is computable by assumption, the metric is also computable.

To check that $\cdot$ is effectively continuous with respect to τ_d, given $\alpha_k \in G$ and $r \in \mathbb{Q}^+$, if $r > 1$, we simply enumerate $G \times G$ as the preimage. If $r \leq 1$, then find all pairs α_i, α_j such that $\alpha_i \cdot \alpha_j = \alpha_k$ and enumerate $B_d(\alpha_i, 1) \times B_d(\alpha_j, 1)$. The proof of the effectively continuity of $^{-1}$ is similar.

Conversely suppose that $G = \{\alpha_i\}_{i\in\omega}$ is a countable group and there is a computable discrete metric d defined on G in which the group operations are effectively continuous with respect to τ_d. Even though d is computable, the isolating radius for each α_i might not be. Nonetheless we (not effectively) fix some rational $r > 0$ for which $B_d(e_G, r)$ which isolates e_G. Since the metric is computable, we can decide given any α_i, whether or not $e_G = \alpha_i$, by computing $d(e_G, \alpha_i)$ to an accuracy of $\frac{r}{2}$. Therefore equality in G is computable if we can show that the group operations are computable.

To compute α_i^{-1}, enumerate the preimage of $B_d(e_G, r)$ under $\cdot$ and wait for (U, U') to show up where $\alpha_i \in U$. Then the center of U' is necessarily the inverse of α_i, as $B_d(e_G, r)$ isolates e_G. Now given α_i, α_j, to compute $\alpha_i \cdot \alpha_j$, search for three basic metric balls U, U', U'' such that $U \cdot U' \cdot (U'')^{-1} \subseteq B_d(e_G, r)$ and where $\alpha_i \in U$ and $\alpha_j \in U'$. Then the center of U'' is necessarily equal to $\alpha_i \cdot \alpha_j$. ⊣

LEMMA 6.2. *A countable discrete group is c.e. presentable iff it admits a right-c.e. Polish presentation.*

PROOF. Suppose that a discrete group G is c.e. presentable, i.e., G is generated by $\{\alpha_i\}_{i\in\omega}$ on which the group operations are computable but the equality relation is c.e. We consider the standard discrete metric defined on the elements of the computable presentation of G, i.e., $d(\alpha_i, \alpha_j) = 0$ iff $\alpha_i = \alpha_j$ and $d(\alpha_i, \alpha_j) = 1$ otherwise. Since equality is c.e. by assumption, the metric is right-c.e. To see that the operations are effectively continuous w.r.t. the topology induced by d, repeat the same procedure as in Lemma 6.1.

Conversely suppose that $G = \{\alpha_i\}_{i\in\omega}$ is a countable group and there is a right-c.e. discrete metric d defined on G in which the group operations are effectively continuous with respect to τ_d. Again we fix some rational $r > 0$ such that $B_d(e_G, r)$ which isolates e_G. To see that the group operations are computable we follow exactly as in Lemma 6.1, noting that the predicate "$\alpha_i \in B_d(\alpha_j, r)$" is still c.e. Since the metric is right-c.e., and the operations are computable, equality in G is c.e. ⊣

COROLLARY 6.3. *There exists a computable topological discrete abelian group* (*thus, right-c.e. Polish*) *that is not topologically isomorphic to any computable Polish group.*

PROOF. Consider the group $G = \bigoplus_{k\in S} \mathbb{Z}_{p_k}$ where S is a Σ^0_2 set that is not c.e., and p_k is the k-th prime number. Then G is c.e. presentable [22, 31] with no computable presentation. By Lemma 6.2, the group admits a right-c.e. (thus, computable topological) presentation. By Lemma 6.1, it is not topologically isomorphic to any computable Polish group. ⊣

6.2. A profinite counterexample. Recall that in Corollary 5.6 we produced a right-c.e. proper Polish presentation which, by effective compatibility, was also effectively locally compact. Can we produce a *computable* (proper) metric, say, in the simplest

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compact case? Note that in the case of a compact Polish group we vacuously have a proper metric. We now prove that the answer is "no."

PROPOSITION 6.4. *There exists a profinite group G that admits an effectively compact right-c.e. Polish presentation but has no computably compact* (*effectively compact computable Polish*) *presentation*.

PROOF. The proof that we outline below resembles similar counter-examples in [15, 27, 44]. However, in our case a bit more care is needed.

We construct G to be isomorphic to the direct product of cyclic groups

$$G_S = \prod_{i \in S} \mathbb{Z}_{p_i},$$

where $S \subseteq \omega$ and $p_1, p_2, \ldots$ is the fixed natural enumeration of all primes.

LEMMA 6.5. *G_S has a computably compact* (*effectively compact and computable Polish*) *presentation iff S is c.e.*

PROOF OF LEMMA. It has been established in [6] that, for a profinite G to be computably compact it is necessary and sufficient that G has "recursive" presentation in the sense of Smith [44]. As was illustrated in [27], for G_S it is also equivalent to computable presentability of the dual discrete group $\bigoplus_{i \in S} \mathbb{Z}_{p_i}$, and the latter is evidently equivalent to S being c.e. ⊣

By the lemma above, it is sufficient to construct an effectively compact right-c.e. Polish (ecrp) presentation of $G_{\overline{K}}$, where K is the halting set. This is done as follows. Fix an enumeration $(K_s)_{s \in \omega}$ of K. We can assume that, for each s, $K_{s+1} \setminus K_s$ has at most one number that we denote $k(s+1)$.

Fix the natural order on the (indices of) the elements of each cyclic $\mathbb{Z}_{p_i}$:

$$0 < 1 < \cdots < p_i - 1,$$

and then (lexicographically) extend this order to $\prod_{i \in \omega} \mathbb{Z}_{p_i}$. Also, fix the natural shortest-path ultra-metric d on $\prod_{i \in \omega} \mathbb{Z}_{p_i}$,

$$d(\xi, \eta) = \inf\{2^{-k} : \xi, \eta \text{ agree up to first } k \text{ coordinates}\}.$$

Observe that $G_{\overline{K}} = \prod_{i \notin K} \mathbb{Z}_{p_k}$ can be viewed as a Π^0_1 class in $\prod_{i \in \omega} \mathbb{Z}_{p_i}$; consider the tuples in $\prod_{i \in \omega} \mathbb{Z}_{p_i}$ having zero projections onto the $\mathbb{Z}_{p_i}$ when $i \in K$. Let H be this Π^0_1 class.

Define

$$\delta_s(\xi, \eta) = \inf_{k \notin K_s} \{2^{-k} : \xi, \eta \text{ agree up to first } k \text{ coordinates}\}.$$

(In particular, we declare two strings equal if they differ only at the j-coordinates for $j \in K_s$.)

It is clear that δ_s induces a Δ^0_2 functional acting on infinite strings; set

$$\delta = \lim_s \delta_s.$$

It is evidently equal to d when restricted to H; this is because $j \in K$ entails that the jth projection of any $\xi \in H$ is 0. On the other hand, we claim that for each

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$\xi \in \prod_{i\in\omega} \mathbb{Z}_{p_i} \setminus H$ there is a string $\eta \in H$ such that

$$\lim_s \delta_s(\xi, \eta) = 0.$$

For that, just take the string η by replacing the ith component of ξ with 0. Also, δ_s is clearly non-decreasing in s, on any input, and therefore δ is a right-c.e., as required. The computable dense set in H with respect to δ is given by strings in $\prod_{i\in\omega} \mathbb{Z}_{p_i}$ having finite support (identified with the respective finite strings). This shows that $(\prod_{i\in\omega} \mathbb{Z}_{p_i}, \delta)$ is a right-c.e. Polish presentation of H. Since $(\prod_{i\in\omega} \mathbb{Z}_{p_i}, \delta)$ is computably compact, and since $\delta \leq d$, the covers witnessing computable compactness of $(\prod_{i\in\omega} \mathbb{Z}_{p_i}, \delta)$ witness the effective compactness of $(\prod_{i\in\omega} \mathbb{Z}_{p_i}, \delta) \cong H$.

It remains to check that the group operations on H are computable with respect to δ. The idea is to use that the group operations $+$ and $-$ in $\prod_{i\in\omega} \mathbb{Z}_{p_i}$ that are (evidently) computable with respect to the natural ultrametric d.

For a finite string σ (which is identified with $\sigma 0^\omega \in \prod_{i\in\omega} \mathbb{Z}_{p_i}$) let

$$D_s(\sigma, 2^{-n}) = \{\xi : \delta_s(\sigma, \xi) < 2^{-n}\},$$

which is the basic open 2^{-n}-ball centred in σ, with respect to δ_s. Since evidently δ_s is non-increasing in s,

$$D_s(\sigma, 2^{-n}) \subseteq D_{s+1}(\sigma, 2^{-n}),$$

for any $s, n \in \omega$ and any string σ. In the claim below, $+$ and $-$ are the group operations of $\prod_{i\in\omega} \mathbb{Z}_{p_i}$. Claim 6.6 guarantees that, once a triple of basic open sets is put into a name for $+$ or $-$ with respect to δ_s, it can be re-used in the name for the same operation with respect to δ_{s+1}.

CLAIM 6.6. *Suppose we have $D_s(\rho, 2^{-k}) + D_s(\tau, 2^{-m}) \subseteq D_s(\sigma, 2^{-n})$. Then for every $t > s$, $D_t(\rho, 2^{-k}) + D_t(\tau, 2^{-m}) \subseteq D_t(\sigma, 2^{-n})$. (The same holds if we replace $+$ with $-$ throughout.)*

PROOF. It is sufficient to check the property for $t = s + 1$ assuming $\delta_s \neq \delta_{s+1}$, i.e., $j = k(s + 1) \in K_{s+1} \setminus K_s$. Without loss of generality, we may assume that the length of ρ is equal to k, and similarly the lengths of τ and σ are equal to m and n, respectively. It also has to be that $n \leq m, k$ since these clopen sets are cosets.

Fix $(a_i)_i \in D_{s+1}(\rho, 2^{-k})$ and $(b_i)_i \in D_{s+1}(\tau, 2^{-m})$. Let $(a'_i)_i$ and $(b'_i)_i$ be strings that differ from $(a_i)_i$ and $(b_i)_i$ only at the jth coordinate, and the choice of a'_j and b'_j guarantees $(a_i)_i \in D_s(\rho, 2^{-k})$ and $(b_i)_i \in D_s(\tau, 2^{-m})$. Since the definition of δ_{s+1} ignores the jth coordinate, but otherwise is equal to δ_s, we have that such a'_j and b'_j exist. By our assumption,

$$(a'_i)_i + (b'_i)_i = (a'_i + b'_i \mod p_i)_i \in D_s(\sigma, 2^{-n}).$$

The two strings $\xi = (a_i + b_i \bmod p_i)_i$ and $\xi' = (a'_i + b'_i \bmod p_i)_i$ may differ only at position j. Since δ_{s+1} ignores this coordinate in its computation, and since $\delta_{s+1} \leq \delta_s$ in general, we have that

$$\delta_{s+1}(\sigma, \xi') = \delta_{s+1}(\sigma, \xi) \leq \delta_s(\sigma, \xi) < 2^{-n},$$

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and thus

$$\xi' = (a_i' + b_i' \mod p_i)_i \in D_{s+1}(\sigma, 2^{-n}),$$

as required. (The case of "–" is similar, mutatis mutandis.) ⊣

As before, fix a finite string σ. In the claim below, $+$ and – denote the group operations in $\prod_{i \in \omega} \mathbb{Z}_{p_i}$. As noted earlier, these operations are computable with respect to d, but they are also computable with respect to δ_s, uniformly in s.

CLAIM 6.7. *The preimage of* $B(\sigma, 2^{-n}) = \{\xi : \delta(\xi, \sigma) < 2^{-n}\}$ *under* $+$ *is equal to*

$$\bigcup_t \{(B(\rho, 2^{-k}), B(\tau, 2^{-m})) : D_t(\rho, 2^{-k}) + D_t(\tau, 2^{-m}) \subseteq D_t(\sigma, 2^{-n}) \textit{ is listed in } N^+_{\delta_s}\},$$

where $N^+_{\delta_s}$ *the name for* $+$ *w.r.t.* δ_t. (*The same holds if we replace* $+$ *with* – *throughout.*)

PROOF. We claim that, for every finite sigma ρ and any n, there exists an s such that

$$B(\sigma, 2^{-n}) = D_s(\sigma, 2^{-n}).$$

This property follows from the definition of $B(\sigma, 2^{-n})$ and the fact that only the change of δ at the coordinates j with $j \le length(\sigma)$ can possibly make $D_{s+1}(\sigma, 2^{-n}) \neq D_s(\sigma, 2^{-n})$. Thus, if $\xi + \eta \in B(\sigma, 2^{-n})$ then for some large enough stage t and some strings ρ, τ,

$$D_t(\rho, 2^{-k}) + D_t(\tau, 2^{-m}) \subseteq D_t(\sigma, 2^{-n}),$$

where $\xi \in D_t(\rho, 2^{-k}) = B(\rho, 2^{-k})$ and $\eta \in D_t(\tau, 2^{-m}) = B(\tau, 2^{-m})$ and

$$B(\rho, 2^{-k}) + B(\tau, 2^{-m}) \subseteq B(\sigma, 2^{-n}).$$

Thus, this pair of open sets will eventually be listed. It shows that the set

$$\bigcup_t \{(B(\rho, 2^{-k}), B(\tau, 2^{-m})) : D_t(\rho, 2^{-k}) + D_t(\tau, 2^{-m}) \subseteq D_t(\sigma, 2^{-n}) \text{ is listed in } N^+_{\delta_s}\}$$

contains the pre-image of $B(\sigma, 2^{-n})$ under $+$.

To see why this set does not exceed the preimage of $B(\sigma, 2^{-n})$ under $+$, suppose we have $D_s(\rho, 2^{-k}) + D_s(\tau, 2^{-m}) \subseteq D_s(\sigma, 2^{-n})$ for some s. Then, by Claim 6.6, $D_t(\rho, 2^{-k}) + D_t(\tau, 2^{-m}) \subseteq D_t(\sigma, 2^{-n})$ for any $t > s$ as well. As we noted above, for a large enough s we have

$$B(\rho, 2^{-k}) = D_t(\rho, 2^{-k}), \; B(\tau, 2^{-m}) = D_t(\tau, 2^{-m}), \text{ and } B(\sigma, 2^{-n}) = D_t(\sigma, 2^{-n});$$

thus, in particular, $D_t(\rho, 2^{-k}) + D_t(\tau, 2^{-m}) \subseteq D_t(\sigma, 2^{-n})$ entails that

$$B(\rho, 2^{-k}) + B(\tau, 2^{-m}) \subseteq B(\sigma, 2^{-n}),$$

holds. (The case of "–" is similar, up to a change of notation.) ⊣

To finish the proof of the proposition, apply Claim 6.7 to list the c.e. open names of $+$ and – with respect to δ. ⊣

§7. Connections to other notions in the literature. We now briefly discuss how the notions of effective presentability of Polish groups studied in the paper are related to other notions of effectiveness in the literature.

In Section 6 we illustrated that, in the discrete case, computable Polish presentability is equivalent to computable presentability in the sense of Mal'cev [26] and Rabin [40]. Also, right-c.e. Polish presentability is equivalent to c.e. presentability for discrete groups. Note that, in the discrete case, all our presentations are vacuously effectively locally compact.

A version of computable (local) compactness seems to be a necessary extra assumption when considering computable topological group presentations of locally compact groups. As illustrated in [27], computable Polish presentations alone do not make Pontryagin duality effective in the compact abelian case. In contrast, the aforementioned [27] and the recent [6, 25] establish effective versions of Pontryagin duality for computably compact connected abelian groups and "recursive" profinite abelian groups. We do not know the following:

QUESTION 7.1. *For profinite groups, is co-c.e. presentability equivalent to effectively compact right-c.e. Polish presentability*?

This is certainly the case for some profinite groups, as exploited (implicitly) in the proof of Proposition 6.4.

In the satellite paper [29] we investigate the especially nice case of computably locally compact Polish groups. Quite interestingly, we show that in the totally disconnected locally compact (tdlc) case, a group admits a presentation like that if and only if it is computably presentable in the sense of [25, 30]. We note that [25, 30] contain several equivalent definitions of computable presentability of a tdlc group, all of which turn out to be equivalent. These equivalent definitions also generalize the profinite and discrete cases discussed above, and additionally make the Pontryagin–van Kampen duality fully effective for tdlc abelian groups whose duals are also tdlc.

Beyond local compactness, the notion of a computable Polish group turned out to be closely related to computable structure theory. Interestingly, many results in computable structure theory can be viewed as a special case of a computable Polish group computably acting on a computable Polish space. Also, typically the more general result requires a simpler proof; see [28]. As noted in [28], many results in [28] can be carried under the weaker assumption that the group is computable topological and admits a c.e. strong (or formal) inclusion; see Remark 4.5 for a discussion. Quite unexpectedly, the proof of our first main result Theorem 1.1 implies that these seemingly strong extra assumptions in [28] can be completely dropped when we talk about computable topological Polish groups; recall Remark 4.5.

Of course, there are other potential notions of computable presentability that could perhaps work for some special subclasses of Polish groups. For instance, we have mentioned left-c.e. (lower semi-computable) Polish spaces; these are defined similarly to right-c.e. Polish spaces, but they seem less well-understood than the latter. For instance, even finding a natural example of such a space that would not be obviously computable Polish is a bit of a challenge. It is known, however, that there is a left-c.e. Polish space not homeomorphic to any computable Polish space [29]. Left-c.e. Polish spaces do not necessarily induce a natural computable topological structure, and thus perhaps are not suitable for representing topological groups in general. However, interestingly, every left-c.e. Polish Stone space is homeomorphic to a computable Polish space [29] and, thus, to a computably compact one [6, 15]. So it could be that the notion is suitable and well-behaved in the context of profinite or tdlc groups. The reason behind it is that, while right-c.e. spaces make formal inclusion c.e., left-c.e. spaces make formal disjointedness c.e., and thus we can effectively split the space into connected components. We omit these definitions.

We mention another notion of computable presentability motivated by research in computable structure theory that we did not include into the diagram in the introduction. Classically, closed subgroups of S_∞ are exactly the automorphism groups of discrete structures; see [11]. Every automorphism group of a discrete *computable* structure is a Π^0_1 (effectively closed) subgroup of a certain natural effective presentation of S_∞; see [12, 30]. However, the converse fails [12, 15]. For instance, it is known that a *compact* (thus, profinite) Π^0_1 subgroup of S_∞ does not have to be topologically isomorphic to a Δ^0_α-Polish group for any fixed computable α [15]. Therefore, already for compact groups, this notion of computable presentability is (much) weaker than the weakest definition of a computable topological group that we study in this paper. Strictly speaking, such presentations are not really computable since one has essentially no access to the evasive domain of the group.

We have discussed the weak notion of an effectively closed subgroup of S_∞. Other weak notions include (hyper)arithmetical presentations of higher degree, such as Δ^0_α-Polish and right- or left-Σ^0_α Polish presentations. Indeed, we have already mentioned Δ^0_α-Polish presentations above.

We believe that most of these definitions can be separated from each other by direct relativisation of the known effective results or using Pontryagin duality and the corresponding results from the discrete abelian case [21], and leave this as an open question. However, the importance and the exact role of these notions in the theory is not yet clear (beyond their use in extreme counter-examples such as the aforementioned one from [15]).

§8. Conclusion: the two main definitions. The results presented in the present paper, combined with various results in [6, 12, 15, 25, 27, 29, 30] some of which have been discussed above, establish a solid foundation for the rapidly emerging general theory of algorithmically presented topological groups. In particular, it appears that the basic definitions of effective presentability are robust and nicely align themselves (via direct equivalence or duality) with the well-established notions that work for profinite and computable groups. At least in the important case of locally compact Polish groups, the overall intuition seems to be as follows:

$$\text{computable Polish} + \text{computably locally compact} \sim \text{computable}$$

and

$$\text{right-c.e. Polish} + \text{effectively locally compact} \sim \text{computably enumerable},$$

where $\sim$ stands for "should be viewed as an adequate generalisation of." The subtle difference between computable compactness and effective compactness was elaborated in the preliminaries.

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There are many open questions that can be attacked in the new theory; e.g., we cite [7] for problems related to computable classification. We state a few.

One can consider complexity as measured by enumeration:

QUESTION 8.1. *Which classes of profinite or tdlc groups admit a Friedberg enumeration*?

One can also measure the complexity of a structure via index sets:

QUESTION 8.2. *What is the complexity of the index set of, say* $SO_3(\mathbb{R})$ *or any sufficiently interesting group, up to topological isomorphism*?

Also, we wonder if Pontryagin–van Kampen duality works for arbitrary computably locally compact abelian groups; this question has been raised in [25]. We leave these (and many other) questions of this sort open for future investigation.

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