GENERALIZED INJECTIVITY AND CHAIN CONDITIONS

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Relationships between injectivity or generalized injectivity and chain conditions on a module category have been studied by several authors. A well-known theorem of Osofsky [**14, 15**] asserts that a ring all of whose cyclic right modules are injective is semisimple Artinian. Osofsky's proofs in [**14, 15**] essentially used homological properties of injective modules, and, later, her arguments were applied by other authors in their studies of rings for which cyclic right modules are quasi-injective, continuous or quasi-continuous (see e.g. [**1, 10, 12**]). Following [**5**] (cf. [**4**]), a module M is called a CS-module if every submodule of M is essential in a direct summand of M. In the recent paper [**17**], B. L. Osofsky and P. F. Smith have proved a very general theorem on cyclic completely CS-modules from which many known results in this area follow rather easily. In another direction, it was proved in [**8**] that a finitely generated quasi-injective module with ACC (respectively DCC) on essential submodules is Noetherian (respectively Artinian). This result was also extended to CS-modules in [**3, 16**], and weak CS-modules in [**19**].

While CS-modules are a generalization of injective modules, finitely generated CS-modules have an interesting property that their closed submodules are finitely generated, which can be regarded as a weak form of the Noetherian condition. This observation led to a study in [**13**] of modules all of whose closed submodules are finitely generated.

In this paper, we will be interested in a class of modules which contains both finitely generated CS-modules and modules with finite uniform dimension. A module M will be called a CEF-module if every closed submodule of M contains a finitely generated essential submodule, or, in other words, if every closed submodule of M is essentially finitely generated. Similarly, a module M is called a CEC-module if every closed submodule of M contains a cyclic essential submodule, or, equivalently, every closed submodule of M is essentially cyclic.

We will show that if M is a finitely generated module such that for every nonzero submodule N of M, M/N and every cyclic submodule of M/N is a direct sum of a CEC-module and a module with finite uniform dimension, then M satisfies ACC on direct summands. A similar result holds also for CEF-modules. Thus we obtain a generalization of the main result in [**9**] on CS-modules. As a consequence, we get also a refinement of the Osofsky-Smith theorem in [**17**]. Further, it is proved that a CS-module M for which $M/\mathrm{Soc}(M)$ satisfies ACC on direct summands is a direct sum of a semisimple module and a module with finite uniform dimension. Consequently, we obtain a partial extension of a result of Camillo and Yousif [**3**] from CS-modules to CEC-modules.

Among examples of CEC-modules, we could mention uniform modules, injective hulls of cyclic modules, or, more generally, any CS-module M with a cyclic essential submodule K. Indeed, if A is a closed submodule in M, then A is a direct summand of M. Thus the projection of K in A is cyclic, and contains $K \cap A$ which clearly is essential in A. Hence A contains a cyclic essential submodule. Similarly, examples of CEF-modules are modules with finite uniform dimension, injective hulls of finitely generated modules, or any CS-module with a finitely generated essential submodule.

Throughout this paper, all rings considered are associative with identity and all modules are unitary right modules. For a module M, $\mathrm{Soc}(M)$ will denote the socle of M, and M is *semisimple* if $M = \mathrm{Soc}(M)$. A submodule N of M is called *essential* in M if $N \cap K \neq 0$ for every nonzero submodule K of M. In this case, M is called an *essential extension* of N. A submodule C is called *closed* in M if C has no proper essential extensions in M. A module M is said to have *finite uniform dimension* if M does not contain an infinite direct sum of nonzero submodules. A module N is called a *subquotient* of a module M if N is a submodule of a quotient of M.

We now state the main result of this paper.

THEOREM 1. *Let M be a finitely generated module such that for every nonzero submodule A of M, M/A and all cyclic submodules of M/A are direct sums of a* CEC-*module and a module with finite uniform dimension. Then M satisfies* ACC *on direct summands.*

To prove this theorem, we will adapt the techniques developed by Osofsky and Smith in **[17]**. First, we prove a lemma, which is of independent interest.

LEMMA 2. *Let N be a* CEF-*module with the infinitely generated essential socle S such that every finitely generated submodule of S is a direct summand of N, and every cyclic submodule of N is a direct sum of a* CEF-*module and a module with finitely generated socle. Then N/S is not a* CEC-*module.*

Proof. Suppose that N/S is a CEC-module. Let $S = \bigoplus_{i=1}^{\infty} S_i$ such that all S_i are infinitely generated. For each i, S_i has a maximal essential extension D_i. Since D_i is closed in N, D_i contains a finitely generated essential submodule. Thus it is clear that $D_i \neq S_i$ for each i, so $D_i' = (D_i + S)/S$ is nonzero for every i. Let A' be a maximal essential extension of $\bigoplus_{i=1}^{\infty} D_i'$ in N/S. Since N/S is a CEC-module, A' contains a cyclic essential submodule E'. There exists a cyclic submodule E of N such that $(E + S)/S = E'$. Since E' is essential in A', $C_i' = E' \cap D_i'$ is nonzero for each i. Let C_i be the inverse image of C_i' in D_i under the canonical map. Then we have $S_i \subset C_i \subseteq D_i$, and clearly C_i is not contained in S. Because $C_i' \subseteq E'$, we have

$$C_i \subseteq E + S = E \oplus T$$

for some submodule T of S. If $C_i \cap E = 0$ for some i, then C_i is isomorphic to a submodule of T, thus C_i is semisimple and so C_i is contained in S, a contradiction. Therefore we have that for each i $C_i \cap E \neq 0$. But C_i is an essential extension of S_i, so it follows that $S_i \cap E \neq 0$ for each i. Then we can take a nonzero simple submodule V_i in $S_i \cap E$ for each i.

Since E is cyclic, by assumption, $E = F \oplus K$, where F is a CEF-module and K has finitely generated socle. It is easy to see that K is finitely generated semisimple. Let $V = \bigoplus_{i=1}^{\infty} V_i$ and $U = F \cap V$. Then $V = U \oplus X$, where X is isomorphic to a submodule of K; hence X is finitely generated. Thus U is infinitely generated. Since F is a CEF-module, and U is semisimple, U has a finitely generated essential extension L in F. Clearly $L \neq U$, thus $L' = (L+S)/S$ is nonzero, and $L' \subset A'$. Now we want to show that $L \cap \left(\bigoplus_{i=1}^{\infty} D_i\right) \subseteq S$. Since $S \subseteq \bigoplus_{i=1}^{\infty} D_i$, this would imply that $L' \cap \left(\bigoplus_{i=1}^{\infty} D_i'\right) = 0$, a contradiction of the fact that $\bigoplus_{i=1}^{\infty} D_i'$ is essential in A'.

In fact, for each n, we have

$$\left(L \cap \bigoplus_{i=1}^{n} D_i\right) \cap S = L \cap \left(\left(\bigoplus_{i=1}^{n} D_i\right) \cap S\right)$$
$$= L \cap \bigoplus_{i=1}^{n} S_i \subseteq \left(\bigoplus_{i=1}^{\infty} V_i\right) \cap \left(\bigoplus_{i=1}^{n} S_i\right) = \bigoplus_{i=1}^{n} V_i.$$

But $\bigoplus_{i=1}^{n} V_i$ is a direct summand of N by assumption; together with the fact that S is essential in N, it implies that $L \cap \bigoplus_{i=1}^{n} D_i \subseteq S$, for each n. Thus we have $L \cap \bigoplus_{i=1}^{\infty} D_i \subseteq S$ which gives us the desired contradiction. This completes the proof of Lemma 2.

LEMMA 3. *Let M be a* CEF-*module such that $M/\mathrm{Soc}(M)$ has finite uniform dimension. Then M has finite uniform dimension.*

Proof. It is enough to show that $S = \mathrm{Soc}(M)$ is finitely generated. Suppose that S is infinitely generated; then we can write $S = \bigoplus_{i=1}^{\infty} S_i$, where each S_i is infinitely generated. Since M is a CEF-module, each S_i has a maximal essential extension D_i which contains a finitely generated essential submodule B_i. Then clearly $D_i \neq S_i$, and $\bigoplus_{i=1}^{\infty} ((D_i + S)/S)$ is an infinite direct sum in M/S, a contradiction.

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Suppose that there exists an infinite ascending chain of direct summands A_i of M:

$$A_1 \subset A_2 \subset \ldots \subset A_i \subset \ldots$$

There is a direct summand B_1 of M such that $M = A_1 \oplus B_1$. Then it is clear that $A_2 = A_1 \oplus (A_2 \cap B_1)$. Thus there is a submodule B_2 of B_1 such that $B_1 = B_2 \oplus (A_2 \cap B_1)$. It follows that $M = A_2 \oplus B_2$. Repeating this argument, we produce an infinite descending chain of direct summands B_i of M with

$$B_1 \supset B_2 \supset \ldots \supset B_i \supset \ldots.$$

Let $B_n = C_{n+1} \oplus B_{n+1}$ with $n \geq 1$, and put $C_1 = A_1$. Then we get an infinite sequence $\{C_n\}$ of direct summands C_n of M, such that

$$M = \left(\bigoplus_{i=1}^{n} C_i\right) \oplus B_n, \quad \text{and} \quad \left(\bigoplus_{j=n+1}^{\infty} C_j\right) \subseteq B_n$$

for each $n \geq 1$.

Since each C_i is finitely generated, C_i contains a maximal submodule X_i. Consider the quotient module $P = M \big/ \bigoplus_{i=1}^{\infty} X_i$. Then we have $S = \left(\bigoplus_{i=1}^{\infty} C_i\right) \big/ \left(\bigoplus_{i=1}^{\infty} X_i\right)$ is a semisimple submodule of P. Note that, by the construction, for each n, $S_n = \bigoplus_{i=1}^{n} (C_i/X_i)$ is a direct summand of P. It follows that every finitely generated submodule of S, being a direct summand of some S_n, must be a direct summand of P.

By hypothesis, $P = P_1 \oplus D$, where P_1 is a CEC-module and D has finite uniform dimension. Let $S' = P_1 \cap S$; then $S = S' \oplus T$ for some submodule T of S. Since T is isomorphic to a submodule of D, T is finitely generated. Thus S' is infinitely generated. Since P_1 is a CEC-module, it is easy to see that S' has a cyclic essential extension L in P_1. Again by hypothesis, $L = N \oplus F$ such that N is a CEC module and F has finite uniform dimension. Let $Q = N \cap S'$; then Q is essential in N. Repeating the above argument, we can show that Q is infinitely generated. It is also clear that every finitely generated submodule of Q is a direct summand of N.

Now we have $N/Q = H \oplus G$, where H is a CEC-module and G has finite uniform dimension. We see that N satisfies the conditions of Lemma 2, thus N/Q cannot be a CEC-module, so G must be nonzero. Since G is cyclic, there is a cyclic submodule N_1 of N such that $(N_1 + Q)/Q \cong G$. Let $Q_1 = N_1 \cap Q$; then $Q_1 = \mathrm{Soc}(N_1)$ and $N_1/Q_1 \cong G$, so N_1/Q_1 has finite uniform dimension. By hypothesis, $N_1 = N_2 \oplus Y$ such that N_2 is a CEC-module and Y has finite uniform dimension. Then

$$N_1/Q_1 \cong N_2/\mathrm{Soc}(N_2) \oplus Y/\mathrm{Soc}(Y).$$

and it follows that $N_2/\mathrm{Soc}(N_2)$ has finite uniform dimension. By Lemma 3, $\mathrm{Soc}(N_2)$ is finitely generated, hence Q_1 is finitely generated. Therefore Q_1 is a direct summand of N, and, since Q_1 is essential in N_1, it follows that $N_1 = Q_1$, so $G = 0$. This contradiction completes the proof of the theorem.

REMARK. We are unable to answer the following question. Let M be a cyclic module such that every cyclic subquotient of M is a direct sum of a CEC-module and a module with finitely generated socle. Does M satisfy ACC on direct summands?

For CEF-modules, the following theorem can be obtained with a proof similar to that of Theorem 1.

THEOREM 4. *Let M be a finitely generated module such that for every nonzero submodule A of M, every finitely generated submodule of M/A is a direct sum of a* CEF-*module and a module with finite uniform dimension. Then M has* ACC *on direct summands.*

Next we will derive some consequences of these results. The first corollary is the main result of [**9**], which is turn is a generalization of [**6**] and [**7**].

COROLLARY 5 (see [**9**]). *Let M be a cyclic module such that every cyclic subquotient of M is a direct sum of a* CS-*module and a module with finite uniform dimension. Then M is a finite direct sum of uniform modules.*

Proof. By Theorem 1, M has ACC on direct summands; thus M is a finite direct sum of indecomposable submodules. Now the result follows from the fact that an indecomposable CS-module is uniform.

The next result can be regarded as a refinement of the Osofsky-Smith theorem in [**17**].

COROLLARY 6. *Let M be a cyclic module such that every cyclic subquotient of M is a* CEC-*module. Then M has* ACC *on direct summands.*

COROLLARY 7. *Let M be a finitely generated module such that every finitely generated subquotient of M is a* CEF-*module. Then M has ACC on direct summands.*

The following result is immediate from Corollaries 6 and 7.

COROLLARY 8. *Let R be a ring for which every cyclic (respectively finitely generated) right module is a* CEC-*module (respectively* CEF-*module). Then every cyclic (respectively finitely generated) right R-module satisfies* ACC *on direct summands.*

If R is a von Neumann regular ring, then every finitely generated right ideal of R is principal. Thus, from the proof of Theorem 1, we obtain

COROLLARY 9. *Let R be a von Neumann regular ring. Then R is semisimple Artinian if and only if every cyclic right R-module is a* CEF-*module.*

In [**14**, Lemma 5], Osofsky proved that if $\{e_i\}_{i=1}^{\infty}$ is an infinite set of orthogonal idempotents in a von Neumann regular right self-injective ring R, then $R\big/\bigoplus_{i=1}^{\infty} e_iR$ is not an injective right R-module. From Lemma 2 we can obtain a related result.

COROLLARY 10. *Let R be a von Neumann regular right self-injective ring. If $S=\mathrm{Soc}(R_R)$ is infinitely generated, then $(R/S)_R$ is not a* CEF-*module.*

Proof. Let E be the injective hull of S in R_R; then the module E_R satisfies the conditions of Lemma 2. Note that every finitely generated submodule of E_R is cyclic. From the proof of Lemma 2, it follows that $(E/S)_R$ is not a CEF-module. It is easy to prove that a direct summand of a CEF-module is again CEF, so we obtain that $(R/S)_R$ is not a CEF-module.

Modules with chain conditions on essential submodules have been studied extensively in recent years (see e.g. [**2, 3, 8, 11, 16, 18, 19**]). Extending [**8**] and [**11**], Camillo and Yousif [**3**] showed that if M is a CS-module such that $M/\mathrm{Soc}(M)$ has finite uniform dimension, then M is a direct sum of a semisimple module and a module with finite uniform dimension. Note that a module with finite uniform dimension has ACC on direct summands. Now we shall extend the above result of Camillo and Yousif to CS-modules M with $M/\mathrm{Soc}(M)$ satisfying ACC on direct summands. For our result, we need some lemmas, the first of which is elementary, so we omit the proof.

LEMMA 11. *For a module M the following conditions are equivalent.*

(a) *M satisfies* ACC *on direct summands.*

(b) *M does not contain an infinite direct sum $\bigoplus_{i=1}^{\infty} A_i$ of submodules A_i, where $\bigoplus_{i=1}^{n} A_i$ is a direct summand of M for each $n \geq 1$.*

LEMMA 12. *Let M be a module and $S=\mathrm{Soc}(M)$.*

(a) *If A and B are submodules of M with $A\cap B=0$, then*

$$((A+S)/S)\cap((B+S)/S)=0.$$

(b) *If A is a direct summand of M, then $(A+S)/S$ is a direct summand of M/S.*

(c) *If $\bigoplus_{i\in I} A_i$ is a direct sum of submodules of M, then $\bigoplus_{i\in I}((A_i+S)/S)$ is also a direct sum of submodules in M/S.*

Proof. (a) Let $f: M\to M/S$ be the canonical map. Suppose that A and B are submodules of M such that $A\cap B=0$. Set $\bar{V}=f(A)\cap f(B)$, then there exists a submodule $V\subseteq A$ such that $f(V)=\bar{V}$. Clearly $V\subseteq B+S=B\oplus T$ for some submodule $T\subseteq S$. Since $V\cap B=0$, it follows that V is isomorphic to a submodule of T. Hence $V\subseteq S$ which implies that $\bar{V}=0$. Therefore we have $f(A)\cap f(B)=0$.

(b) Let A be a direct summand of M. Then $M=A\oplus B$ for some submodule B of M. Clearly $M/S=f(A)+f(B)$. By (a) we have $f(A)\cap f(B)=0$. Thus $f(A)$ is a direct summand of M/S.

(c) This is an immediate consequence of (a).

Following Smith **[19]**, a module M will be called *almost semisimple* if M has essential socle and every finitely generated semisimple submodule of M is closed in M.

PROPOSITION 13. *Let M be a* CS*-module such that $M/\text{Soc}(M)$ has* ACC *on direct summands. Then M is a direct sum of a semisimple module and a module with finite uniform dimension.*

Proof. Let $S=\text{Soc}(M)$. Then $M=M_1\oplus M_2$, where S is essential in M_1. Clearly M_2 is isomorphic to a direct summand of M/S, thus by hypothesis M_2 has ACC on direct summands. Hence M_2 is a finite direct sum of indecomposable submodules, and, since M_2 is CS, M_2 is a finite direct sum of uniform modules. Therefore, without loss of generality, we may assume that M has essential socle.

Now we show that M is a direct sum of an almost semisimple module and a module with finite uniform dimension. If M is not almost semisimple, there exists a finitely generated submodule S_1 of S such that S_1 is not closed in M. Then S_1 is essential in a direct summand A_1 of M, and $A_1\neq S_1$. Let $M=A_1\oplus B_1$. If B_1 is not almost semisimple, B_1 contains a finitely generated semisimple submodule S_2 which is not closed in B_1. Thus S_2 is essential in a direct summand A_2 of B_1, and $A_2\neq S_2$. Note that $A_1\oplus A_2$ is a direct summand of M. Repeat this process; it produces a direct sum $A_1\oplus A_2\oplus A_3\oplus\dots$ of submodules A_i of M such that A_i has finite uniform dimension and A_i is not semisimple for each $i\geq 1$, and furthermore $\bigoplus_{i=1}^{n} A_i$ is a direct summand of M for all $n\geq 1$. By Lemma 12,

$$((A_1+S)/S)\oplus((A_2+S)/S)\oplus\dots$$

is a direct sum in M/S, and $\bigoplus_{i=1}^{n}((A_i+S)/S)$ is a direct summand of M/S for all $n\geq 1$. By Lemma 11, this process must stop. Therefore M is a direct sum of an almost semisimple module N and a module F with finitely generated essential socle. Clearly N is a CS-module and $N/\text{Soc}(N)$ has ACC on direct summands.

Next we show that N is semisimple. Let E be a finitely generated submodule of N. Then E is essential in a direct summand H of N. Suppose that $T=\text{Soc}(E)$ is infinitely generated. We claim that T contains an infinitely generated submodule which is closed in

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H. Assume that it is not so. Take an infinitely generated submodule T_1 of T such that T/T_1 is infinitely generated. Then $H = C_1 \oplus D_1$, where T_1 is essential in C_1. Clearly $\mathrm{Soc}(D_1)$ is infinitely generated. Take an infinitely generated submodule T_2 of $\mathrm{Soc}(D_1)$ such that $(\mathrm{Soc}(D_1))/T_2$ is infinitely generated. Then T_2 is essential in a direct summand C_2 of H. Continuing in this manner, we get an infinite direct sum $C_1 \oplus C_2 \oplus C_3 \oplus \ldots$ of H such that C_i is not semisimple and $\bigoplus_{i=1}^{n} C_i$ is a direct summand of H for all $n \geq 1$. Since $H/\mathrm{Soc}(H)$ has also ACC on direct summands, Lemma 12 gives us a contradiction. Thus T contains an infinitely generated submodule K which is closed in H. Then K is a direct summand of H, and hence of E, so K is finitely generated. This contradiction shows that $T = \mathrm{Soc}(E)$ must be finitely generated. Because N is CS and almost semisimple, T is a direct summand of N. But T is essential in E, so we get that $T = E$, thus E is semisimple. Therefore N is semisimple which completes the proof.

Remark. In **[19]** Smith introduced weak CS-modules as modules in which every semisimple submodule is essential in a direct summand. By **[19**, Corollary 2.7], if M is a weak CS-module such that $M/\mathrm{Soc}(M)$ has finite uniform dimension, then M is a direct sum of a semisimple module and a module with finite uniform dimension. Also, Osofsky **[16]** studied CS-modules satisfying $\aleph$-chain conditions on essential submodules, for an infinite cardinal $\aleph$. Thus, it might be interesting to investigate CS or weak CS-modules M such that $M/\mathrm{Soc}(M)$ satisfies $\aleph$-chain conditions on direct summands, for an infinite cardinal $\aleph$.

Extending the Osofsky-Smith theorem **[17]**, Camillo and Yousif **[3]** proved that if M is a cyclic CS-module such that all cyclic singular subquotients of M are CS-modules, then M has finite uniform dimension. As an application of Proposition 13, we can now get a partial generalization of this last result.

Proposition 14. *Let M be a CS-module with the essential socle S such that M/S is cyclic and all cyclic singular subquotients of M are CEC-modules. Then M is a direct sum of a semisimple module and a module with finite uniform dimension.*

Proof. Clearly every cyclic subquotient of M/S is a CEC-module. Thus, by Corollary 6, M/S has ACC on direct summands. The result follows now by Proposition 13.

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