Proceedings of the Edinburgh Mathematical Society (2011) **54**, 149–160
DOI:10.1017/S0013091509000807   First published online 11 November 2010

CLOSED EINSTEIN–WEYL STRUCTURES ON COMPACT SASAKIAN AND COSYMPLECTIC MANIFOLDS

PAOLA MATZEU

Dipartimento di Matematica e Informatica, Università degli Studi di Cagliari, Viale Merello 92, 09124 Cagliari, Italy (matzeu@unica.it)

(Received 12 June 2009)

Dedicated to the memory of Krzysztof Galicki

Abstract We study closed Einstein–Weyl structures on compact K-contact, Sasakian and cosymplectic manifolds. In particular we prove that compact Sasakian and cosymplectic manifolds endowed with a closed Einstein–Weyl structure are η-Einstein.

Keywords: K-contact manifolds; Sasakian manifolds; cosymplectic manifolds; Einstein–Weyl structures

2010 *Mathematics subject classification:* Primary 53C15; 53C25

1. Introduction

A *Weyl structure* on a manifold M with conformal class $[g]$ is a torsion-free connection $\nabla^{[g]}$ such that parallel translation induces conformal transformations. Hence, corresponding to every $g \in [g]$ there exists a unique 1-form θ on M such that the given Weyl structure $\nabla^{[g]}$ satisfies the following conformally invariant equation: $\nabla^{[g]}g = -2\theta \otimes g$. We shall call a manifold M endowed with such a structure a *Weyl manifold*, denoting it by $(M, [g], \nabla^{[g]})$. In what follows, unless stated otherwise, θ is always associated to the metric g and B denotes its g-dual vector field: $g(B, X) = \theta(X)$ for every $X \in \mathcal{X}(M)$.

The Ricci tensor of a Weyl connection is usually non-symmetric. So, to define Einstein-like structures in conformal geometry one has to refer to the symmetrized Ricci tensor of the Weyl connection $\nabla^{[g]}$. If this is proportional to the metrics in $[g]$, the structure is called *Einstein–Weyl*. In particular, a Weyl structure is said to be closed when $d\theta = 0$ for every $g \in [g]$. Closed (respectively, exact) Weyl structures are locally (respectively, globally) the Levi-Civita connections of compatible metrics. Hence, a closed Einstein–Weyl structure admits locally (but not necessarily globally) compatible Einstein metrics.

A careful study of closed Einstein–Weyl structures on compact manifolds has been made by Gauduchon [**7**], who found that these have very special properties.

Closed Weyl structures naturally arise in complex and quaternionic geometry, where Weyl and Einstein–Weyl conditions are rather well understood [**15**, **16**].

Conversely, for an almost-contact metric structure, the conformal changes of the metric are not so natural, since they give rise to a metric which is no more compatible with the almost-contact structure. The problem of the compatibility between Weyl and almost-contact geometries has been studied in [**11**], where it was proved that locally conformal cosymplectic manifolds admit a naturally defined conformally invariant Weyl structure. Moreover, in [**5**, **8**, **10**, **12**, **13**] several examples of Einstein–Weyl structures on almost-contact manifolds have been constructed together with some existence conditions. In particular, in [**8**] it has been proved that every complete K-contact manifold admitting two different Einstein–Weyl structures with θ and $-\theta$ as associated 1-forms is Sasakian. We also remark that the results cited showed that often the existence of Einstein–Weyl structures is equivalent to affirming that the almost-contact manifolds are η-Einstein.

However, in spite of their incompatibility, it is clear that Einstein–Weyl structures on almost-contact manifolds imply strong relations between the two structures.

Here we study closed Einstein–Weyl structures on the main classes of almost-contact metric manifolds as K-contact, Sasakian and cosymplectic structures, again finding strict connections between Weyl and η-Einstein geometry. The outline of the paper is as follows. In the next section, having described the objects we work with, we prove a characterization for closed Weyl structures on almost-contact manifolds. In § 3 we reach our main achievements. More precisely, in Theorem 3.5 we determine the Ricci curvature tensor of a compact K-contact manifold endowed with a closed Einstein–Weyl structure. In Theorem 3.6 we prove an integrability result: a compact K-contact manifold whose metric belongs to a closed Einstein–Weyl structure is Sasakian if and only if it is η-Einstein. So, in the light of results of [**5**] on η-Einstein Sasakian manifolds, we can affirm that there exist compact Sasakian manifolds that do not admit closed Einstein–Weyl structures. Finally, in Theorem 4.2 we obtain an analogous result to that of Theorem 3.6 for compact cosymplectic manifolds with closed Einstein–Weyl structures.

2. Preliminaries

Let M be a differentiable manifold of odd dimension $2n+1$. An almost-contact structure on M consists in a field of endomorphisms of the tangent bundle φ, a vector field ξ and a 1-form η satisfying the following relations [**1**]:

$$\left.\begin{aligned} \varphi^2 = -I + \eta\otimes\xi, \quad \eta(\xi) = 1, \\ \varphi(\xi) = 0, \quad \eta\circ\varphi = 0, \quad \operatorname{rank}\varphi = 2n. \end{aligned}\right\} \tag{2.1}$$

A Riemannian metric g on M is called compatible with the almost-contact structure if

$$g(X,\xi) = \eta(X), \qquad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for every $X, Y \in \mathfrak{X}(M)$. An almost-contact structure together with a compatible metric is called an *almost-contact metric structure* and $(M, \varphi, \xi, \eta, g)$ is called an almost-contact metric manifold.

In the following, we shall call *horizontal* any vector field on $(M, \varphi, \xi, \eta, g)$ orthogonal to ξ. Furthermore, we will denote by $\dot{X}$ the horizontal part of a vector field $X \in \mathfrak{X}(M)$, recalling that we can always write $X = \dot{X} + \eta(X)\xi$.

We also remark that an almost-contact structure (φ, ξ, η) satisfying the equation $N_\varphi + d\eta \otimes \xi = 0$, where N_φ denotes the Nijenhuis tensor of φ, is said to be *normal*. For properties, geometric interpretations and examples (which include the odd Euclidean spheres, some odd tori, the total spaces of some circle bundles over compact symplectic manifolds), we refer the reader to [**1**].

Suppose now that the almost-contact metric manifold $(M, \varphi, \xi, \eta, g)$ also carries a Weyl structure $\nabla^{[g]}$. In this case we shall say that *the metric* g *represents the Weyl structure*. The connection $\nabla^{[g]}$ and the Levi-Civita connection D^g of g are then connected by the well-known formula [**9**]

$$\nabla^{[g]}_X Y = D^g_X Y + \theta(X)Y + \theta(Y)X - g(X,Y)B \tag{2.2}$$

for every $X, Y \in \mathfrak{X}(M)$. Equation (2.2) together with the relation $\nabla^{[g]} g = -2\theta \otimes g$ (see § 1) are invariant for Weyl transformations

$$g' = \mathrm{e}^{2f} g, \qquad \theta' = \theta - df \tag{2.3}$$

with $f \in C^\infty(M)$. Moreover, (2.2) implies the following relation between the curvature tensor field $R_{[g]}$ of the Weyl connection $\nabla^{[g]}$ and the Riemannian curvature tensor field R_g of D^g [**9**]:

$$R_{[g]}(X,Y)Z = R_g(X,Y)Z + \Sigma_g(X,Y)Z - \Sigma_g(Y,X)Z, \tag{2.4}$$

where

$$\begin{aligned}\Sigma_g(X,Y)Z = (D^g_X\theta)(Y)Z + (D^g_X\theta)(Z)Y - g(Y,Z)D^g_X B \\ - g(Y,Z)|\theta|^2 X - g(X,Z)\theta(Y)B + \theta(Y)\theta(Z)X.\end{aligned} \tag{2.5}$$

Consequently, the relation between the Ricci curvatures $\rho(R_{[g]})$ and $\rho(R_g)$ is given by

$$\begin{aligned}\rho(R_{[g]})(Y,Z) = \rho(R_g)(Y,Z) - 2n(D^g_Z\theta)(Y) + (D^g_Y\theta)(Z) \\ + (2n-1)\theta(Z)\theta(Y) + (\delta\theta - (2n-1)|\theta|^2)g(Y,Z),\end{aligned} \tag{2.6}$$

where $X, Y, Z \in \mathfrak{X}(M)$ and $\delta\theta$ denotes the codifferential of θ with respect to g.

In [**7**] Gauduchon proved that, when M is compact, the conformal class $[g]$ contains a unique (up to homothety) metric g_0 such that the corresponding form θ_0 is g_0-coclosed. We shall refer to this metric as the *Gauduchon metric*. Moreover, he also showed that the 1-form θ_0 of the metric g_0 of closed, non-exact, Einstein–Weyl structures defined on compact manifolds of dimension at least 3 is parallel with respect to the Levi-Civita connection D^{g_0} of g_0: $D^{g_0}\theta_0 = 0$. In particular, θ_0 is closed and coclosed, and hence it is g_0-harmonic and the curvature tensor, and then the Ricci tensor, of such a structure vanish identically. Finally, the parallelism of the 1-form θ_0 with respect to D^{g_0} also implies that its pointwise norm $|\theta_0|$ with respect to g_0 is constant, while (2.6) gives the simple relation

$$\rho(R_{g_0}) + (2n-1)\theta \otimes \theta = (2n-1)g_0, \tag{2.7}$$

https://doi.org/10.1017/S0013091509000807 Published online by Cambridge University Press

where $\rho(R_{g_0})$ denotes the Ricci tensor field of the Gauduchon metric g_0. It is important to remark now that because, for an almost-contact metric manifold, conformal changes of the metric give rise to metrics which are no longer compatible with the almost-contact structure, in general it is not possible to consider in these manifolds the Gauduchon metric as the contact metric (see, for example, [**8**, Corollary 2 and Theorem 2]).

Now we see that the almost-contact conditions (2.1) together with (2.2) imply further relations for a Weyl structure $\nabla^{[g]}$ defined on an almost-contact metric manifold $(M, \varphi, \xi, \eta, g)$. In fact, the conformally invariant equations

$$\left.\begin{aligned} \eta(\nabla^{[g]}_X \xi) &= \theta(X), \\ (\nabla^{[g]}_X \eta)(Y) &= -2\theta(X)\eta(Y) + g(\nabla^{[g]}_X \xi, Y) \end{aligned}\right\} \tag{2.8}$$

hold for all $X, Y \in \mathfrak{X}(M)$. Although simple and easy to obtain, these relations are fundamental for the Weyl structure $\nabla^{[g]}$, which acquires some special properties from those of the almost-contact metric structure (φ, ξ, η, g). In particular, (2.8) gives rise to the following useful characterization of closed Weyl structures defined on almost-contact metric manifolds.

Proposition 2.1. *Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost-contact metric manifold. Then the Weyl structure $\nabla^{[g]}$ on M with 1-form θ associated to the metric $g \in [g]$ is closed if and only if $\eta(R_{[g]}(X, Y)\xi) = 0$ for all vector fields $X, Y \in \mathfrak{X}(M)$.*

Proof. As usual, let θ be the 1-form associated to the metric $g \in [g]$. Then for all vector fields $X, Y \in \mathfrak{X}(M)$ we can write

$$\begin{aligned} d\theta(X, Y) &= X(\theta(Y)) - Y(\theta(X)) - \theta[X, Y] \\ &= X(\eta(\nabla^{[g]}_Y \xi)) - Y(\eta(\nabla^{[g]}_X \xi)) - \eta(\nabla^{[g]}_{[X,Y]} \xi) \\ &= \eta(\nabla^{[g]}_X \nabla^{[g]}_Y \xi) - \eta(\nabla^{[g]}_Y \nabla^{[g]}_X \xi) - \eta(\nabla^{[g]}_{[X,Y]} \xi) \\ &\qquad + (\nabla^{[g]}_X \eta)(\nabla^{[g]}_Y \xi) - (\nabla^{[g]}_X \eta)(\nabla^{[g]}_Y \xi), \end{aligned} \tag{2.9}$$

obtaining, directly from (2.8), $d\theta(X, Y) = \eta(R_{[g]}(X, Y)\xi)$ and then the claim. □

3. Closed Einstein–Weyl structures on *K*-contact and Sasakian manifolds

We consider in this section contact metric manifolds $(M, \varphi, \xi, \eta, g)$ where the 1-form η is a contact form: then (φ, ξ, η, g) is said to be a *contact metric structure* on M. In this case we say that the structure (φ, ξ, η, g) is K-*contact* if ξ is a Killing vector field of g. Any normal K-contact structure is called *Sasakian*. The K-contact manifolds are surely the most important among the almost-contact ones. They have been carefully examined by Boyer and Galicki in several papers [**2**, **3**, **5**]. In particular, in [**5**] Boyer *et al.* studied the η-Einstein condition on Sasakian manifolds, constructing some obstructions to the existence of η-Einstein structures on this kind of manifold. In addition, they discussed some relations between Einstein–Weyl and η-Einstein K-contact or Sasakian structures.

https://doi.org/10.1017/S0013091509000807 Published online by Cambridge University Press

We will present here only the main results and theorems proved in [**5**], referring the reader to the cited paper for more explanations and proofs.

Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold and let $\mathcal{D}$ be the sub-bundle defined by the equation $\eta = 0$. Thus, $\mathcal{D}$ is just the normal bundle to the foliation $\mathcal{F}_\xi$ generated on M by ξ and inherits in a natural way from the Sasakian structure of M both a complex structure $J = \varphi|_{\mathcal{D}}$ and a symplectic structure $d\eta$. In this way, $(\mathcal{D}, J, d\eta)$ gives M a transverse Kähler structure with Kähler form $d\eta$ and metric $g_{\mathcal{D}}$ defined by $g_{\mathcal{D}}(X, Y) = d\eta(X, JY)$, related to the Sasakian metric g by $g = g_{\mathcal{D}} \oplus \eta \otimes \eta$. We also remark that the 2-form $d\eta$ is a basic form on M, since it satisfies the relations $\xi \rfloor d\eta = 0$ and $\mathcal{L}_\xi d\eta = 0$ (see, for example, [**18**]). In the case when the transverse Kähler structure of the Sasakian manifold M is also Einstein, then M has an η-Einstein metric, i.e. its Ricci curvature tensor is given by $\rho(R_g) = \lambda g + \nu \eta \otimes \eta$ for some functions λ, ν. In [**14**] it has been proved that in the case of Sasakian, and more generally of K-contact, η-Einstein manifolds of dimension 5 or higher, the functions λ, ν are actually constant and satisfy the relation $\lambda + \nu = 2n$: they are called the *Einstein constants* of M. Then, η-Einstein Sasakian geometry is strictly connected with Einstein–Kähler geometry and provides a generalization of the more familiar Sasakian–Einstein condition (obviously obtained when $\nu = 0$) which is not significant for these structures: in fact, Sasakian–Einstein metrics are necessarily positive with Einstein constant equal to $\dim(M) - 1$.

Starting from the Sasakian structure (φ, ξ, η, g) of M, it is possible to construct infinitely many Sasakian metrics considering special deformations of (φ, ξ, η, g). We now focus our attention on deformations of (φ, ξ, η, g) which fix $\mathcal{F}_\xi$ but deform $\mathcal{D}$ (called *type II* deformations in [**5**]). More explicitly, we wish to deform (φ, ξ, η, g) through Sasakian structures that have the same fundamental basic cohomology class up to a scale. In fact, we first consider a deformation of the structure (φ, ξ, η, g) obtained by adding to η a continuous one-parameter family of 1-forms ζ_t, basic with respect to the foliation $\mathcal{F}_\xi$ and such that the 1-form $\eta_t = \eta + \zeta_t$ satisfies for all $t \in [0, 1]$ the conditions

$$\eta_0 = \eta, \qquad \zeta_0 = 0, \qquad \eta_t \wedge (d\eta_t)^n \neq 0. \tag{3.1}$$

Then, if we define

$$\left.\begin{aligned} \varphi_t &= \varphi - \xi \otimes \zeta_t \circ \varphi, \\ g_t &= d\eta_t \circ (\varphi \otimes I_t) + \eta_t \otimes \eta_t \end{aligned}\right\} \tag{3.2}$$

for every $t \in [0, 1]$ and every basic 1-form ζ_t for which $d\zeta_t$ is of type $(1, 1)$ and if (3.1) holds, then $(\varphi_t, \xi, \eta_t, g_t)$ gives a continuous one-parameter family of Sasakian structures on M belonging to the same underlying contact structure as η [**4**]. We remark that, if (φ, ξ, η, g) and $(\varphi', \xi', \eta', g')$ are two Sasakian structures related by a deformation of type II, they are homologous in the sense that $[d\eta]_B = [d\eta']_B$ [**6**]. In what follows, we denote by $\mathcal{F}$ the set of all the Sasakian structures on M with the same foliation $\mathcal{F}_\xi$. Considering now the first Chern class $c_1(\mathcal{D}) \in H^2(M, \mathbb{Z})$ of the contact complex sub-bundle $\mathcal{D}$, we say that $c_1(\mathcal{D})$ is a torsion class if and only if there exists a real number a such that $c_1(\mathcal{D}) = a[d\eta]_B$. In particular, the following claim has been proved.

Proposition 3.1. *Let (φ,ξ,η,g) be a Sasakian structure on a given $(2n+1)$-dimensional manifold M with underlying contact bundle $\mathcal{D}$. If $\mathcal{F}$ admits a Sasakian η-Einstein structure, then $c_1(\mathcal{D})$ is a torsion class [**5**].*

Hence, a non-torsion $c_1(\mathcal{D})$ is the obstruction to the existence of a Sasakian η-Einstein structure. Moreover, since $c_1(\mathcal{D})$ is related to the second Stiefel–Whitney class $w_2(M)$, Proposition 3.1 also implies the following.

Theorem 3.2. *Let M be a non-spin manifold (i.e. $w_2(M) \neq 0$) with $H_1(M,\mathbb{Z})$ torsion free. Then M does not admit a Sasakian η-Einstein structure.*

Finally, starting from these results, an example of a Sasakian manifold that does not admit a Sasakian η-Einstein metric has been constructed in [**5**].

On the other hand, several results have been obtained in [**5**] concerning the existence of Einstein–Weyl structure on K-contact and Sasakian manifolds too. We recall the following.

Theorem 3.3. *Let (M,φ,ξ,η,g) be a Sasakian manifold of dimension $2n+1 \geqslant 5$ and let θ be a 1-form on M. Then M admits two Einstein–Weyl structures with θ and $-\theta$ as associated 1-forms to $g \in [g]$ if and only if this is η-Einstein with Einstein constants (λ,ν) such that $\nu < 0$ and $\theta = \pm\mu\eta$, where μ is a constant given by $\mu^2 = -\nu/(2n-1)$.*

Finally, in a recent paper [**8**], it was proved that Theorem 3.3 can be extended to the class of all K-contact manifolds (see also [**5**]) and, in particular, the following.

Theorem 3.4 (Ghosh [8]). *Let (M,φ,ξ,η,g) be a complete K-contact manifold of dimension $2n+1 \geqslant 5$ and let θ be a 1-form on M. Suppose that g represents two Einstein–Weyl structures on M with associated 1-forms θ and $-\theta$. Then (M,φ,ξ,η,g) is Sasakian.*

To now study closed Einstein–Weyl structures on compact K-contact and Sasakian manifolds (M,φ,ξ,η,g), we recall some special properties of these spaces. First, the relation $D^g_X\xi = \varphi X$ is always satisfied from all vector fields X, while the equality $\rho(R_g)(\xi,\xi) = 2n$ concerning the Ricci curvature tensor of M also holds on $(2n+1)$-dimensional K-contact manifolds. In addition, if the structure is Sasakian, the Ricci tensor obeys the further equation $\rho(R_g)(X,\xi) = 2n\eta(X)$ for every $X \in \mathfrak{X}(M)$. Finally, since, as we have already noted, the Weyl curvature tensor $R_{[g]}$ and the Weyl Ricci tensor $\rho(R_{[g]})$ of closed Einstein–Weyl structures defined on compact manifolds vanish identically, from (2.4) and (2.6) we obtain that, in this case, the curvature tensor and the Ricci tensor of (M,φ,ξ,η,g) satisfy the equalities

$$R_g(X,Y)Z = \Sigma_g(Y,X)Z - \Sigma_g(X,Y)Z, \tag{3.3}$$

$$\rho(R_g)(X,Y) = (2n-1)(D^g_X\theta)(Y) - (2n-1)\theta(X)\theta(Y) - (\delta\theta - (2n-1)|\theta|^2)g(X,Y) \tag{3.4}$$

for all $X,Y,Z \in \mathfrak{X}(M)$. We can then state the following.

https://doi.org/10.1017/S0013091509000807 Published online by Cambridge University Press

Theorem 3.5. *Let $(M, \varphi, \xi, \eta, g)$ be a compact $(2n+1)$-dimensional K-contact manifold, with $2n+1 \geqslant 3$. If g represents a closed Einstein–Weyl structure $\nabla^{[g]}$ with associated 1-form θ, then the Ricci curvature tensor of M is given by*

$$\begin{aligned}\rho(R_g)(X,Y) = ((2n-1)|\theta|^2 - 2\delta\theta - 1)g(X,Y) + (2n-1)\eta(Y)(D^g_X\theta)(\xi) \\ + (2n-1)\eta(X)(D^g_Y\theta)(\xi) - (2n-1)\theta(\xi)\eta(Y)\theta(X) \\ - (2n-1)\theta(\xi)\eta(X)\theta(Y) + (1-2n+(2n-1)|\theta|^2)\eta(X)\eta(Y)\end{aligned} \tag{3.5}$$

for all $X, Y \in \mathfrak{X}(M)$.

Proof. First, considering (2.5) for all vector fields X, we have

$$\left.\begin{aligned}\Sigma_g(X,\xi)\xi &= 2(D^g_X\theta)(\xi)\xi - D^g_X B - |\theta|^2 X - \theta(\xi)\eta(X)B + (\theta(\xi))^2 X, \\ \Sigma_g(\xi,X)\xi &= (D^g_\xi\theta)(X)\xi + (D^g_\xi\theta)(\xi)X - \eta(X)D^g_\xi B - \eta(X)|\theta|^2\xi \\ &\qquad - \theta(X)B + \theta(X)\theta(\xi)\xi.\end{aligned}\right\} \tag{3.6}$$

Then, since on a K-contact manifold the equality $R_g(X,\xi)\xi = X - \eta(X)\xi$ always holds and θ is closed, from (3.3) and (3.6) for every $X \in \mathfrak{X}(M)$ we get

$$\begin{aligned}D^g_X B = (1 - |\theta|^2 + (\theta(\xi))^2 - (D^g_\xi\theta)(\xi))X + \theta(X)B \\ + ((D^g_X\theta)(\xi) - \theta(X)\theta(\xi) - \eta(X) + \eta(X)|\theta|^2)\xi + \eta(X)D^g_\xi B - \eta(X)\theta(\xi)B,\end{aligned} \tag{3.7}$$

which, for horizontal vector fields X, Y implies

$$g(D^g_X B, Y) = (D^g_X\theta)(Y) = (1 - |\theta|^2 + (\theta(\xi))^2 - (D^g_\xi\theta)(\xi))g(X,Y) + \theta(X)\theta(Y). \tag{3.8}$$

Now, to find the expression of the curvature Ricci tensor of M for all $X, Y \in \mathfrak{X}(M)$ we use the following decomposition:

$$\begin{aligned}\rho(R_g)(X,Y) = \rho(R_g)(\dot X, \dot Y) + \eta(X)\rho(R_g)(\xi, \dot Y) \\ + \eta(Y)\rho(R_g)(\dot X, \xi) + \eta(X)\eta(Y)\rho(R_g)(\xi,\xi).\end{aligned} \tag{3.9}$$

Then, taking into account that $\rho(R_g)(\xi,\xi) = 2n$, from (3.9), (3.4) and (3.8) we get the relations

$$\begin{aligned}\rho(R_g)(X,Y) = (2n-1)(1 - 2|\theta|^2 + (\theta(\xi))^2 - (D^g_\xi\theta)(\xi))g(\dot X, \dot Y) \\ + (2n-1)\eta(X)(D^g_{\dot Y}\theta)(\xi) - (2n-1)\theta(\xi)\eta(X)\theta(\dot Y) \\ - (2n-1)\theta(\xi)\eta(Y)\theta(\dot X) + (2n-1)\eta(Y)(D^g_{\dot X}\theta)(\xi) + 2n\eta(X)\eta(Y),\end{aligned} \tag{3.10}$$

$$(2n-1)(D^g_\xi\theta)(\xi) = 2n + (2n-1)(\theta(\xi))^2 + \delta\theta - (2n-1)|\theta|^2, \tag{3.11}$$

and, finally, by substituting (3.11) into (3.10), recalling that $\dot X = X - \eta(X)\xi$, we obtain the claim of the theorem. □

Theorem 3.6. *Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional compact K-contact manifold, with $2n+1 \geqslant 3$, admitting a closed Einstein–Weyl structure $\nabla^{[g]}$ and let θ be the associated 1-form. Then M is Sasakian if and only if it is η-Einstein. Moreover, in this case, $\theta(\xi) \neq 0$ and, if $2n+1 \geqslant 5$, $(2n-1)|\theta|^2 - 2\delta\theta$ is constant.*

Proof. In fact, suppose that $(M, \varphi, \xi, \eta, g)$ is a compact Sasakian manifold endowed with a closed Einstein–Weyl structure $\nabla^{[g]}$. Thus, in particular, its curvature Ricci tensor $\rho(R_g)$ satisfies (3.5). So, comparing (3.5) with the relation $\rho(R_g)(X, \xi) = 2n\eta(X)$, which holds for every $X \in \mathfrak{X}(M)$, for X horizontal vector field and $Y = \xi$ we get

$$(D^g_X \theta)(\xi) = \theta(X)\theta(\xi). \tag{3.12}$$

Now, from (3.12), we see that, when $\theta(\xi) = 0$, we have $(D^g_X \theta)(\xi) = -\theta(\varphi X) = 0$ for every horizontal vector field X, so that the 1-form θ vanishes identically on M. As a consequence, if (φ, ξ, η, g) is Sasakian, $\theta(\xi)$ must be different from zero. Moreover, by substituting (3.12) into (3.10) and taking account of (3.9) and (3.11), finally we obtain the following equation:

$$\rho(R_g)(X, Y) = (\sigma - 1)g(X, Y) + (2n + 1 - \sigma)\eta(X)\eta(Y), \tag{3.13}$$

where we put $\sigma = (2n-1)|\theta|^2 - 2\delta\theta$, proving the first part of the theorem.

On the contrary, suppose now that $(M, \varphi, \xi, \eta, g)$ is a compact K-contact manifold satisfying the hypothesis of Theorem 3.5. Then its Ricci curvature tensor field is given by (3.5) so that, in order for M to be η-Einstein, for all $X, Y \in \mathfrak{X}(M)$ we must have

$$\eta(Y)(D^g_X \theta)(\xi) + \eta(X)(D^g_Y \theta)(\xi) - \theta(\xi)\eta(Y)\theta(X) - \theta(\xi)\eta(X)\theta(Y) = \lambda\eta(X)\eta(Y) \tag{3.14}$$

for some function λ on M. Considering the above equation for a horizontal vector field X and for $Y = \xi$ again we find the relation (3.12), obtaining that also in this case we need $\theta(\xi) \neq 0$. We will now prove that $(M, \varphi, \xi, \eta, g)$ is a Sasakian manifold showing that its curvature tensor field for all $X, Y \in \mathfrak{X}(M)$ is given by $R_g(X, Y)\xi = \eta(Y)X - \eta(X)Y$, an equation which characterizes the K-contact normal manifolds. We then consider (3.3) to find the following equation for the curvature of M:

$$\begin{aligned} R_g(X, Y)\xi &= \Sigma_g(Y, X)\xi - \Sigma_g(X, Y)\xi \\ &= (D^g_Y \theta)(\xi)X - \eta(X)D^g_Y B - \eta(X)|\theta|^2 Y - \eta(Y)\theta(X)B + \theta(\xi)\theta(X)Y \\ &\quad - (D^g_X \theta)(\xi)Y + \eta(Y)D^g_X B + \eta(Y)|\theta|^2 X + \eta(X)\theta(Y)B - \theta(\xi)\theta(Y)X. \end{aligned} \tag{3.15}$$

Now, substituting the expression (3.7) into (3.15) for $D^g_X B$ and $D^g_Y B$, after a long computation, (3.15) reduces to

$$\begin{aligned} R_g(X, Y)\xi = \eta(Y)X - \eta(X)Y + (D^g_{\dot{X}} \theta)(\xi)\dot{Y} - (D^g_{\dot{Y}} \theta)(\xi)\dot{X} \\ + \theta(\dot{Y})\theta(\xi)\dot{X} - \theta(\dot{Y})\theta(\xi)\dot{X}. \end{aligned} \tag{3.16}$$

https://doi.org/10.1017/S0013091509000807 Published online by Cambridge University Press

Finally, taking into account that (3.14) implies (3.12) and supposing that either X or Y is horizontal, we obtain for R_g the equality we are looking for, proving that the K-contact structure is actually Sasakian. Finally, we note that, since the Ricci curvature tensor $\rho(R_g)$ in this case reduces to (3.13), $(2n-1)|\theta|^2 - 2\delta\theta$ must be constant when $2n+1 \geqslant 5$, which concludes the proof. □

Since Boyer *et al*. proved in [**5**] that a Sasakian manifold of dimension 3 is η-Einstein if and only if it has constant φ-sectional curvature, we can state the following.

Corollary 3.7. *Let $(M, \varphi, \xi, \eta, g)$ be a compact three-dimensional Sasakian manifold endowed with a closed Einstein–Weyl structure $\nabla^{[g]}$. Then M has constant φ-sectional curvature.*

As is well known, the Sasakian space forms provide examples of Sasakian η-Einstein spaces. We also remark that in [**13**] Narita showed that on every Sasakian manifold with constant φ-sectional curvature $c \geqslant 1$ there exists an Einstein–Weyl structure. More generally, taking into account the cited results on the existence of η-Einstein structure on a Sasakian manifold $(M, \varphi, \xi, \eta, g)$, from Theorem 3.6 we get the following.

Corollary 3.8. *Let $(M, \varphi, \xi, \eta, g)$ be a compact Sasakian manifold satisfying the hypotheses of Theorems 3.1 and 3.2. Then no Sasakian metric can represent a closed Einstein–Weyl structure.*

Remark 3.9. We remark that from Theorem 3.6 it is also possible to deduce that in a compact Sasakian manifold $(M, \varphi, \xi, \eta, g)$ the metric g compatible with the structure cannot be the Gauduchon metric of a closed Einstein–Weyl structure $\nabla^{[g]}$. In fact, in that case, the corresponding 1-form θ would be g-coclosed, closed and thus g-harmonic on M. However, Tachibana proved in [**17**] that every harmonic 1-form on compact K-contact manifolds must be orthogonal to η so that, as we showed in Theorem 3.6, θ vanishes identically on M. Moreover, the properties of θ (in particular, its parallelism with respect to D^g) and Tachibana's result easily permit us to extend this observation on every K-contact manifold (see also [**5**]).

Corollary 3.10. *Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional compact K-contact manifold, with $2n+1 \geqslant 3$, satisfying the hypothesis of Theorem 3.5. Then its scalar curvature is given by $s_g = 2n((2n-1)|\theta|^2 - 2\delta\theta)$. Moreover, when $\theta(\xi) = 0$, if the scalar curvature s_g is constant, then $|\theta|$ is constant too, and θ is a harmonic form on M.*

Proof. The expression for s_g is obtained by substituting (3.11) into the trace of (3.5). On the other hand, when $\theta(\xi) = 0$, from (3.4) and the relation $\rho(R_g)(\xi, \xi) = 2n$ we obtain $(2n-1)|\theta|^2 - \delta\theta = 2n$, and the following equivalent expressions for the scalar curvature: $s_g = 2n(4n - (2n-1)|\theta|^2) = 2n(2n - \delta\theta)$. Consequently, supposing s_g is constant, both $|\theta|$ and $\delta\theta$ must be constant too. In particular, since θ is closed, θ is a harmonic form on M. □

4. Closed Einstein–Weyl structures on cosymplectic manifolds

A normal almost-contact metric structure (φ, ξ, η, g) on M is called *cosymplectic* if the fundamental 2-form Ω defined by $\Omega(X, Y) = 2g(\varphi X, Y)$, $X, Y \in \mathfrak{X}(M)$, and the 1-form η are closed on M [**1**]. It is possible to prove that if D^g is the Levi-Civita connection of g, the cosymplectic structure is also characterized by the relations

$$D^g_X \varphi = 0, \qquad D^g_X \eta = 0, \qquad D^g_X \xi = 0. \tag{4.1}$$

Together with (4.1), we recall that the curvature tensor field R_g and the Ricci tensor field $\rho(R_g)$ of a cosymplectic manifold satisfy the equalities $R_g(X, Y)\xi = 0$ and $\rho(R_g)(X, \xi) = 0$ for all vector fields X, Y on M. In particular, the Ricci tensor of a cosymplectic η-Einstein structure is simply given by $\rho(R_g) = \sigma(g - \eta \otimes \eta)$, where $\sigma \in C^\infty(M)$. In [**10**] the following has been proved.

Theorem 4.1. *Every $(2n + 1)$-dimensional cosymplectic manifold $(M, \varphi, \xi, \eta, g)$ of constant φ-sectional curvature $c > 0$ admits two Ricci-flat Weyl structures where the 1-forms associated to the metric $g \in [g]$ are $\pm\theta = \pm\lambda\eta$ respectively, with $\lambda^2 = 2c/(2n-1)$.*

The next result generalizes Theorem 4.1.

Theorem 4.2. *Let $(M, \varphi, \xi, \eta, g)$ be a compact $(2n + 1)$-dimensional, $2n + 1 \geqslant 3$, cosymplectic manifold. Suppose that M admits a closed Einstein–Weyl structure $\nabla^{[g]}$. Then M is η-Einstein.*

Proof. We notice first that, since $\nabla^{[g]}$ is flat and Ricci-flat, if θ again denotes the 1-form associated to $g \in [g]$, the curvature tensor field R_g and the Ricci curvature tensor field $\rho(R_g)$ of M are respectively given by (3.3) and (3.4). Now, by using the equality $\rho(R_g)(X, \xi) = 0$, from (3.4) we obtain

$$(2n - 1)(D^g_\xi \theta)(\xi) = (2n - 1)(\theta(\xi))^2 + \delta\theta - (2n - 1)|\theta|^2. \tag{4.2}$$

On the other hand, since we also have $R_g(X, Y)\xi = 0$ for all $X, Y \in \mathfrak{X}(M)$, for every horizontal vector field X formulae (2.5) and (3.3) imply

$$D^g_X B = ((\theta(\xi))^2 - (D^g_\xi \theta)(\xi) - |\theta|^2)X + \theta(X)B, \tag{4.3}$$

from which we obtain

$$g(D^g_X B, Y) = (D^g_X \theta)(Y) = ((\theta(\xi))^2 - (D^g_\xi \theta)(\xi) - |\theta|^2)g(X, Y) + \theta(X)\theta(Y), \tag{4.4}$$

where we still consider Y as a horizontal vector field. Finally, substituting (4.2) and (4.4) into (3.9), we find that the Ricci curvature tensor field of the cosymplectic structure for X and Y such that $\eta(X) = \eta(Y) = 0$ is given by

$$\rho(R_g)(X, Y) = \sigma g(X, Y), \tag{4.5}$$

where $\sigma = (2n - 1)|\theta|^2 - 2\delta\theta$. As a consequence,

$$\rho(R_g) = \sigma(g - \eta \otimes \eta) \tag{4.6}$$

gives the Ricci tensor of M. This proves the theorem. □

Remark 4.3. We notice that, in the hypothesis of the previous theorem, when $\theta(\xi) = 0$, from (4.4) we obtain that $g(D^g_X B, B) = \frac{1}{2}X(|\theta|^2) = 0$ for all vector fields on M orthogonal to ξ. But, θ being closed, we have in particular that $d\theta(\xi, B) = 0$ and $(D^g_\xi\theta)(B) = (D^g_B\theta)(\xi) = 0$. Then, since $(D^g_\xi\theta)(B) = \xi(\theta(B)) - \theta(D^g_\xi B) = \frac{1}{2}\xi(|\theta|^2) = 0$, we get that $|\theta|$ is constant on M.

Corollary 4.4. *Let $(M, \varphi, \xi, \eta, g)$ be a compact cosymplectic manifold and let $\nabla^{[g]}$ be a closed Einstein–Weyl structure on M. Then the almost-contact metric g is the Gauduchon metric of $\nabla^{[g]}$ if and only if the* 1*-form associated to the metric $g \in [g]$ is given by $\theta = \lambda\eta$ for some constant λ.*

Proof. In fact, if g is the Gauduchon metric, the 1-form θ is g-coclosed, closed and thus g-harmonic on M. In particular, θ is parallel and (4.2) implies that $((\theta)(\xi))^2 = |\theta|^2$, so that $\theta(X) = 0$ for every horizontal vector field X. Then $\theta = \lambda\eta$ for some constant λ because of the parallelism of θ. □

Corollary 4.5. *Let $(M, \varphi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional η-Einstein cosymplectic manifold, where $2n+1 \geqslant 3$. Suppose $\rho(R_g) = \lambda(g - \eta \otimes \eta)$ for some constant $\lambda > 0$. Then M admits two closed Einstein–Weyl structures, $\nabla_1^{[g]}$ and $\nabla_2^{[g]}$, with associated* 1*-forms respectively given by $\theta_1 = \sqrt{(2n - 1)\lambda}\eta$ and $\theta_2 = -\sqrt{(2n - 1)\lambda}\eta$.*

Proof. We directly obtain the claim by comparing (3.4) with the expression $\rho(R_g) = \lambda(g - \eta \otimes \eta)$. □

References

1. D. E. Blair, *Riemannian geometry on contact and symplectic manifolds*, Progress in Mathematics, Volume 203 (Birkhäuser, 2001).
2. C. P. Boyer and K. Galicki, On Sasakian–Einstein geometry, *Int. J. Math.* **11** (2000), 873–909.
3. C. P. Boyer and K. Galicki, Einstein manifolds and contact geometry, *Proc. Am. Math. Soc.* **129** (2001), 2419–2430.
4. C. P. Boyer and K. Galicki, Einstein metrics on rational homology spheres, *J. Diff. Geom.* **74** (2006), 353–362.
5. C. P. Boyer, K. Galicki and P. Matzeu, On eta-Einstein Sasakian geometry, *Commun. Math. Phys.* **262** (2006), 177–208.
6. C. P. Boyer, K. Galicki and M. Nakamaye, On the geometry of Sasakian–Einstein 5-manifolds, *Math. Annalen* **325** (2003), 485–524.
7. P. Gauduchon, Structures de Weyl–Einstein, espaces de twisteurs et variétés de type $S^1 \times S^3$, *J. Reine Angew. Math.* **469** (1995), 1–50.
8. A. Ghosh, Einstein–Weyl structures on contact metric manifolds, *Annals Global Analysis Geom.* **35** (2009), 431–441.
9. T. Higa, Weyl manifolds and Einstein–Weyl manifolds, *Commun. Math. Univ. Saint Pauli* **42** (1993), 143–160.
10. P. Matzeu, Some examples of Einstein–Weyl structures on almost contact manifolds, *Class. Quant. Grav.* **17** (2000), 1–9.
11. P. Matzeu, Almost contact Einstein–Weyl structures, *Manuscr. Math.* **108** (2002), 275–288.

12. F. Narita, Riemannian submersions and Riemannian manifolds with Einstein–Weyl structures, *Geom. Dedicata* **65** (1997), 103–116.
13. F. Narita, Einstein–Weyl structures on almost contact metric manifolds, *Tsukuba J. Math.* **22** (1998), 87–98.
14. M. Okumura, Some remarks on space with a certain contact structure, *Tohoku Math. J.* **14** (1962), 135–145.
15. L. Ornea and P. Piccinni, Compact hyperhermitian Weyl and quaternionic Hermitian Weyl manifolds, *Annals Global Analysis Geom.* **16**(4) (1998), 383–398.
16. H. Pedersen, Y. S. Poon and A. Swann, The Einstein–Weyl equations in complex and quaternionic geometry, *Diff. Geom. Applic.* **3** (1993), 309–321.
17. S. Tachibana, On harmonic tensors in compact Sasakian spaces, *Tohoku Math. J.* **17** (1965), 271–284.
18. P. Tondeur, *Geometry of foliations*, Monographs in Mathematics, Volume 90 (Birkhäuser, 1997).