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THE ANNIHILATOR OF TENSOR SPACE IN THE q-ROOK MONOID ALGEBRA

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Abstract

In this paper, we give an explicit construction of a quasi-idempotent in the q-rook monoid algebra $R_n(q)$ and show that it generates the whole annihilator of the tensor space $U^{\otimes n}$ in $R_n(q)$.

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1. Introduction

The *q-rook monoid algebra* $R_n(q)$ (see Section 2.1 for a precise definition), was first studied by Solomon [15] as the Iwahori–Hecke algebra for the monoid of matrices over a finite field. Then the representation theory of q-rook monoid algebras and their specialisation analogues (with $q = 1$) was taken up in [1, 4, 5, 16]. Paget in [13] considered the modular representation theory of q-rook monoid algebras and proved that the q-rook monoid algebra $R_n(q)$ (where q may be a unit root) is a cellular algebra in the sense of Graham and Lehrer [3] (see [2] for the case of $q = 1$).

In [17], Solomon defined an action of $R_n(q)$ on the tensor space $U^{\otimes n}$, where $U = L(0) \oplus L(\varepsilon_1)$ is the direct sum of the trivial and natural module for the quantum general linear group $U_q(\mathfrak{gl}_m)$. Halverson in [5] found a new presentation of $R_n(q)$ and used it to show that Solomon's action of $R_n(q)$ on the tensor space $U^{\otimes n}$ can be extended to a Schur–Weyl duality as follows.

Theorem 1.1 [5, Corollary 4.3]. *The map $\varphi : R_n(q) \to \mathrm{End}_{U_q(\mathfrak{gl}_m)}(U^{\otimes n})$ is a surjective algebra homomorphism and, if $m \geq n$, then φ is an isomorphism.*

When $m < n$, the algebra homomorphism φ is in general not injective. Therefore it is natural to ask how to describe the kernel of the homomorphism φ, that is, the annihilator of $U^{\otimes n}$ in the algebra $R_n(q)$. The purpose of this article is to answer the question. Furthermore, we characterise the generators of $\mathrm{Ker}(\varphi)$ at an integral level so as to be compatible with the cellular structure of $R_n(q)$ and $\mathrm{End}_{U_q(\mathfrak{gl}_m)}(U^{\otimes n})$. In other words, the generators of $\mathrm{Ker}(\varphi)$ belong to a $\mathbb{Z}[q, q^{-1}]$-lattice of $R_n(q)$.

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In the invariant theory of classical and quantum groups, characterising the annihilator of a tensor power of the natural module of a classical or quantum group in a Hecke algebra, Brauer algebra, or Birman–Murakami–Wenzl (BMW) algebra is one formulation of the second fundamental theorem of invariant theory (see [11] and the references therein for a detailed description of this topic). Recently, Hu and the author [8] proved the second fundamental theorem for symplectic groups and Lehrer and Zhang [10] gave the second fundamental theorem for orthogonal groups, taking advantage of a different formulation of the invariant theory. It is surprising to some extent that in both the symplectic and orthogonal cases and their quantised versions, the annihilator of n-tensor space in a specialised Brauer algebra or BMW algebra is generated by an explicitly described quasi-idempotent. Motivated by these results, we have found that the annihilator of tensor space $U^{\otimes n}$ in a rook monoid algebra (the case $q = 1$ in the present paper) is also generated by a quasi-idempotent [18]. We shall construct a quasi-idempotent Φ_{m+1} (see Section 3) in $\operatorname{Ker}\varphi$ and prove the following result.

Theorem 1.2. *With the above notation, if $m < n$, then* $\operatorname{Ann}_{R_n(q)}(U^{\otimes n}) = \langle \Phi_{m+1} \rangle$.

On the other hand, Halverson and Ram in [6] proved that the q-rook monoid algebra $R_n(q)$ is a quotient of the Hecke algebra of type B. From this point of view, they showed that the Schur–Weyl duality for $R_n(q)$ (Theorem 1.1) comes from a Schur–Weyl duality for cyclotomic Hecke algebras studied in [7, 14]. Another motivation of this paper is to try to build a bridge to characterise the annihilator of tensor space in a cyclotomic Hecke algebra.

Note that one of the main differences between q-rook monoid algebras and the Hecke algebras, Brauer algebras and BMW algebras is that the q-rook monoid algebra $R_n(q)$ generally cannot be realised as a diagram algebra except in the case of $q = 1$ (see [5, Remark 4.4]). Therefore our proof of Theorem 1.2 differs from that in [8, 11, 18] and we will view $R_n(q)$ as a module of the Hecke algebra of a symmetric group.

2. Preliminaries

2.1. The q-rook monoid. Let q be an indeterminate. Halverson [5] defined the q-rook monoid algebra $R_n(q)$ to be the unital associative $\mathbb{C}(q)$-algebra generated by $T_1, T_2, \ldots, T_{n-1}$ and $P_1, P_2, \ldots, P_n$ subject to the relations:

(A1) $T_i^2 = (q - q^{-1})T_i + 1,$ for $1 \le i \le n-1$,

(A2) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$ for $1 \le i \le n-2$,

(A3) $T_i T_j = T_j T_i,$ for $|i - j| > 1$,

(R1) $P_i^2 = P_i,$ for $1 \le i \le n$,

(R2) $P_i P_j = P_j P_i,$ for $1 \le i, j \le n$,

(R3) $P_i T_j = T_j P_i,$ for $1 \le i < j \le n-1$,

(R4) $P_i T_j = T_j P_i = q P_i,$ for $1 \le j < i \le n$,

(R5) $P_{i+1} = q P_i T_i^{-1} P_i = q P_i T_i P_i - (q^2 - 1) P_i,$ for $1 \le i \le n-1$.

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Note that our definition of $R_n(q)$ is slightly different from the definition in [5]. However, it is equivalent (see [6, Remark 1.2]). Halverson gave a basis of $R_n(q)$ which we now recall. Throughout this paper, we identify the symmetric group $\mathfrak{S}_n$ with the group of *left* permutations on the set $\{1, 2, \ldots, n\}$. For $\sigma \in \mathfrak{S}_n$ with reduced expression $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$ define $T_\sigma := T_{i_1} T_{i_2} \cdots T_{i_k}$. Then T_σ is well defined because of the braid relations (A2) and (A3). Furthermore, the subalgebra generated by $T_1, T_2, \ldots, T_{n-1}$, denoted by $H_n(q)$, is isomorphic to an Iwahori–Hecke algebra of type A (see [5, Corollary 3.4]).

For an integer r with $0 \leq r \leq n$, define

$$\mathcal{D}_r := \{d \in \mathfrak{S}_n \mid d(1) < d(2) < \cdots < d(r), d(r+1) < \cdots < d(n)\}.$$

Note that $\mathcal{D}_0 = \{1\}$ and $\mathcal{D}_r$ is the set of distinguished left coset representatives of the parabolic subgroup $\mathfrak{S}_{(r,n-r)}$ in $\mathfrak{S}_n$. Write $\Omega_r := \{(d_1, d_2, \sigma) \mid d_1, d_2 \in \mathcal{D}_r, \sigma \in \mathfrak{S}_{\{r+1,\ldots,n\}}\}$ and $\Omega := \bigcup_{r=0}^{n} \Omega_r$. For $(d_1, d_2, \sigma) \in \Omega_r$, define

$$T_{(d_1,d_2,\sigma)} := T_{d_1} P_r T_\sigma T_{d_2}^{-1}.$$

When $r = 0$, we interpret $P_0 = 1$. For $d \in \mathcal{D}_r$, if we assume that $a_i = d(i)$ for $1 \leq i \leq r$, then there is a reduced expression

$$d = (s_{a_1-1} \cdots s_2 s_1)(s_{a_2-1} \cdots s_3 s_2) \cdots (s_{a_r-1} \cdots s_{r+1} s_r).$$

Hence our notation coincides with that in [5, Section 2].

Lemma 2.1 [5, Theorem 2.1 and Corollary 2.2]. *The set $\{T_{(d_1,d_2,\sigma)} \mid (d_1, d_2, \sigma) \in \Omega\}$ forms a basis of $R_n(q)$.*

As foreshadowed in the introduction, we want to characterise the generators of $\mathrm{Ker}(\varphi)$ at an integral level so as to be compatible with the cellular structure of $R_n(q)$ and $\mathrm{End}_{U_q(\mathfrak{gl}_m)}(U^{\otimes n})$. We shall use a slightly different basis of $R_n(q)$ to that in Lemma 2.1. Let $*$ be the involution, an anti-automorphism of order 2, of $R_n(q)$ defined on the generators by

$$T_i^* := T_i, \quad P_j^* := P_j \quad \text{for } 1 \leq i \leq n-1,\ 1 \leq j \leq n.$$

The proof of the following lemma is similar to that of [13, Proposition 3] and hence we omit it here.

Lemma 2.2. *The set $\{T_{d_1} P_r T_\sigma T_{d_2}^* \mid (d_1, d_2, \sigma) \in \Omega\}$ forms a basis of $R_n(q)$.*

2.2. The classical case ($q = 1$). In this subsection, we recall the main results of [18] for later use. Let R_n be the set of all $n \times n$ matrices that contain *at most* one entry equal to 1 in each row and column and zeros elsewhere. With the operation of matrix multiplication, R_n has the structure of a monoid. The monoid R_n is known both as the *rook monoid* and the *symmetric inverse semigroup* [15]. The following presentation of R_n is much more helpful. The rook monoid R_n is generated by $s_1, s_2, \dots, s_{n-1}$ and $p_1, p_2, \dots, p_n$ subject to the following relations:

$$\begin{aligned} s_i^2 &= 1 && \text{for } 1 \le i \le n-1, \\ s_i s_j &= s_j s_i && \text{for } |i-j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for } 1 \le i \le n-2, \\ p_i^2 &= p_i && \text{for } 1 \le i \le n, \\ p_i p_j &= p_j p_i && \text{for } i \ne j, \\ s_i p_i &= p_{i+1} s_i && \text{for } 1 \le i \le n-1, \\ s_i p_j &= p_j s_i && \text{for } j \ne i, i+1, \\ p_i s_i p_i &= p_i p_{i+1} && \text{for } 1 \le i \le n-1. \end{aligned}$$

From this presentation, it is clear that the q-rook monoid algebra $R_n(q)$ is indeed a q-analogue of the rook monoid algebra $\mathbb{C}R_n$. Notice, when we take the specialisation $q \to 1$, that $\lim_{q\to 1} P_j = p_1 p_2 \cdots p_j$ for each $1 \le j \le n$.

Let V be an m-dimensional vector space over the field $\mathbb{C}$. Let $U_1 = \mathbb{C} \oplus V$ and $\mathrm{GL}(V)$ denote the general linear group over V. The following analogue of Theorem 1.1 was proved by Solomon [16, Theorem 5.10 and Corollary 5.18].

Proposition 2.3. *The map* $\varphi_1 : \mathbb{C}R_n \to \mathrm{End}_{\mathrm{GL}(V)}(U_1^{\otimes n})$ *is a surjective algebra homomorphism and, if* $m \ge n$, *then* φ *is an isomorphism.*

For any positive integer $k \le n$, the natural map $s_i \mapsto s_i, p_j \mapsto p_j$ for all $1 \le i \le k-1$ and $1 \le j \le k$ extends to an algebra embedding from $\mathbb{C}R_k$ into $\mathbb{C}R_n$. In [18, Section 4], when $m < n$, we defined a quasi-idempotent

$$Y_{m+1} = \sum_{\sigma \in \mathfrak{S}_{m+1}} (-1)^{\ell(\sigma)} \sigma - \sum_{(d_1, d_2, \sigma) \in \Omega_1} (-1)^{\ell(d_1) + \ell(\sigma) + \ell(d_2)} d_1 p_1 \sigma d_2^{-1} \in \mathbb{C}R_{m+1}.$$

Proposition 2.4 [18, Theorem 1.2]. *If* $m < n$, *then* $\mathrm{Ann}_{\mathbb{C}R_n}(U_1^{\otimes n}) = \langle Y_{m+1} \rangle$.

2.3. Specialisations. We now relate the quantised case to the classical ($q = 1$) case and then find a way to construct the generators of $\mathrm{Ker}(\varphi)$ at an integral level. Let $\mathcal{A}_q$ be the subring of $\mathbb{C}(q)$ consisting of the rational functions with no pole at $q = 1$. The evaluation map $\psi_1 : \mathcal{A}_q \to \mathbb{C}$ taking q to 1 is a $\mathbb{C}$-algebra homomorphism.

Let $R_n(\mathcal{A}_q)$ be the $\mathcal{A}_q$-span of the set $\{T_{d_1} P_r T_\sigma T^*_{d_2} \mid (d_1, d_2, \sigma) \in \Omega\}$. Then $R_n(\mathcal{A}_q)$ is an $\mathcal{A}_q$-subalgebra of $R_n(q)$ and $R_n(q) = \mathbb{C}(q) \otimes_\iota R_n(\mathcal{A}_q)$, where ι is the inclusion of $\mathcal{A}_q$ into $\mathbb{C}(q)$ (see the cellular structure of a q-rook monoid algebra in [13]). On the other hand, since $U = L(0) \oplus L(\varepsilon_1)$ is the direct sum of the trivial and natural module for $U_q(\mathfrak{gl}_m)$, both $U_q(\mathfrak{gl}_m)$ and $U^{\otimes n}$ have $\mathcal{A}_q$-forms $U_{\mathcal{A}_q}(\mathfrak{gl}_m)$ and $U^{\otimes n}_{\mathcal{A}_q}$, such that $U_{\mathcal{A}_q}(\mathfrak{gl}_m)$ acts on $U^{\otimes n}_{\mathcal{A}_q}$. We can therefore take the specialisation $\lim_{q\to 1} := \mathbb{C} \otimes_{\psi_1} -$, for all the $\mathcal{A}_q$-modules just mentioned. It is well known that $\lim_{q\to 1} U_{\mathcal{A}_q}(\mathfrak{gl}_m) = U(\mathfrak{gl}_m)$, the universal enveloping algebra of $\mathfrak{gl}_m$ over $\mathbb{C}$. Clearly $\lim_{q\to 1} R_n(\mathcal{A}_q) = \mathbb{C}R_n$. We refer to [9] for more details of the specialisation of quantum groups.

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The following proposition indicates a way to construct the generators of $\mathrm{Ker}(\varphi)$. The proof is similar to that in [11, Theorem 8.2].

Proposition 2.5. *With the above notation, let Φ be an idempotent in $\mathbb{C}R_n$ such that the ideal $\langle\Phi\rangle = \mathrm{Ker}(\varphi_1)$. Assume that $\Phi_q \in R_n(\mathcal{A}_q)$ is such that:*

(1) $\Phi_q^2 = f(q)\Phi_q$, *where* $f(q) \in \mathcal{A}_q$;
(2) $\lim_{q\to 1} \Phi_q = c\Phi$, *where* $c \neq 0$.

Then Φ_q generates the ideal $\mathrm{Ker}(\varphi)$.

Proof. It follows from $\lim_{q\to1}\langle\Phi_q\rangle = \langle\Phi\rangle$ that $\dim_{\mathbb{C}(q)}\langle\Phi_q\rangle \geq \dim_{\mathbb{C}}\langle\Phi\rangle$. Here $\langle\Phi_q\rangle$ is the ideal in $R_n(q)$ generated by Φ_q. Hence, if $\Phi_q \in \mathrm{Ker}(\varphi)$,

$$\begin{aligned}\dim_{\mathbb{C}} \mathbb{C}R_n/\langle\Phi\rangle &\geq \dim_{\mathbb{C}(q)} R_n(q)/\langle\Phi_q\rangle \\ &\geq \dim_{\mathbb{C}(q)} R_n(q)/\,\mathrm{Ker}(\varphi) \\ &= \dim_{\mathbb{C}(q)} \mathrm{End}_{U_q(\mathfrak{gl}_m)}(U^{\otimes n}) = \dim_{\mathbb{C}} \mathbb{C}R_n/\langle\Phi\rangle.\end{aligned}$$

We now prove $\Phi_q \in \mathrm{Ker}(\varphi)$, that is, $\Phi_q U^{\otimes n} = 0$. In fact, we only need to prove $\Phi_q U^{\otimes n}_{\mathcal{A}_q} = 0$. Note that $\lim_{q\to1} \Phi_q U^{\otimes n}_{\mathcal{A}_q} = c\Phi U^{\otimes n}_1 = 0$ and hence $\Phi_q U^{\otimes n}_{\mathcal{A}_q} \subseteq (q-1)U^{\otimes n}_{\mathcal{A}_q}$. We use a recursive procedure to show that $\Phi_q U^{\otimes n}_{\mathcal{A}_q} \subseteq (q-1)^i U^{\otimes n}_{\mathcal{A}_q}$ for each positive integer i, which in turn implies that $\Phi_q U^{\otimes n}_{\mathcal{A}_q} = 0$. Assume that $\Phi_q U^{\otimes n}_{\mathcal{A}_q} \subseteq (q-1)^i U^{\otimes n}_{\mathcal{A}_q}$ for some positive integer i. Then $f(q)\Phi_q U^{\otimes n}_{\mathcal{A}_q} = \Phi_q^2 U^{\otimes n}_{\mathcal{A}_q} \subseteq (q-1)^{i+1} U^{\otimes n}_{\mathcal{A}_q}$ by the inductive hypothesis. But $f(q)$ is not divisible by $q-1$ in $\mathcal{A}_q$, since $\lim_{q\to1} \Phi_q^2 = c^2\Phi = f(1)\Phi \neq 0$. In other words, $f(q)$ is invertible in $\mathcal{A}_q$. Therefore $\Phi_q U^{\otimes n}_{\mathcal{A}_q} \subseteq (q-1)^{i+1} U^{\otimes n}_{\mathcal{A}_q}$ and this completes the proof of the proposition. □

3. Proof of Theorem 1.2

By Propositions 2.5 and 2.4, to construct the generators of $\mathrm{Ker}(\varphi)$, we only need to construct a q-analogue of Y_{m+1}. In other words, we need to construct an element $\Phi_{m+1} \in R_{m+1}(q)$ having the one-dimensional sign representation of $R_{m+1}(q)$ (see [18, Section 3]), that is,

$$T_i\Phi_{m+1} = \Phi_{m+1}T_i = (-q)^{-1}\Phi_{m+1} \quad \text{and} \quad P_j\Phi_{m+1} = \Phi_{m+1}P_j = 0$$

for all $1 \leq i \leq m$ and $1 \leq j \leq m+1$.

Since we work on the field $\mathbb{C}(q)$, the q-rook monoid algebra $R_n(q)$ is semisimple [17]. By the representation theory of $R_n(q)$ [5, 13], there exists an element $\Phi_n \in R_n(q)$ for $n \geq 2$ such that $T_i\Phi_n = \Phi_nT_i = (-q)^{-1}\Phi_n$ and $P_j\Phi_n = \Phi_nP_j = 0$ for all $1 \leq i \leq n-1$ and $1 \leq j \leq n$.

Lemma 3.1. *The element Φ_n can be taken of the form*

$$\Phi_n = \sum_{\sigma\in\mathfrak{S}_n} (-q)^{-\ell(\sigma)}T_\sigma + \sum_{r=1}^{n} \sum_{(d_1,d_2,\sigma)\in\Omega_r} C_{(d_1,d_2,\sigma)}(-q)^{-\ell(d_1)-\ell(\sigma)-\ell(d_2)}T_{d_1}P_rT_\sigma T^*_{d_2},$$

where $C_{(d_1,d_2,\sigma)} \in \mathbb{C}(q)$.

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PROOF. For $0 \le r \le n$, let $R_n^{(r)}$ be the two-sided ideal of $R_n(q)$ generated by P_r. This gives a filtration

$$R_n(q) = R_n^{(0)} \supset R_n^{(1)} \supset R_n^{(2)} \supset \cdots \supset R_n^{(n)} \supset 0$$

of two-sided ideals. It is clear that there is an algebra epimorphism

$$\theta : R_n(q) \twoheadrightarrow R_n(q)/R_n^{(1)} \cong H_n(q),$$

where $H_n(q)$, generated by $T_1, T_2, \ldots, T_{n-1}$, is isomorphic to an Iwahori–Hecke algebra of type A. Since the algebras $R_n(q)$ and $H_n(q)$ are both semisimple, the image $\theta(\Phi_n)$ must correspond to the Young anti-symmetriser of $H_n(q)$. Then the lemma follows from Lemma 2.2 and the well-known representation theory of the Iwahori–Hecke algebra $H_n(q)$. □

Since $R_n(q)$ generally cannot be realised as a diagram algebra except in the case $q = 1$ (see [5, Remark 4.4]), we find another way to describe Φ_n different from the methods in [8, 11, 18]. Note that the Iwahori–Hecke algebra $H_n(q)$ is a subalgebra of $R_n(q)$ by [5, Corollary 3.4]. Hence $R_n(q)$ can be viewed as a left $H_n(q)$-module in the natural manner. Define

$$R_n^{[r]} := \mathbb{C}(q)\text{- Span}\{T_{d_1} P_r T_\sigma T^*_{d_2} \mid (d_1, d_2, \sigma) \in \Omega_r\}$$

for $0 \le r \le n$. The following technical lemma aims to give some explicit structure constants.

LEMMA 3.2. *The space $R_n^{[r]}$ is an $H_n(q)$-submodule of $R_n(q)$ for each r with $0 \le r \le n$.*

PROOF. For any $(d_1, d_2, \sigma) \in \Omega_r$, we only need to prove $T_i T_{d_1} P_r T_\sigma T^*_{d_2} \in R_n^{[r]}$ for each $1 \le i \le n-1$. Since $\mathcal{D}_r$ is the set of distinguished left coset representatives of $\mathfrak{S}_{(r,n-r)}$ in $\mathfrak{S}_n$, there exists a sequence of positive integers $1 \le a_1 < a_2 < \cdots < a_r \le n$ such that

$$T_{d_1} = (T_{a_1-1} \cdots T_2 T_1)(T_{a_2-1} \cdots T_3 T_2) \cdots (T_{a_r-1} \cdots T_{r+1} T_r).$$

Then four cases arise.

Case 1. $i, i+1 \notin \{a_1, a_2, \ldots, a_r\}$. Then $d_1(j) = i$ with $j > r$. Moreover,

$$\begin{aligned} T_i T_{d_1} P_r T_\sigma T^*_{d_2} &= T_{d_1} T_j P_r T_\sigma T^*_{d_2} \\ &= T_{d_1} P_r (T_j T_\sigma) T^*_{d_2} \qquad \text{(by relation (R3))} \\ &= \begin{cases} T_{d_1} P_r T_{s_j\sigma} T^*_{d_2} & \text{if } \ell(s_j\sigma) = \ell(\sigma) + 1, \\ (q - q^{-1}) T_{d_1} P_r T_\sigma T^*_{d_2} + T_{d_1} P_r T_{s_j\sigma} T^*_{d_2} & \text{if } \ell(s_j\sigma) = \ell(\sigma) - 1. \end{cases} \end{aligned}$$

Case 2. $i \in \{a_1, a_2, \ldots, a_r\}$ and $i+1 \notin \{a_1, a_2, \ldots, a_r\}$. Then $s_i d_1 \in \mathcal{D}_r$ and $\ell(s_i d_1) = \ell(d_1) + 1$. Hence

$$T_i T_{d_1} P_r T_\sigma T^*_{d_2} = T_{s_i d_1} P_r T_\sigma T^*_{d_2}.$$

Case 3. $i \notin \{a_1, a_2, \ldots, a_r\}$ and $i+1 \in \{a_1, a_2, \ldots, a_r\}$. Then $s_i d_1 \in \mathcal{D}_r$ and $\ell(s_i d_1) = \ell(d_1) - 1$. Hence

$$T_i T_{d_1} P_r T_\sigma T^*_{d_2} = (q - q^{-1}) T_{d_1} P_r T_\sigma T^*_{d_2} + T_{s_i d_1} P_r T_\sigma T^*_{d_2}.$$

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Case 4. $i, i+1 \in \{a_1, a_2, \ldots, a_r\}$. Then $d_1(j) = i$ with $j < r$. From relation (R4),

$$T_i T_{d_1} P_r T_\sigma T_{d_2}^* = T_{d_1} T_j P_r T_\sigma T_{d_2}^* = q T_{d_1} P_r T_\sigma T_{d_2}^*.$$

In each case, $T_i T_{d_1} P_r T_\sigma T_{d_2}^*$ is a linear combination of the basis elements belonging to the space $R_n^{[r]}$, and hence this completes the proof of the lemma. □

Let us now calculate the coefficients $C_{(d_1,d_2,\sigma)}$ in Lemma 3.1. The following lemma is well known for symmetric groups.

Lemma 3.3. *Let r be an integer with $0 \le r \le n$. There exists a unique element $w_0 \in \mathcal{D}_r$ of maximal length $r(n-r)$. If $s_{i_{r(n-r)}} \cdots s_{i_2} s_{i_1}$ is a reduced expression of w_0, then for any integer j with $0 \le j \le r(n-r)$, there is $s_{i_j} \cdots s_{i_2} s_{i_1} \in \mathcal{D}_r$. Conversely, for any $d \in \mathcal{D}_r$, there exists a reduced expression $s_{i_{r(n-r)}} \cdots s_{i_2} s_{i_1}$ of w_0 such that $d = s_{i_j} \cdots s_{i_2} s_{i_1}$ for some j with $0 \le j \le r(n-r)$.*

For an arbitrary element $a \in R_n(q)$, we say that $T_{d_1} P_r T_\sigma T_{d_2}^*$ is involved in a, if $T_{d_1} P_r T_\sigma T_{d_2}^*$ appears with nonzero coefficient when writing a as a linear combination of the basis in Lemma 2.2.

Lemma 3.4. *For any r with $1 \le r \le n$ and any $(d_1, d_2, \sigma_1), (d_3, d_4, \sigma_2) \in \Omega_r$, we have $C_{(d_1,d_2,\sigma_1)} = C_{(d_3,d_4,\sigma_2)}$. In particular, the element Φ_n can be taken of the form*

$$\Phi_n = \sum_{\sigma \in \mathfrak{S}_n} (-q)^{-\ell(\sigma)} T_\sigma + \sum_{r=1}^{n} c_r \sum_{(d_1,d_2,\sigma) \in \Omega_r} (-q)^{-\ell(d_1)-\ell(\sigma)-\ell(d_2)} T_{d_1} P_r T_\sigma T_{d_2}^*,$$

where $c_r \in \mathbb{C}(q)$.

Proof. We first claim that $C_{(d_1,d_2,\sigma)} = C_{(d_3,d_2,\sigma)}$. By Lemma 3.3, it suffices to prove that

$$C_{(d_1,d_2,\sigma)} = C_{(s_i d_1,d_2,\sigma)}$$

whenever $s_i d_1 \in \mathcal{D}_r$ with $\ell(s_i d_1) = \ell(d_1) + 1$. Compare the coefficients of $T_{d_1} P_r T_\sigma T_{d_2}^*$ on both sides of the equality $T_i \Phi_n = (-q)^{-1} \Phi_n$. For any $(d_5, d_6, w) \in \Omega_s$, if $T_{d_1} P_r T_\sigma T_{d_2}^*$ is involved in $T_i T_{d_5} P_s T_w T_{d_6}^*$, then $s = r$ by Lemma 3.2. Furthermore, if $T_{d_1} P_r T_\sigma T_{d_2}^*$ is involved in $T_i T_{d_5} P_r T_w T_{d_6}^*$, it follows from the proof of Lemma 3.2 that $d_5 = d_1$ or $s_i d_5 = d_1$. However, if $d_5 = d_1$, then $T_i T_{d_5} P_r T_w T_{d_6}^* = T_{s_i d_1} P_r T_w T_{d_6}^*$ since $s_i d_1 \in \mathcal{D}_r$ with $\ell(s_i d_1) = \ell(d_1) + 1$, a contradiction. Hence we must have $s_i d_5 = d_1$ and then

$$\begin{aligned} T_i T_{d_5} P_r T_w T_{d_6}^* &= T_i T_{s_i d_1} P_r T_w T_{d_6}^* = T_i^2 T_{d_1} P_r T_w T_{d_6}^* \\ &= (q - q^{-1}) T_{s_i d_1} P_r T_w T_{d_6}^* + T_{d_1} P_r T_w T_{d_6}^*. \end{aligned}$$

This yields $(d_5, d_6, w) = (s_i d_1, d_2, \sigma)$. Now, the coefficient of $T_{d_1} P_r T_\sigma T_{d_2}^*$ in $T_i \Phi_n$ is $C_{(s_i d_1,d_2,\sigma)} (-q)^{-\ell(d_1)-1-\ell(\sigma)-\ell(d_2)}$. Comparing with the coefficient of $T_{d_1} P_r T_\sigma T_{d_2}^*$ in $(-q)^{-1} \Phi_n$, we have $C_{(d_1,d_2,\sigma)} = C_{(s_i d_1,d_2,\sigma)}$ and hence the claim is proved.

Using Lemma 3.1, we see that $\Phi_n^* = \Phi_n$. Combining this fact and the above claim,

$$C_{(d_1,d_2,\sigma)} = C_{(1,d_2,\sigma)} = C_{(1,1,\sigma)}$$

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for all $(d_1, d_2, \sigma) \in \Omega_r$ and $1 \le r \le n$. Therefore, to prove the lemma, it suffices to prove $C_{(1,1,\sigma_1)} = C_{(1,1,\sigma_2)}$ for all $\sigma_1, \sigma_2 \in \mathfrak{S}_{\{r+1,r+2,\ldots,n\}}$. Equivalently, it is enough to show that $C_{(1,1,s_i\sigma)} = C_{(1,1,\sigma)}$ for any $\sigma \in \mathfrak{S}_{\{r+1,r+2,\ldots,n\}}$ and $r+1 \le i < n$ satisfying $\ell(s_i\sigma) = \ell(\sigma) + 1$. Compare the coefficients of P_rT_σ on both sides of the equality $T_i\Phi_n = (-q)^{-1}\Phi_n$. For any $(d_5, d_6, w) \in \Omega_s$, if P_rT_σ is involved in $T_iT_{d_5}P_sT_wT^*_{d_6}$, then $s = r$ by Lemma 3.2. Furthermore, if P_rT_σ is involved in $T_iT_{d_5}P_rT_wT^*_{d_6}$, it follows from the proof of Lemma 3.2 that $d_5 = 1$ (the identity element of the symmetric group $\mathfrak{S}_n$), that is, $\ell(d_5) = 0$) or $d_5 = s_i$. However, $d_5 = s_i$ with $r+1 \le i < n$ contradicts the condition $d_5 \in \mathcal{D}_r$. Hence we must have $d_5 = 1$. Then, by relation (R3) and calculations in $H_n(q)$,

$$\begin{aligned} T_iP_rT_wT^*_{d_6} &= P_rT_iT_wT^*_{d_6} \\ &= \begin{cases} P_rT_{s_iw}T^*_{d_6} & \text{if } \ell(s_iw) = \ell(w)+1, \\ (q-q^{-1})P_rT_wT^*_{d_6} + P_rT_{s_iw}T^*_{d_6} & \text{if } \ell(s_iw) = \ell(w)-1. \end{cases} \end{aligned}$$

This yields $(d_5, d_6, w) = (1, 1, \sigma)$ or $(d_5, d_6, w) = (1, 1, s_i\sigma)$. If $(d_5, d_6, w) = (1, 1, \sigma)$, then $T_iT_{d_5}P_rT_wT^*_{d_6} = T_iP_rT_\sigma = P_rT_{s_i\sigma}$, since $\ell(s_i\sigma) = \ell(\sigma)+1$, a contradiction. Hence $(d_5, d_6, w) = (1, 1, s_i\sigma)$ and the coefficient of P_rT_σ in $T_i\Phi_n$ is $C_{(1,1,s_i\sigma)}(-q)^{-\ell(\sigma)-1}$. Comparing with the coefficient of P_rT_σ in $(-q)^{-1}\Phi_n$, we have $C_{(1,1,\sigma)} = C_{(1,1,s_i\sigma)}$ and this completes the proof of the lemma. □

Lemma 3.5. *With the above notation,* $c_2 = c_3 = \cdots = c_n = 0$.

Proof. By Lemma 3.4, the element Φ_n can be taken of the form

$$\Phi_n = \sum_{\sigma\in\mathfrak{S}_n} (-q)^{-\ell(\sigma)}T_\sigma + \sum_{r=1}^{n} c_r \sum_{(d_1,d_2,\sigma)\in\Omega_r} (-q)^{-\ell(d_1)-\ell(\sigma)-\ell(d_2)} T_{d_1}P_rT_\sigma T^*_{d_2},$$

where $c_r \in \mathbb{C}(q)$. To compute the coefficients c_r with $r \ge 2$, our strategy is to compare the coefficients of P_r on both sides of $T_1\Phi_n = (-q)^{-1}\Phi_n$.

Assume $(d_1, d_2, w) \in \Omega_s$ and P_r is involved in $T_1T_{d_1}P_sT_wT^*_{d_2}$. Then Lemma 3.2 implies that $s = r$. Furthermore, if P_r is involved in $T_1T_{d_1}P_rT_wT^*_{d_2}$, it follows from the proof of Lemma 3.2 that $d_1 = 1$ (the identity element of the symmetric group $\mathfrak{S}_n$), that is, $\ell(d_1) = 0$ or $d_1 = s_1$. But $s_1 \notin \mathcal{D}_r$ because $r \ge 2$. Hence $d_1 = 1$ and

$$T_1T_{d_1}P_rT_wT^*_{d_2} = T_1P_rT_wT^*_{d_2} = qP_rT_wT^*_{d_2},$$

where the second equality follows from relation (R4). Therefore, P_r is involved in $T_1T_{d_1}P_rT_wT^*_{d_2}$ if and only if $(d_1, d_2, w) = (1, 1, 1)$. In this case, the coefficient of P_r in $T_i\Phi_n$ is qc_r. Comparing with the coefficient of P_r in $(-q)^{-1}\Phi_n$, we have $qc_r = (-q)^{-1}c_r$, which implies that $c_r = 0$ since q is an indeterminate. □

Lemma 3.6. *With the above notation,* $c_1 = -q^{2(n-1)}$.

Proof. To compute the coefficient c_1, our strategy is to compare the coefficients of P_1 on both sides of $P_1\Phi_n = 0$.

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We first find the $w \in \mathfrak{S}_n$ for which P_1 is involved in $P_1 T_w^*$. For any $w \in \mathfrak{S}_n$, we can write $w = s_{i-1} \cdots s_2 s_1 \sigma$ with $1 \le i \le n$ and $\sigma \in \mathfrak{S}_{\{2,\ldots,n\}}$. Now

$$P_1 T_w^* = P_1 T_{w^{-1}} = P_1 T_{\sigma^{-1}} (T_1 T_2 \cdots T_{i-1}),$$

which is an element in the set $\{T_{d_1} P_r T_\sigma T_{d_2}^* \mid (d_1, d_2, \sigma) \in \Omega\}$. Therefore, P_1 is involved in $P_1 T_w^*$ if and only if $w = 1$, the identity element of the symmetric group $\mathfrak{S}_n$. Hence P_1 is involved in $P_1 T_w = P_1 T_{w^{-1}}^*$ if and only if $w = 1$.

Next, we find the $(d_1, d_2, w) \in \Omega_1$ for which P_1 is involved in $P_1 T_{d_1} P_1 T_w T_{d_2}^*$. If $\ell(d_1) = 0$, then $(d_1, d_2, w) = (1, 1, 1)$. If $\ell(d_1) > 0$, we have $T_{d_1} = T_{i-1} \cdots T_2 T_1$ for some $2 \le i \le n$. It follows from relations (R3) and (R5) that

$$\begin{aligned} P_1 T_{d_1} P_1 T_w T_{d_2}^* &= T_{i-1} \cdots T_2 (P_1 T_1 P_1) T_w T_{d_2}^* \\ &= q^{-1} T_{i-1} \cdots T_2 P_2 T_w T_{d_2}^* + (q - q^{-1}) P_1 T_{i-1} \cdots T_2 T_w T_{d_2}^*. \end{aligned}$$

In this case, P_1 is only involved in the term $P_1 T_{i-1} \cdots T_2 T_w T_{d_2}^*$. By calculations in the Iwahori–Hecke algebra $H_n(q)$ (see, for example, [12, Proposition 1.16]), P_1 is involved in $P_1 T_{i-1} \cdots T_2 T_w T_{d_2}^*$ if and only if $w = s_2 s_3 \cdots s_{i-1}$ and $d_2 = 1$. Here, for $i = 2$, we take $w = 1$. Therefore, P_1 is involved in $P_1 T_{d_1} P_1 T_w T_{d_2}^*$ with $\ell(d_1) > 0$ if and only if $(d_1, d_2, w) = (s_{i-1} \cdots s_2 s_1, 1, s_2 \cdots s_{i-1})$ with $2 \le i \le n$.

By the above argument, the coefficient of P_1 in $P_1 \Phi_n$ is

$$\begin{aligned} 1 + c_1 \Big(1 + \sum_{i=2}^{n} (-q)^{-(i-1)-(i-2)} (q - q^{-1}) \Big) &= 1 + c_1 \Big(1 + (1 - q^2) \sum_{i=1}^{n-1} q^{-2i} \Big) \\ &= 1 + c_1 q^{-2(n-1)}. \end{aligned}$$

Thus $P_1 \Phi_n = 0$ implies that $c_1 = -q^{2(n-1)}$. □

We now turn to the proof of the main result of this paper. For any positive integer $k \le n$, the natural map $T_i \mapsto T_i, P_j \mapsto P_j$ for all $1 \le i \le k - 1$ and $1 \le j \le k$ can be extended to an algebra embedding from $R_k(q)$ into $R_n(q)$. From this point of view, when $m < n$ (where $m = \dim L(\varepsilon_1)$),

$$\Phi_{m+1} = \sum_{\sigma \in \mathfrak{S}_{m+1}} (-q)^{-\ell(\sigma)} T_\sigma - q^{2m} \sum_{(d_1, d_2, \sigma) \in \Omega_1} (-q)^{-\ell(d_1) - \ell(\sigma) - \ell(d_2)} T_{d_1} P_r T_\sigma T_{d_2}^* \in R_n(q).$$

Proof of Theorem 1.2. By Proposition 2.4, the element $\Phi := Y_{m+1}/(m+1)!$ is an idempotent such that $\langle \Phi \rangle = \mathrm{Ker}(\varphi_1)$. Assume $\Phi_q = \Phi_{m+1}$, which belongs to the lattice $\mathbb{Z}[q, q^{-1}]$-Span$\{T_{d_1} P_r T_\sigma T_{d_2}^* \mid (d_1, d_2, \sigma) \in \Omega\}$. Note that $\Phi_q^2 = \sum_{\sigma \in \mathfrak{S}_{m+1}} q^{-2\ell(\sigma)} \Phi_q$ and $\lim_{q \to 1} \Phi_q = (m+1)! \Phi$. Thus Proposition 2.5 completes the proof of the theorem. □

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