COMPOSITIO MATHEMATICA

Kähler–Einstein metrics with positive curvature near an isolated log terminal singularity

Vincent Guedj and Antonio Trusiani

With an appendix by Sébastien Boucksom.

Compositio Math. **161** (2025), 714–755.

doi: 10.1112/S0010437X24007619

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

Compositio Math. 161 (2025) 714–755
doi:10.1112/S0010437X24007619

Kähler–Einstein metrics with positive curvature near an isolated log terminal singularity

Vincent Guedj and Antonio Trusiani

With an appendix by Sébastien Boucksom.

Abstract

We analyze the existence of Kähler–Einstein metrics of positive curvature in the neighborhood of a germ of a log terminal singularity (X, p). This boils down to solving a Dirichlet problem for certain complex Monge–Ampère equations. We establish a Moser–Trudinger inequality $(MT)_\gamma$ in subcritical regimes $\gamma < \gamma_{\mathrm{crit}}(X, p)$ and show the existence of smooth solutions in those cases. We show that the expected critical exponent $\tilde{\gamma}_{\mathrm{crit}}(X, p) = ((n+1)/n)\widehat{\mathrm{vol}}(X, p)^{1/n}$ can be expressed in terms of the normalized volume, an important algebraic invariant of the singularity.

Contents

Received 13 June 2023, accepted in final form 24 September 2024.
2020 Mathematics Subject Classification 32W20, 32U05, 32Q15, 35A23 (primary).
Keywords: Dirichlet problem; Kähler–Einstein metrics; log-terminal singularity; complex Monge–Ampère equation; normalized volume.

1. Introduction

Let (X, p) be a germ of an isolated singularity. We analyze the existence of local Kähler–Einstein metrics of positive curvature in a neighborhood of p. It follows from [BBEGZ19, Proposition 3.8] that the singularity has to be *log terminal*, a relatively mild type of singularity that plays a central role in birational geometry. We refer the reader to Definition 2.8 for a precise formulation and simply indicate here that a prototypical example is the vertex of the affine cone over a Fano manifold. Consider indeed

$$X = \{z \in \mathbb{C}^{n+1},\ P(z) = 0\},$$

P a homogeneous polynomial of degree $d \in \mathbb{N}^*$ so that $H = \{[z] \in \mathbb{CP}^n,\ P(z) = 0\}$ is a smooth hypersurface of the complex projective space. Then $(X, 0)$ is log terminal if and only if H is Fano (which is equivalent here to $d < n+1$). Thus, log terminal singularities can be seen as a local analogue of Fano varieties.

Given a local embedding $(X, p) \hookrightarrow (\mathbb{C}^N, 0)$, constructing such a local Kähler–Einstein metric boils down to solve a complex Monge–Ampère equation

$$(MA)_{\gamma,\phi,\Omega} \begin{cases} (dd^c\varphi)^n = \dfrac{e^{-\gamma\varphi}\, d\mu_p}{\int_\Omega e^{-\gamma\varphi}\, d\mu_p}, \\ \varphi_{|\partial\Omega} = \phi, \end{cases}$$

where Ω is a smooth neighborhood of p, ϕ is a smooth boundary data, μ_p is an adapted volume form (see Definition 2.9), and $\gamma > 0$ is a parameter. We seek for a solution $\varphi \in \mathcal{C}^\infty(\Omega \setminus \{p\}) \cap \mathcal{C}^0(\overline{\Omega})$ which is strictly plurisubharmonic in $\Omega \setminus \{p\}$, so that $\omega_{\mathrm{KE}} := dd^c\varphi$ is a Kähler form in $\Omega \setminus \{p\}$ satisfying the Einstein equation

$$\mathrm{Ric}(\omega_{\mathrm{KE}}) = \gamma\omega_{\mathrm{KE}}.$$

An important motivation comes from the global study of positively curved Kähler–Einstein metrics ω_{KE} on $\mathbb{Q}$-Fano varieties. Such canonical singular metrics have been constructed in [BBEGZ19] and further studied in [BBJ21, LTW21, Li22], extending the resolution of the Yau–Tian–Donaldson conjecture [CDS15] to this singular context. Despite recent important progress [HS17, Dru18, HP19, BGL22], the geometry of these singular metrics remains mysterious and one needs to better understand the asymptotic behavior of ω_{KE} near the singularities.

We restrict the metric ω_{KE} to a neighborhood of p and wish to analyze the behavior of its local potentials $\omega_{\mathrm{KE}} = dd^c\varphi_{\mathrm{KE}}$ near p. The latter solve a Monge–Ampère equation $(MA)_{\gamma,\phi,\Omega}$, as can be seen by locally trivializing a representative of the first Chern class (after an appropriate rescaling). The boundary data are thus given by the solution $\varphi_{\mathrm{KE}} = \phi$ itself.

Studying the family of equations $(MA)_{\gamma,\phi,\Omega}$ we will give evidence that:

– the possibility of solving $(MA)_{\gamma,\phi,\Omega}$ should be independent of Ω and ϕ;

https://doi.org/10.1112/S0010437X24007619

– the largest exponent $\gamma_{\rm crit}(X,p)$ for which we can solve $(MA)_{\gamma,\phi,\Omega}$ should only depend on the algebraic nature of the log terminal singularity.

Following earlier works dealing with the case of compact Kähler varieties or the local smooth setting [BBGZ13, GKY13, BB22, BBEGZ19], we develop a variational approach to solve these equations. A crucial role is played by

$$E_\phi(\varphi) = \frac{1}{n+1}\sum_{j=0}^{n}\int_\Omega (\varphi-\phi_0)(dd^c\varphi)^j\wedge(dd^c\phi_0)^{n-j},$$

the Monge–Ampère energy of φ relative to a plurisubharmonic extension ϕ_0 of ϕ. This energy is a primitive of the Monge–Ampère operator and a building block of the functional F_γ whose Euler–Lagrange equation is $(MA)_{\gamma,\phi,\Omega}$,

$$\varphi\in\mathcal{T}_\phi(\Omega)\mapsto F_\gamma(\varphi) = E_\phi(\varphi)+\frac{1}{\gamma}\log\int_\Omega e^{-\gamma\varphi}\,d\mu_p\in\mathbb{R}.$$

Here $\mathcal{T}_\phi(\Omega)$ denotes the set of all plurisubharmonic functions φ in Ω which are continuous on $\overline{\Omega}$ and such that $\varphi_{|\partial\Omega}=\phi$.

In order to solve $(MA)_{\gamma,\phi,\Omega}$ one can try and extremize F_γ by showing that it is a proper functional. Our first main result in this direction (Theorem 5.1) is the following Moser–Trudinger-type inequality.

Theorem A. *For any $0<\gamma<((n+1)/n)\alpha(X,\mu_p)$, there exists $C_\gamma>0$ such that*

$$\left(\int_\Omega e^{-\gamma\varphi}\,d\mu_p\right)^{1/\gamma}\le C_\gamma\exp(-E_\phi(\varphi)),\qquad (MT_\gamma)$$

for all $\varphi\in\mathcal{T}_\phi(\Omega)$.

The *alpha invariant* of the singularity (X,p) is defined by

$$\alpha(X,\mu_p):=\sup\left\{\alpha>0,\ \sup_{\varphi\in\mathcal{F}_1(\Omega)}\int_\Omega e^{-\alpha\varphi}\,d\mu_p<+\infty\right\},$$

where $\mathcal{F}_1(\Omega)$ denotes the set of plurisubharmonic functions φ with ϕ-boundary values, whose Monge–Ampère mass is bounded by $\int_\Omega (dd^c\varphi)^n\le 1$.

When (X,p) is smooth, Theorem A has been obtained independently in [BB22, GKY13] with $\alpha(X,\mu_p)=n$ (the normalizations and methods are quite different in these two works, but they eventually produce the same critical exponent).

We introduce

$$\gamma_{\rm crit}(X,p):=\sup\{\gamma>0 \text{ such that } (MT_\gamma) \text{ holds}\}.$$

While Theorem A provides a lower bound for $\gamma_{\rm crit}(X,p)$, we provide an upper bound in Theorem 4.5, which yields

$$\frac{n+1}{n}\alpha(X,\mu_p)\le\gamma_{\rm crit}(X,p)\le\frac{n+1}{n}\widehat{\rm vol}(X,p)^{1/n},$$

where $\widehat{\rm vol}(X,p)$ denotes the *normalized volume* of the singularity (X,p). This is an algebraic invariant of the singularity at p introduced by Chi Li in [Li18], which has recently played a key role in the algebraic understanding of the moduli space of K-stable Fano varieties (see [Blu18, Liu18, LWX21, LXZ22] and the references therein); we refer to Definition 2.15 for a precise definition.

When p is smooth then $\alpha(X,\mu_p)=\widehat{\mathrm{vol}}(X,p)^{1/n}=n$ by [ACKPZ09, Dem09]. It is tempting to conjecture that the equality $\alpha(X,\mu_p)=\widehat{\mathrm{vol}}(X,p)^{1/n}$ always holds. We establish in § 5 the following partial bounds on $\alpha(X,\mu_p)$.

Theorem B. *The following inequalities hold:*

$$\frac{n}{\mathrm{mult}(X,p)^{1-1/n}}\frac{\mathrm{lct}(X,p)}{1+\mathrm{lct}(X,p)}\leq\alpha(X,\mu_p)\leq\widehat{\mathrm{vol}}(X,p)^{1/n}.$$

Moreover, $\alpha(X,\mu_p)=\widehat{\mathrm{vol}}(X,p)^{1/n}$ if (X,p) is an admissible singularity.

Here $\mathrm{mult}(X,p)$ denotes the algebraic multiplicity of (X,p), while $\mathrm{lct}(X,p)$ is its log canonical threshold (see Definition 2.12). Having $\alpha(X,\mu_p)$ bounded from below is quite involved; we show that $\alpha(X,\mu_p)=\widehat{\mathrm{vol}}(X,p)^{1/n}$ when $n=2$, but our lower-bound is not sharp when $n\geq 3$ unless (X,p) is an *admissible singularity*, a notion introduced in [LTW21]. The vertex of the affine cone over a smooth Fano manifold is an example of admissible singularity (see § 5).

Using analytic Green functions and Demailly's comparison theorem, we provide in Propositions 5.6 and 5.8 evidence for the equality $\alpha(X,\mu_p)=\widehat{\mathrm{vol}}(X,p)^{1/n}$. Appendix A uses an algebraic approach based on [BdFF12], to establish a stronger result than Proposition 5.8.

We note in Lemma 3.13 that if (MT_γ) holds, then F_γ is *coercive* (a strong quantitative version of properness). When $\gamma<\gamma_{\mathrm{crit}}(X,p)$, we then further show the existence of smooth solutions to $(MA)_{\gamma,\phi,\Omega}$.

Theorem C. *If $\gamma<\gamma_{\mathrm{crit}}(X,p)$, then there exists a plurisubharmonic function $\varphi\in\mathcal{C}^\infty(\Omega\setminus\{p\})$ which is continuous in $\overline{\Omega}$ with $\varphi_{|\partial\Omega}=\phi$, and such that*

$$(dd^c\varphi)^n=\frac{e^{-\gamma\varphi}\,d\mu_p}{\int_\Omega e^{-\gamma\varphi}\,d\mu_p}\quad \text{in } \Omega.$$

We expect the solution to be unique, at least when Ω is a generic Stein neighborhood of p. We refer the reader to [GKY13, BB22] for partial results in this direction when p is a smooth point.

2. Preliminaries

2.1 Analysis on singular spaces

Let X be a reduced complex analytic space of pure dimension $n\geq 1$. We let X_{reg} denote the complex manifold of regular points of X and $X_{\mathrm{sing}}:=X\setminus X_{\mathrm{reg}}$ be the set of singular points; this is an analytic subset of X of complex codimension ≥1. We always assume in this article that:

- $X_{\mathrm{sing}}=\{p\}$ consists of a single isolated point;
- X_{reg} is locally irreducible at p;
- U is a fixed neighborhood of p and $j:U\hookrightarrow\mathbb{C}^N$ is a local embedding onto an analytic subset of $\mathbb{C}^N$ for some $N\geq 1$.

As we are interested in the asymptotic behavior of Kähler–Einstein potentials near the singular point p, we shall identify X with $j(U)$ in the following.

2.1.1 *Plurisubharmonic functions.* Using the local embedding j, it is possible to define the spaces of smooth forms on X as restriction of smooth forms of $\mathbb{C}^N$. The notion of currents on X

https://doi.org/10.1112/S0010437X24007619

is defined by duality; the operators ∂ and $\bar{\partial}$, d, d^c and dd^c are also well defined by duality (see [Dem85] for more details).

Here $d = \partial + \overline{\partial}$ and $d^c = (1/4i\pi)(\partial - \overline{\partial})$ are real operators and $dd^c = (i/2\pi)\partial\overline{\partial}$. With this normalization the function $z \in \mathbb{C}^n \mapsto \rho_{FS}(z) = \log[1 + |z|^2] \in \mathbb{R}$ is smooth and plurisubharmonic (psh for short) in $\mathbb{C}^n$, with

$$\int_{\mathbb{C}^n} (dd^c \rho_{FS})^n = 1.$$

Definition 2.1. We say that a function $u : X \longrightarrow \mathbb{R} \cup \{-\infty\}$ is psh on X if it is the restriction of a psh function of $\mathbb{C}^N$.

We let $PSH(X)$ denote the set of all psh functions on X that are not identically $-\infty$.

Recall that u is called weakly psh on X if it is locally bounded from above on X and its restriction to X_{reg} is psh. One can extend it to X by $u^*(p) := \limsup_{X_{\mathrm{reg}} \ni y \to p} u(y)$. Since X is locally irreducible, it follows from the work of Fornæss and Narasimhan [FN80] that u is weakly psh if and only if u^* is psh (see [Dem85, Corollary 1.11]).

If $u \in PSH(X)$, then u is upper semi-continuous on X and locally integrable with respect to the volume form

$$dV_X := \omega_{\mathrm{eucl}}^n \wedge [X].$$

Here $[X]$ denotes the current of integration along X and $\omega_{\mathrm{eucl}} := \sum_{j=1}^N i\, dz_j \wedge d\overline{z_j}$ is the euclidean Kähler form. In particular, $dd^c u$ is a well-defined current of bidegree $(1,1)$ which is positive.

2.1.2 *Pseudoconvex domains and boundary data.* Following [FN80] we say that X is Stein if it admits a $\mathcal{C}^2$-smooth strongly psh exhaustion.

Definition 2.2. A domain $\Omega \Subset X$ is strongly pseudoconvex if it admits a negative $\mathcal{C}^2$-smooth strongly psh exhaustion, i.e. a function ρ strongly psh in a neighborhood Ω' of $\overline{\Omega}$ such that $\Omega := \{x \in \Omega'; \rho(x) < 0\}$, $d\rho \neq 0$ on $\partial\Omega$, and for any $c < 0$,

$$\Omega_c := \{x \in \Omega'; \rho(x) < c\} \Subset \Omega$$

is relatively compact.

We are interested in solving a Dirichlet problem for some complex Monge–Ampère equations in a bounded strongly pseudoconvex domain $\Omega = \{\rho < 0\}$, with given boundary data $\phi \in \mathcal{C}^\infty(\partial\Omega)$.

Definition 2.3. Given $\phi \in \mathcal{C}^\infty(\partial\Omega)$, we fix ϕ_0 a psh function in Ω which is $\mathcal{C}^\infty$-smooth near $\overline{\Omega}$ and such that $\phi_0 | \partial\Omega = \phi$.

Such an extension can be obtained as follows: we pick $\tilde{\phi}$ an arbitrary $\mathcal{C}^2$-smooth extension to $\overline{\Omega}$, and then consider $\phi_0 := \tilde{\phi} + A\rho$, for A so large that ϕ_0 is $\mathcal{C}^2$-smooth and psh in $\overline{\Omega}$. All quantities introduced in the remainder of the paper are essentially independent of the particular choice of the extension.

2.1.3 *Monge–Ampère operators.* The complex Monge–Ampère operator $(dd^c\cdot)^n$ acts on a smooth psh functions φ. When $X = \mathbb{C}^n$, it boils down to

$$(dd^c\varphi)^n = c_n \det\left(\frac{\partial^2 \varphi}{\partial z_j \overline{\partial} z_k}\right)\omega_{\mathrm{eucl}}^n,$$

where $c_n > 0$ is a normalizing constant.

2.1.4 *Bounded functions.* Following [BT82] this operator can be extended to the class $PSH(X)\cap L^\infty_{\text{loc}}$ by using approximation by smooth psh functions: given $\varphi\in PSH(X)\cap L^\infty_{\text{loc}}$, there exists a unique positive Radon measure μ_φ on X such that for *any* sequence (φ_j) of smooth psh functions decreasing to φ, one has

$$\mu_\varphi = \lim (dd^c\varphi_j)^n,$$

where the limit holds in the weak sense. One then sets $(dd^c\varphi)^n := \mu_\varphi$.

Definition 2.4. We set

$$\mathcal{T}^\infty_\phi(\Omega) := \{\varphi\in SPSH(\Omega)\cap\mathcal{C}^\infty(\overline{\Omega}) : \varphi_{|\partial\Omega}=\phi\},$$

where $SPSH(\Omega)$ is the set of strictly psh functions, and

$$\mathcal{T}_\phi(\Omega) := \left\{\varphi\in PSH(\Omega)\cap\mathcal{C}^0(\overline{\Omega}) : \varphi_{|\partial\Omega}=\phi,\ \int_\Omega (dd^c\varphi)^n < +\infty\right\},$$

This latter class has been introduced by Cegrell in [Ceg98]; it can be used as a psh version of test functions (in the sense of distributions), as well as a building block for finite-energy classes of mildly unbounded functions.

Lemma 2.5. *Any $\varphi\in\mathcal{T}_\phi(\Omega)$ is a quasi-decreasing limit of functions in $\mathcal{T}^\infty_\phi(\Omega)$.*

Proof. Fix a local embedding $X\hookrightarrow\mathbb{C}^N$. A function $\varphi\in\mathcal{T}_\phi(\Omega)$ is the restriction of an ambient continuous psh function ψ. We use standard convolution in $\mathbb{C}^N$ to find a sequence of smooth strictly psh functions ψ_j decreasing to ψ. Consider $\varphi_j=\psi_{j|X}-\varepsilon_j$, where $0<\varepsilon_j$ goes to zero so that $\varphi_j<\phi_0$ near $\partial\Omega$ (the functions $\psi_{j|X}$ uniformly converge to φ by continuity). Set $\tilde{\varphi}_j := \tilde{\max}(\varphi_j, A_j\rho+\phi_0)$, where $\tilde{\max}$ is a regularized maximum, then $\varphi_j\in\mathcal{T}^\infty_\phi(\Omega)$ converges to φ as $A_j\to+\infty$. ☐

2.1.5 *Mildly unbounded functions.* The complex Monge–Ampère operator can be defined for mildly unbounded psh functions. We refer the reader to [Ceg04, Blo06] for the case of smooth domains in $\mathbb{C}^n$; their analysis easily extends to our context.

Definition 2.6. We let $\mathcal{F}(\Omega)$ denote the set of all functions $\varphi\in PSH(\Omega)$ which are decreasing limit of a sequence of functions $\varphi_j\in\mathcal{T}_\phi(\Omega)$ such that

$$\sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty.$$

The operator $(dd^c\cdot)^n$ is well defined on $\mathcal{F}(\Omega)$, continuous along monotonic sequences, and yields Radon measures $(dd^c\varphi)^n$ which have finite mass in Ω. We endow $\mathcal{F}(\Omega)$ with the L^1-topology. Let us stress that the operator $\varphi\mapsto(dd^c\varphi)^n$ is *not* continuous for the L^1-topology, but the class $\mathcal{F}(\Omega)$ enjoys the following useful compactness property.

Proposition 2.7. *The set $\mathcal{F}_1(\Omega)=\{\varphi\in\mathcal{F}(\Omega);\ \int_\Omega(dd^c\varphi)^n\le 1\}$ is compact.*

This is shown in [Zer09, Observation A.3] for smooth domains, and the same proof applies in our mildly singular context. Let us stress that the Monge–Ampère operator cannot be defined for all psh functions: there is, for example, no reasonable way to make sense of $(dd^c\log|z_1|)^n$. A consequence of Proposition 2.7 is that one cannot approximate such a function by a decreasing sequence of psh functions with prescribed boundary values and uniformly bounded Monge–Ampère masses.

2.2 Adapted volume form

2.2.1 *Log terminal singularities.* Let Y be a connected normal complex variety such that K_Y is $\mathbb{Q}$-Cartier near $p \in Y$. One can consider the dd^c-cohomology class of $-K_Y$, denoted by $c_1(Y)$.

Given a log-resolution $\pi : \tilde{Y} \to Y$ of (Y, p), there exists a unique $\mathbb{Q}$-divisor $\sum_i a_i E_i$ whose push-forward to Y is 0 and with

$$K_{\tilde{Y}} = \pi^*(K_Y) + \sum_i a_i E_i.$$

Definition 2.8. The coefficient $a_i \in \mathbb{Q}$ is the *discrepancy* of Y along E_j. One says that p is a *log terminal* singularity if $a_j > -1$ for all j.

It is classical that this condition is independent of the choice of resolution. In the remainder of this article we assume that:

- the singularity $(X, 0)$ is log terminal;
- $Y = \Omega$ is a strongly pseudoconvex neighborhood of $0 = p \in X$;
- the canonical bundle K_Ω is $\mathbb{Q}$-Cartier and $rK_\Omega = 0$ for some $r \in \mathbb{N}$.

Definition 2.9 [EGZ09, Definition 6.5]. Fix σ a nowhere-vanishing holomorphic section of rK_Ω, and h a smooth hermitian metric of K_Ω, then

$$\mu_p = \lambda \frac{(c_n \sigma \wedge \bar{\sigma})^{1/r}}{|\sigma|_{h^r}^{2/r}}$$

is an *adapted measure*, where $\lambda > 0$ is a positive normalizing constant.

Observe that μ_p is independent of the choice of σ, and

$$dd^c \log \mu_p = -\Theta_h(K_\Omega)$$

is the curvature of h, as follows from the Poincaré–Lelong formula.

The measure μ_p has finite mass by [EGZ09, Lemma 6.4]: let $\pi : \tilde{\Omega} \to \Omega$ be a resolution of $(\Omega, 0)$, then

$$\pi^* \mu_p = \lambda \prod_{j=1}^{M} |s_{E_j}|^{2a_j} \, dV_{\tilde{\Omega}},$$

where $dV_{\tilde{\Omega}}$ is a smooth volume form on $\tilde{\Omega}$, $E_1, \dots, E_M$ are exceptional divisors, s_{E_j} are holomorphic sections such that $E_j = (s_{E_j} = 0)$, and

$$rK_{\tilde{\Omega}} = \pi^*(rK_\Omega) + r \sum_{j=1}^{M} a_j E_j = r \sum_{j=1}^{M} a_j E_j.$$

Thus $\tilde{f} = \prod_{j=1}^{M} |s_{E_j}|^{2a_j}$ belongs to $L^s(dV_{\tilde{\Omega}})$ for some $s > 1$, as p is log terminal.

Definition 2.10. We choose $\lambda = \lambda_\Omega$ so that μ_p is a probability measure in Ω.

The results to follow are independent of this (convenient) normalization.

https://doi.org/10.1112/S0010437X24007619

2.2.2 *Ricci curvature.* Let ω be a positive closed current of bidegree $(1,1)$ in Ω with bounded local potentials. Its top power ω^n is well defined as explained in § 2.1.3. If ω^n is absolutely continuous with respect to dV_X, then we set

$$\mathrm{Ric}(\omega) := -dd^c \log \omega^n.$$

Definition 2.11. We say that ω is a Kähler–Einstein metric if it satisfies

$$\mathrm{Ric}(\omega) = \gamma\omega$$

for some $\gamma \in \mathbb{R}$.

In this article, we are mainly interested in the case when $\gamma > 0$. We choose the hermitian metric $h \equiv 1$ for K_Ω, so that $\Theta_h = 0$. Since

$$\mathrm{Ric}(\omega) = \mathrm{Ric}(\mu_p) - dd^c \log(\omega^n/\mu_p),$$

the above Kähler–Einstein equation is equivalent, writing $\omega = dd^c\varphi$, to

$$(dd^c\varphi)^n = e^{-\gamma\varphi} e^w \mu_p,$$

where w is a pluriharmonic function in Ω. Changing φ in $\varphi - w/\gamma$ and then φ in $t\varphi$ (observe that $\mathrm{Ric}(t\omega) = \mathrm{Ric}(\omega)$ for any $t > 0$), we can normalize ω by $\int_\Omega \omega^n = 1$ and reduce to

$$(dd^c\varphi)^n = \frac{e^{-\gamma\varphi}\mu_p}{\int_\Omega e^{-\gamma\varphi}\mu_p}.$$

Seeking for a Kähler–Einstein metric thus leads one to solve $(MA)_{\gamma,\phi,\Omega}$.

Conversely solving $(MA)_{\gamma,\phi,\Omega}$ will produce a Kähler–Einstein metric $\omega = dd^c\varphi$, if we can establish enough regularity of the solution φ.

2.2.3 *Log canonical threshold.* We consider the density $f = \mu_p/dV_X$. It is related to the density $\tilde{f}$ in a resolution by

$$\pi^*\mu_p = f \circ \pi \cdot \pi^* \, dV_X = \tilde{f} \, dV_{\tilde{\Omega}}.$$

An analytic expression for f is obtained as follows. Recall that $dV_X = \omega_{\mathrm{eucl}}^n \wedge [X]$, where ω_{eucl} denotes the euclidean Kähler form on $\mathbb{C}^N$. Set $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_n}$, where $1 \le i_1 < \cdots < i_n \le N$. There exists germs of holomorphic functions $f_I \in \mathcal{O}_{\Omega,0}$ such that $(dz_I)^r = f_I\sigma$ since σ is a local generator of rK_X. In particular, the volume form $dV_X := \omega_{\mathrm{eucl}}^n \wedge [\Omega]$ is comparable to $(\sum_I |f_I|^{2/r})\mu_p$, i.e.

$$\mu_p = f \, dV_X, \quad \text{with } f \sim \left(\sum_I |f_I|^{2/r}\right)^{-1}.$$

The germs of holomorphic functions f_I generate an ideal $\mathcal{I}_p^r$, where $\mathcal{I}^r$ is an ideal sheaf associated to the singularities of (X,p). In particular,

$$\pi^{-1}\mathcal{I}^r \cdot \mathcal{O}_{\tilde{\Omega}} = \mathcal{O}_{\tilde{\Omega}}\left(-r\sum_{j=1}^M b_j E_j\right)$$

for coefficients $b_j \in \mathbb{N}$ such that $f \circ \pi \sim \prod_{j=1}^M |s_{E_j}|^{-2b_j}$.

Definition 2.12. The *log canonical threshold* of (X,p) is given by

$$\mathrm{lct}(X,p) := \inf_{j \in 1,\dots,M} \frac{a_j + 1}{b_j}.$$

https://doi.org/10.1112/S0010437X24007619

We let the reader check that the definition is independent of the choice of resolution, and that $\mathrm{lct}(X,\mathcal{I})\in(0,n]$. One can equivalently use the following point of view: if $\mathcal{I}$ is a general ideal sheaf,

$$\mathrm{lct}(X,\mathcal{I}) := \inf_{E/X} \frac{A_X(E)}{\mathrm{ord}_E(\mathcal{I})} \tag{2.1}$$

where $A_X(E) := 1 + \mathrm{ord}_E(K_{Y/X})$ is the log-discrepancy of E, and the infimum is over all prime divisors E on resolutions Y of X. When $\mathcal{I}$ is supported at p we can restrict in (2.1) to consider prime divisors centered at p.

Example 2.13. The ordinary double point (ODP) $X=\{z\in\mathbb{C}^{n+1},\ \sum_{j=0}^n z_j^2=0\}$ is the simplest isolated log terminal singularity which is not a quotient singularity when $n\geq 3$ (when $n=2$, log terminal singularities are precisely the singularities of the form $X=\mathbb{C}^2/G$, $G\subset GL(2,\mathbb{C})$ a finite subgroup).

In this case $\mathcal{I}^2=(z_1^2,\dots,z_n^2)$. Indeed the n-forms

$$\sigma_j := \frac{(dz_0\wedge\cdots\wedge\widehat{dz_j}\wedge\cdots\wedge dz_n)^2}{z_j^2} = -\frac{(dz_0\wedge\cdots\wedge\widehat{dz_j}\wedge\cdots\wedge dz_n)^2}{\sum_{k\neq j} z_k^2},$$

defined on $U_j := \{z_j\neq 0\}$, glue together to give a local generator σ of $2K_X$ (note that $\sum_{j=0}^n z_j\, dz_j = 0$). In particular, $|f_I|^{2/r}=|z_j|^2$ where $j=[0,n]\setminus I$, $r=2$ and

$$\mu_p \sim \frac{1}{\sum_{j=0}^n |z_j|^2}\, dV_X.$$

If $\pi:\mathrm{Bl}_0\mathbb{C}^{n+1}\to\mathbb{C}^{n+1}$ denotes the blow-up at 0, E the exceptional divisor, and F the restriction of E to Y, the strict transform of X, we obtain

$$\pi^{-1}\mathcal{I}^2\cdot\mathcal{O}_Y=\mathcal{O}_Y(-2F) \quad\text{and}\quad \pi^*\mu_p=|s_F|^{2(n-2)}\, dV_Y$$

for a smooth volume form dV_Y. Thus, $\mathrm{lct}(X,p)=\mathrm{lct}(X,\mathcal{I})=n-1$.

We will need the following result which connects $\mathrm{lct}(X,p)$ and the integrability properties of the density $f=\mu_p/dV_X$.

Lemma 2.14. *The density $f=\mu_p/dV_X$ belongs to $L^r(dV_X)$ for $r<1+\mathrm{lct}(X,p)$.*

Proof. Let $\pi:\tilde{\Omega}\to\Omega$ be a resolution of the singularity. Recall that

$$f\circ\pi\sim\prod_{j=1}^M |s_{E_j}|^{-2b_j} \quad\text{and}\quad \tilde{f}=\prod_{j=1}^M |s_{E_j}|^{2a_j}, \quad\text{hence}\quad \pi^*dV_X\sim\prod_{j=1}^M |s_{E_j}|^{2(a_j+b_j)}\, dV_{\tilde{\Omega}}.$$

It follows that $\int_\Omega f^r\, dV_X\sim\int_{\tilde{\Omega}}\prod_{j=1}^M |s_{E_j}|^{2(a_j+b_j)-2rb_j}\, dV_{\tilde{\Omega}}<+\infty$ if and only if $r<(1+a_j+b_j)/b_j$ for all j, which yields the statement since $\mathrm{lct}(X,p)=\inf_j((1+a_j)/b_j)$. □

2.3 Normalized volume

The *(Hilbert–Samuel) multiplicity* of an ideal $\mathcal{I}$ supported at p is defined as

$$\mathrm{e}(X,\mathcal{I}) := \lim_{m\to+\infty}\frac{l(\mathcal{O}_{X,p}/\mathcal{I}^m)}{m^n/n!}$$

where l denotes the length of an Artinian module.

Given a divisor E over X centered at p, the volume of E over $p\in X$ is

$$\mathrm{vol}_{X,p}(E) := \lim_{m\to+\infty}\frac{l(\mathcal{O}_{X,p}/\mathfrak{a}_m(E))}{m^n/n!}$$

where $\mathfrak{a}_m(E) := \{f\in\mathcal{O}_{X,p} : \mathrm{ord}_E(f)\geq m\}$ (see [ELS03]).

https://doi.org/10.1112/S0010437X24007619

Definition 2.15 [Li18]. The *normalized volume of* $p \in X$ is

$$\widehat{\mathrm{vol}}(X,p) := \inf_{E/X} \widehat{\mathrm{vol}}_{X,p}(E),$$

where the infimum runs over all prime divisors E over X centered at p, and

$$\widehat{\mathrm{vol}}_{X,p}(E) := A_X(E)^n \cdot \mathrm{vol}_{X,p}(E)$$

is the *normalized volume of* E *over* $(x \in X)$.

We shall need the following important result.

Theorem 2.16 [Liu18, Theorem 27]. *Let* (X,p) *be a log terminal singularity of complex dimension* $\dim_{\mathbb{C}} X = n$. *Then*

$$\widehat{\mathrm{vol}}(X,p) = \inf_{\mathcal{I}\text{ supported at }p} \mathrm{lct}(X,\mathcal{I})^n \cdot \mathrm{e}(X,\mathcal{I}).$$

Observe that the quantity $\mathrm{lct}(X,\mathcal{I})^n \cdot \mathrm{e}(X,\mathcal{I})$ is invariant under rescaling $\mathcal{I} \to \mathcal{I}^r$, $r \in \mathbb{N}$. One can actually only consider coherent ideal sheaves supported at p. Indeed any ideal $\mathcal{I}$ supported at p is associated to a closed subscheme Z such that $\mathrm{Supp}\, Z = \{p\}$ (see [Har77, Corollary II.5.10]), while any ideal associated to a closed subscheme is coherent [Har77, Proposition II.5.9].

Example 2.17. Consider again $X = \{z \in \mathbb{C}^{n+1},\ \sum_{j=0}^n z_j^2 = 0\}$. Recall that $\mathcal{I}^2 = (z_1^2, \ldots, z_n^2)$ is the ideal sheaf associated to the adapted measure, and that the ideal $\mathcal{I}^2$ corresponds to $2F$ where F is the exceptional divisor in the blow-up at p. In particular, $A_X(F) = n-1$.

We observe here that $e(X,\mathcal{I}^2) = 2^{n+1}$ and $\widehat{\mathrm{vol}}_{X,p}(F) = 2(n-1)^n$ since

$$l(\mathcal{O}_{X,p}/\mathcal{I}^{2m}) = l(\mathcal{O}_{X,p}/\mathfrak{a}_{2m}(F)) = 2^{n+1}\frac{m^n}{n!} + O(m^{n-1}).$$

In [Li18, Example 5.3] it is further shown that F is a minimizer for the normalized volume of $p \in X$, i.e. that $\widehat{\mathrm{vol}}(X,p) = 2(n-1)^n$.

3. A variational approach

A variational approach for solving degenerate complex Monge–Ampère equations has been developed in [BBGZ13] in the context of compact Kähler manifolds. It notably applies to the construction of singular Kähler–Einstein metrics of non-positive curvature. This has been partially adapted to smooth pseudoconvex domains of $\mathbb{C}^n$ in [ACC12].

The case of positive curvature is notoriously more difficult, as illustrated by the resolution of the Yau–Tian–Donaldson conjecture by Chen, Donaldson and Sun [CDS15]. It has been treated extensively in [BBEGZ19], and eventually lead to an alternative solution of the Yau–Tian–Donaldson conjecture for Fano varieties [BBJ21, LTW21, Li22]. Adapting [BBEGZ19] to our local singular context, we develop in this section a variational approach for solving the equation

$$(MA)_{\gamma,\phi,\Omega} \begin{cases} (dd^c\varphi)^n = \dfrac{e^{-\gamma\varphi}\, d\mu_p}{\int_\Omega e^{-\gamma\varphi}\, d\mu_p}, \\ \varphi_{|\partial\Omega} = \phi. \end{cases} \tag{3.1}$$

3.1 Monge–Ampère energy

3.1.1 *Smooth tests.* Fix $\Omega = \{\rho < 0\}$ and ϕ as described previously, and

$$\mathcal{T}_\phi^\infty(\Omega) = \{\varphi \in SPSH(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega}) : \varphi_{|\partial\Omega} \equiv \phi\}.$$

Recall that $\phi_0 \in \mathcal{C}^\infty(\bar\Omega) \cap PSH(\Omega)$ denotes a smooth psh extension of ϕ to $\bar\Omega$. We set $\omega := dd^c\phi_0$. This is a semi-positive form, which can be assumed to be Kähler. However, if $\phi \equiv 0$, we can equally well take $\phi_0 \equiv 0$ and get $\omega \equiv 0$.

DEFINITION 3.1. We call $E_\phi(\varphi) := (1/(n+1)) \sum_{j=0}^n \int_\Omega (\varphi - \phi_0)(dd^c\varphi)^j \wedge (dd^c\phi_0)^{n-j}$ the ϕ-relative Monge–Ampère energy of $\varphi \in \mathcal{T}_\phi^\infty(\Omega)$.

While the formula depends on the choice of ϕ_0, it follows from Lemma 3.2 that the difference of two such relative energies is constant:

$$E_{\phi_1}(\varphi) - E_{\phi_0}(\varphi) = E_{\phi_1}(\phi_0).$$

For $\phi_0 = 0$, the formula reduces to $E(\varphi) := E_0(\varphi) = (1/(n+1)) \int_\Omega \varphi (dd^c\varphi)^n$.

This definition is motivated by the fact the E_ϕ is a primitive of the Monge–Ampère operator for smooth psh functions with ϕ-boundary values.

LEMMA 3.2. *Fix $\varphi \in \mathcal{T}_\phi^\infty(\Omega)$, $v \in \mathcal{D}(\Omega)$. Then $\varphi + tv \in \mathcal{T}_\phi^\infty(\Omega)$ for t small, and*

$$\frac{d}{dt}_{|t=0} E_\phi(\varphi + tv) = \int_\Omega v (dd^c\varphi)^n.$$

In particular, $\varphi \mapsto E_\phi(\varphi)$ is increasing.

Here $\mathcal{D}(\Omega)$ denotes the space of smooth functions with compact support in Ω.

Proof. Fix $\varphi \in \mathcal{T}_\phi^\infty(\Omega)$ and $v \in \mathcal{D}(\Omega)$. Since v is smooth with compact support, the function $\pm v + C\rho$ is psh for $C > 0$ large enough, while $\varphi - \varepsilon\rho$ is psh for $\varepsilon > 0$ small enough. It follows that $\varphi + tv$ is psh for t small enough.

Set $\omega = dd^c\phi_0$. The function $\psi_t = \varphi - \phi_0 + tv$ has zero boundary values, and

$$E_\phi(\varphi + tv) = \frac{1}{n+1} \sum_{j=0}^n \int_\Omega \psi_t (\omega + dd^c\psi_t)^j \wedge \omega^{n-j}.$$

It follows from Stokes theorem, as all functions involved in the integration by parts are identically zero on $\partial\Omega$, that

$$\begin{aligned}
&(n+1)\frac{d}{dt} E_\phi(\varphi + tv) \\
&= \sum_{j=0}^n \int_\Omega \dot\psi_t (\omega + dd^c\psi_t)^j \wedge \omega^{n-j} + \sum_{j=1}^n \int_\Omega j\psi_t \, dd^c\dot\psi_t \wedge (\omega + dd^c\psi_t)^{j-1} \wedge \omega^{n-j} \\
&= \sum_{j=0}^n \int_\Omega \dot\psi_t (\omega + dd^c\psi_t)^j \wedge \omega^{n-j} + \sum_{j=1}^n \int_\Omega j\dot\psi_t \, dd^c\psi_t \wedge (\omega + dd^c\psi_t)^{j-1} \wedge \omega^{n-j} \\
&= \sum_{j=0}^n \int_\Omega (j+1)\dot\psi_t (\omega + dd^c\psi_t)^j \wedge \omega^{n-j} - \sum_{j=1}^n \int_\Omega j\dot\psi_t (\omega + dd^c\psi_t)^{j-1} \wedge \omega^{n-j+1} \\
&= (n+1) \int_\Omega \dot\psi_t (\omega + dd^c\psi_t)^n,
\end{aligned}$$

writing $dd^c\psi_t = (\omega + dd^c\psi_t) - \omega$ in the third line, and then distributing and relabelling so as to obtain a telescopic series. The formula follows for $t = 0$.

In short, the derivative of E_ϕ is the complex Monge–Ampère operator $(dd^c\varphi)^n$ which is a positive measure. It follows that $\varphi \mapsto E_\phi(\varphi)$ is increasing. □

3.1.2 *Continuous setting.* The previous result extends to the case of continuous psh functions that are not necessarily strictly psh. Recall that

$$\mathcal{T}_\phi(\Omega) := \left\{ \varphi \in PSH(\Omega) \cap C^0(\bar{\Omega}),\ \varphi_{|\partial\Omega} = \phi \text{ and } \int_\Omega (dd^c\varphi)^n < +\infty \right\}.$$

We would like to extend Lemma 3.2 to this less-regular setting. As $\varphi + tv$ is not necessarily psh, we need to project it onto the cone of all psh functions. The following result will thus be useful.

Lemma 3.3. *Fix $\varphi \in \mathcal{T}_\phi(\Omega)$ and $f \in \mathcal{D}(\Omega)$. Then $P(\varphi + f) \in \mathcal{T}_\phi(\Omega)$ where*

$$P(\varphi + f) := \sup\{\psi \in PSH(\Omega),\ \psi \le \varphi + f\}.$$

Moreover, $(dd^c P(\varphi + f))^n$ is supported on the contact set $\{P(\varphi + f) = \varphi + f\}$.

Proof. Since $\varphi + f$ is bounded and continuous, it is classical to check that the envelope $P(\varphi + f)$ is a well-defined psh function. As f has compact support, one moreover checks that $P(\varphi + f)$ is continuous on $\partial\Omega$ with $P(\varphi + f)_{|\partial\Omega} = \varphi_{|\partial\Omega} = \phi$.

Solving Dirichlet problems in small 'balls' not containing the singular point, it follows from a balayage argument that the Monge–Ampère measure of the envelope $(dd^c P(\varphi + f))^n$ is supported on the contact set $\{P(\varphi + f) = \varphi + f\}$. □

We extend $E_\phi(\cdot)$ to $\mathcal{T}_\phi(\Omega)$ by monotonicity, setting

$$E_\phi(\varphi) := \inf\{E_\phi(\psi),\ \psi \in \mathcal{T}_\phi^\infty(\Omega) \text{ and } \varphi \le \psi\}.$$

It has been observed by Berman and Boucksom (in the setting of compact Kähler manifolds [BB10]) that $E_\phi \circ P$ is still differentiable, with $(E_\phi \circ P)' = E_\phi' \circ P$. This result extends to our local singular setting.

Proposition 3.4. *Fix $\varphi \in \mathcal{T}_\phi(\Omega)$ and $f \in \mathcal{D}(\Omega)$. Then $t \to E_\phi(P(\varphi + tf))$ is differentiable and*

$$\frac{d}{dt}_{|t=0} E_\phi(P(\varphi + tf)) = \int_\Omega f (dd^c\varphi)^n.$$

Proof. The proof is very similar to that in the compact case, we provide it as a courtesy to the reader. Set $\varphi_t := P(\varphi + tf)$. By Lemma 3.5 we have

$$\int_\Omega (\varphi_t - \varphi)(dd^c\varphi_t)^n \le E_\phi(\varphi_t) - E_\phi(\varphi) \le \int_\Omega (\varphi_t - \varphi)(dd^c\varphi)^n. \tag{3.2}$$

Since $\varphi_t - \varphi \le tf$, the second inequality yields

$$\limsup_{t \to 0^+} \frac{E_\phi(\varphi_t) - E_\phi(\varphi)}{t} \le \int_X f (dd^c\varphi)^n,$$

and $\liminf_{t \to 0^-} ((E_\phi(\varphi_t) - E_\phi(\varphi))/t) \ge \int_X f (dd^c\varphi)^n$.

It follows from Lemma 3.3 that $(dd^c\varphi_t)^n$ is supported on $\{\varphi_t = \varphi + tf\}$, hence the first inequality in (3.2) yields

$$\int_\Omega \frac{\varphi_t - \varphi}{t} (dd^c\varphi_t)^n = \int_\Omega f (dd^c\varphi_t)^n.$$

Now $(dd^c\varphi_t)^n \to (dd^c\varphi)^n$ weakly since $\varphi_t \to \varphi$ uniformly, therefore

$$\liminf_{t \to 0^+} \frac{E_\phi(\varphi_t) - E_\phi(\varphi)}{t} \ge \liminf_{t \to 0^+} \int_\Omega f (dd^c\varphi_t)^n = \int_\Omega f (dd^c\varphi)^n,$$

https://doi.org/10.1112/S0010437X24007619

and

$$\limsup_{t\to 0^-}\frac{E_\phi(\varphi_t)-E_\phi(\varphi)}{t}\le \limsup_{t\to 0^-}\int_\Omega f(dd^c\varphi_t)^n=\int_\Omega f(dd^c\varphi)^n. \qquad \square$$

Lemma 3.5. *For any $\varphi_1,\varphi_2\in\mathcal{T}_\phi(\Omega)$,*

$$\int_\Omega(\varphi_1-\varphi_2)(dd^c\varphi_1)^n\le E_\phi(\varphi_1)-E_\phi(\varphi_2)\le\int_\Omega(\varphi_1-\varphi_2)(dd^c\varphi_2)^n, \tag{3.3}$$

while if $\varphi_1\le\varphi_2$, then

$$E_\phi(\varphi_1)-E_\phi(\varphi_2)\le\frac{1}{n+1}\int_X(\varphi_1-\varphi_2)(dd^c\varphi_1)^n. \tag{3.4}$$

The energy is continuous along decreasing sequence in $\mathcal{T}_\phi(\Omega)$.

Proof. It follows from Stokes theorem that

$$E_\phi(\varphi_1)-E_\phi(\varphi_2)=\frac{1}{n+1}\sum_{j=0}^n\int_\Omega(\varphi_1-\varphi_2)(dd^c\varphi_1)^j\wedge(dd^c\varphi_2)^{n-j}$$

and

$$\int_\Omega(\varphi_1-\varphi_2)(dd^c\varphi_1)^{j+1}\wedge(dd^c\varphi_2)^{n-j-1}\le\int_\Omega(\varphi_1-\varphi_2)(dd^c\varphi_1)^j\wedge(dd^c\varphi_2)^{n-j},$$

for any $j=0,\dots,n-1$. The desired inequalities follow.

Let $\varphi_j\in\mathcal{T}_\phi(\Omega)$ be a decreasing sequence converging to $\varphi\in\mathcal{T}_\phi(\Omega)$. We obtain

$$0\le E_\phi(\varphi_j)-E_\phi(\varphi)\le\int_\Omega(\varphi_j-\varphi)(dd^c\varphi)^n\to 0$$

as $j\to+\infty$ by the monotone convergence theorem. $\square$

3.1.3 *Finite energy class.* Let $PSH_\phi(\Omega)$ denote the set of decreasing limits of functions in $\mathcal{T}_\phi(\Omega)$. We extend E_ϕ to $PSH_\phi(\Omega)$ by monotonicity, setting

$$E_\phi(\varphi):=\inf\{E_\phi(\psi),\ \psi\in\mathcal{T}_\phi(\Omega)\text{ and }\varphi\le\psi\}.$$

Definition 3.6. We set $\mathcal{E}^1(\Omega):=\{\varphi\in PSH_\phi(\Omega);\ E_\phi(\varphi)>-\infty\}$.

This ‘finite energy class’ has been introduced and studied intensively by Cegrell for smooth domains of $\mathbb{C}^n$. His analysis extends to our mildly singular context. We summarize here the key facts that we shall need.

Theorem 3.7 (Cegrell). *The complex Monge–Ampère operator $(dd^c\cdot)^n$ and the energy E_ϕ are well defined on the class $\mathcal{E}^1(\Omega)$. Moreover:*

- *functions in $\mathcal{E}^1(\Omega)$ have zero Lelong numbers;*
- *the sets $\mathcal{G}_b(\Omega)=\{\varphi\in\mathcal{E}^1(\Omega),\ -b\le E_\phi(\varphi)\}$ are compact for all $b\in\mathbb{R}$;*
- *Lemma 3.5 holds if $\varphi_1,\varphi_2\in\mathcal{E}^1(\Omega)$;*
- *if μ is a non-pluripolar probability measure such that $\mathcal{E}^1(\Omega)\subset L^1(\mu)$, then there exists a unique function $v\in\mathcal{E}^1(\Omega)\cap\mathcal{F}_1(\Omega)$ such that $\mu=(dd^cv)^n$.*

We refer the reader to [Ceg98, Theorems 3.8, 7.2 and 8.2] for the proof of these results when Ω is smooth.

https://doi.org/10.1112/S0010437X24007619

3.2 Ding functional

3.2.1 *Euler–Lagrange equation.* The Ding functional is

$$F_\gamma(\varphi) := E_\phi(\varphi) + \frac{1}{\gamma} \log \int_\Omega e^{-\gamma\varphi}\, d\mu_p.$$

PROPOSITION 3.8. *If φ maximizes F_γ over $\mathcal{T}_\phi(\Omega)$, then φ solves the complex Monge–Ampère equation (3.1).*

Proof. Assume that φ maximizes F_γ over $\mathcal{T}_\phi(\Omega)$, fix $f \in \mathcal{D}(\Omega)$, and set $\varphi_t := P(\varphi + tf)$. Then

$$E_\phi(\varphi_t) + \frac{1}{\gamma} \log \int_\Omega e^{-\gamma(\varphi+tf)}\, d\mu_p \le F_\gamma(\varphi_t) \le F_\gamma(\varphi),$$

i.e. the function $t \to E_\phi(\varphi_t) + (1/\gamma) \log \int_\Omega e^{-\gamma(\varphi+tf)}\, d\mu_p$ reaches its maximum at $t = 0$. Combining Proposition 3.4 and Lemma 3.9, we obtain

$$0 = \frac{d}{dt}\left(E_\phi(\varphi_t) + \frac{1}{\gamma} \log \int_\Omega e^{-\gamma(\varphi+tf)}\, d\mu_p \right) = \int_\Omega f \left((dd^c\varphi)^n - \frac{e^{-\gamma\varphi}\, d\mu_p}{\int_\Omega e^{-\gamma\varphi}\, d\mu_p} \right),$$

i.e. φ solves (3.1). □

LEMMA 3.9. *Fix $\varphi \in \mathcal{T}_\phi(\Omega)$, $f \in \mathcal{D}(\Omega)$, and set $\psi_t := \varphi + tf$. Then*

$$\frac{d}{dt}\left(\log \int_\Omega e^{-\gamma\psi_t}\, d\mu_p \right)_{|t=0} = -\gamma \frac{\int_\Omega f e^{-\gamma\varphi}\, d\mu_p}{\int_\Omega e^{-\gamma\varphi}\, d\mu_p}.$$

Proof. By the chain rule, it is enough to observe that

$$\frac{\int_\Omega e^{-\gamma\psi_t}\, d\mu_p - \int_\Omega e^{-\gamma\varphi}\, d\mu_p}{t} = -\int_\Omega e^{-\gamma\varphi} \left(\frac{1 - e^{-t\gamma f}}{t} \right) d\mu_p$$

and to apply the Lebesgue dominated convergence theorem to conclude. □

3.2.2 *Coercivity.* In order to solve (3.1), one is lead to try and maximize F_γ. We show in § 6 that when F_γ is *coercive*, the complex Monge–Ampère equation (3.1) admits a solution $\varphi \in \mathcal{T}_\phi(\Omega)$ which is smooth away from p.

DEFINITION 3.10. The functional F_γ is coercive if there exists $A, B > 0$ such that

$$F_\gamma(\varphi) \le A E_\phi(\varphi) + B$$

for all $\varphi \in \mathcal{T}_\phi(\Omega)$.

We observe in Lemma 3.13 that $E_\phi(\varphi) \le C(\phi_0)$ is bounded from above, uniformly in $\varphi \in \mathcal{T}_\phi(\Omega)$. In particular, if F_γ is coercive with *slope* $A > 0$, then it is coercive for any $A' \in (0, A]$. We can thus assume, without loss of generality, that $A \in (0, 1)$. The coercivity property is then equivalent to

$$\frac{1}{\gamma} \log \int_\Omega e^{-\gamma\varphi}\, d\mu_p \le (1 - A)(-E_\phi(\varphi)) + B,$$

or, equivalently, to the following Moser–Trudinger inequality

$$\left(\int_X e^{-\gamma\varphi}\, d\mu_p \right)^{1/\gamma} \le C e^{(1-A)(-E_\phi(\varphi))}.$$

We summarize these observations in the following.

Proposition 3.11. *Fix $\gamma > 0$. The following properties are equivalent:*

(i) *F_γ is coercive;*

(ii) *there exists $C_\gamma > 0$ and $a \in (0,1)$ such that for all $\varphi \in \mathcal{T}_\phi(\Omega)$,*

$$\left(\int_\Omega e^{-\gamma\varphi}\, d\mu_p \right)^{1/\gamma} \leq C_\gamma e^{-aE_\phi(\varphi)}.$$

It follows from Hölder inequality (and the normalization $\mu_p(\Omega) = 1$) that if

$$\left(\int_\Omega e^{-\gamma\varphi}\, d\mu_p \right)^{1/\gamma} \leq C e^{-E_\phi(\varphi)} \tag{3.5}$$

holds for $\gamma > 0$, then it also holds for $\gamma' < \gamma$. We thus introduce the following critical exponent.

Definition 3.12. We set

$$\gamma_{\mathrm{crit}}(X,p) := \sup\{\gamma > 0,\ (3.5) \text{ holds for all } \varphi \in \mathcal{T}_\phi(\Omega)\}.$$

Lemma 3.13. *The functional E_ϕ is bounded from above on $\mathcal{T}_\phi(\Omega)$. Moreover:*

– *if F_γ is coercive, then $\gamma \leq \gamma_{\mathrm{crit}}(X,p)$;*
– *conversely, if $\gamma < \gamma_{\mathrm{crit}}(X,p)$, then F_γ is coercive.*

Proof. Consider $\tilde{\phi}_0 = P(\phi) := \sup\{\psi, \psi \in \mathcal{T}_\phi(\Omega)\}$. This is the largest psh function in Ω such that $\tilde{\phi}_0 = \phi$ on $\partial\Omega$. The reader can check that it is continuous on $\overline{\Omega}$ and satisfies $(dd^c\tilde{\phi}_0)^n = 0$ in Ω. If $\varphi \in \mathcal{T}_\phi(\Omega)$, then $\varphi \leq \tilde{\phi}_0$, hence $E_{\tilde{\phi}_0}(\varphi) \leq 0$. Thus,

$$E_{\phi_0}(\varphi) = E_{\tilde{\phi}_0}(\varphi) + E_{\phi_0}(\tilde{\phi}_0) \leq E_{\phi_0}(\tilde{\phi}_0),$$

hence $E_{\phi_0}(\varphi)$ is uniformly bounded from above independently of the choice of ϕ_0.

Similarly, the coercivity of F_γ or the inequality (3.5) do not depend on the choice of ϕ_0. In the remainder of this proof we thus assume that $\phi_0 = P(\phi)$. Since $E_\phi(\varphi) \leq 0$ in this case, it follows from Proposition 3.11 that if F_γ is coercive, then (3.5) holds, hence $\gamma \leq \gamma_{\mathrm{crit}}(X,p)$.

Conversely, assume $\gamma < \gamma_{\mathrm{crit}}(X,p)$. Fix $\gamma < \gamma' < \gamma_{\mathrm{crit}}(X,p)$ and $\lambda = \gamma/\gamma' < 1$. We can assume that λ is close to 1. We assume first that $\phi \equiv 0$. For $\varphi \in \mathcal{T}_0(\Omega)$ we observe that $\lambda\varphi \in \mathcal{T}_0(\Omega)$, with $E_0(\lambda\varphi) = \lambda^{n+1}E_0(\varphi)$. The Moser–Trudinger (3.5) applied to $(\gamma', \lambda\varphi)$ thus yields

$$\left(\int_\Omega e^{-\gamma\varphi}\, d\mu_p \right)^{1/\gamma} \leq C_{\gamma'} e^{-\lambda^n E_0(\varphi)},$$

so that F_γ is coercive.

We now treat the general case, replacing the condition $\phi \equiv 0$ by $(dd^c\phi_0)^n \equiv 0$. For $\varphi \in \mathcal{T}_\phi(\Omega)$ we observe that $\varphi_\lambda = \lambda\varphi + (1-\lambda)\phi_0 \in \mathcal{T}_\phi(\Omega)$, with $\varphi_\lambda - \phi_0 = \lambda(\varphi - \phi_0) \leq 0$ and

$$\begin{aligned}(n+1)E_{\phi_0}(\varphi_\lambda) &= \lambda \sum_{j=0}^{n-1} \int_\Omega (\varphi - \phi_0)(dd^c\varphi_\lambda)^j \wedge (dd^c\phi_0)^{n-j} \\ &= \lambda \sum_{k=1}^{n} \sum_{j=k}^{n} \binom{j}{k} \lambda^k (1-\lambda)^{j-k} \int_\Omega (\varphi - \phi_0)(dd^c\varphi)^k \wedge (dd^c\phi_0)^{n-k}.\end{aligned}$$

Now $\sum_{j=k}^n \binom{j}{k}\lambda^k(1-\lambda)^{j-k} \leq a < 1$ for all $1 \leq k \leq n$, since $\lambda < 1$ can be chosen arbitrarily close to 1. Thus, $E_{\phi_0}(\varphi_\lambda) \geq a\lambda E_{\phi_0}(\varphi)$ and the result follows as previously by applying the Moser–Trudinger inequality (3.5) to φ_λ. □

https://doi.org/10.1112/S0010437X24007619

3.3 Invariance

We give here some evidence that the critical exponent $\gamma_{\text{crit}}(X, p)$ should be independent of the domain Ω and the boundary values ϕ.

3.3.1 *Enlarging the domain.* We first reduce to the case of zero boundary values.

Proposition 3.14. *Let Ω_2 be a smooth strongly pseudoconvex domain containing $\overline{\Omega}$. If the Moser–Trudinger inequality holds for $(\gamma, \Omega_2, 0)$, then it holds for (γ, Ω, ϕ).*

Proof. Consider indeed $\varphi \in \mathcal{T}_\phi(\Omega)$ and set

$$\varphi_2 := \sup\{u \in \mathcal{T}_0(\Omega_2),\ \text{such that } u \leq \varphi \text{ in } \Omega\}.$$

The family $\mathcal{F}$ of such functions is non-empty, as it contains $A\rho_2$ for some large $A > 1$, where ρ_2 is a psh defining function for Ω_2. Moreover, $\mathcal{F}$ is uniformly bounded from above by 0, so the upper envelope φ_2 is well defined and psh, as $\mathcal{F}$ is compact. Finally, $\varphi_2 \geq A\rho_2$, hence φ_2 has zero boundary values, and φ_2 is lower semi-continuous, as an envelope of continuous functions, thus $\varphi_2 \in \mathcal{T}_0(\Omega_2)$.

Since $\varphi_2 \leq \varphi$ in Ω, we observe that

$$\int_\Omega e^{-\gamma\varphi}\, d\mu_p \leq \int_{\Omega_2} e^{-\gamma\varphi_2}\, d\mu_p.$$

Our claim will follow if we can show that, on the other hand, $E_0(\varphi_2) \geq E_\phi(\varphi)$.

If φ is smooth one can show, by adapting standard techniques, that:

- φ_2 is $\mathcal{C}^{1,\bar{1}}$-smooth in $\Omega_2 \setminus \{p\}$;
- $(dd^c\varphi_2)^n = 0$ in $\Omega_2 \setminus \Omega$ and $(dd^c\varphi_2)^n = \mathbf{1}_{\{\varphi_2=\varphi\}}(dd^c\varphi)^n$ in $\overline{\Omega}$.

Assuming $\phi \geq 0$ and $\phi_0 = \sup\{\psi,\ \psi \in \mathcal{T}_\phi(\Omega)\}$, we infer

$$\begin{aligned} E_0(\varphi_2) &= \frac{1}{n+1}\int_\Omega \varphi_2(dd^c\varphi_2)^n = \frac{1}{n+1}\int_\Omega \mathbf{1}_{\{\varphi_2=\varphi\}}\varphi(dd^c\varphi)^n \\ &\geq \frac{1}{n+1}\int_\Omega \varphi(dd^c\varphi)^n \geq \frac{1}{n+1}\int_\Omega (\varphi-\phi)(dd^c\varphi)^n \geq E_\phi(\varphi). \end{aligned}$$

To get rid of the assumption $\phi \geq 0$, we observe that the Moser–Trudinger inequality holds for given boundary data ϕ if and only if does so for $\phi + c$, for any $c \in \mathbb{R}$ (by changing φ in $\varphi + c$).

Using Lemma 2.5, one can uniformly approximate φ by a sequence of smooth $\varphi_j \in \mathcal{T}_\phi(\Omega)$. The corresponding sequence $\varphi_{2,j}$ uniformly converges to φ_2, and we obtain the desired inequality by passing to the limit in $E_0(\varphi_{2,j}) \geq E_\phi(\varphi_j)$. □

3.3.2 *Rescaling.* We now assume that $\phi = 0$ and reformulate the coercivity property after an appropriate rescaling. Observe that for any $\lambda > 0$, the map

$$\varphi \in \mathcal{T}_0(\Omega) \mapsto \lambda\varphi \in \mathcal{T}_0(\Omega)$$

is a homeomorphism. This allows us to reformulate the Moser–Trudinger inequality.

Proposition 3.15. *The following statements are equivalent:*

(a) *F_γ is coercive;*
(b) *there exists $C > 0$, $B \in (0, 1)$ such that for all $\varphi \in \mathcal{T}_0(\Omega)$, $\int_\Omega e^{-\varphi}\, d\mu_p \leq Ce^{-(B/\gamma^n)E_0(\varphi)}$.*

In particular, we can define the following critical exponent.

https://doi.org/10.1112/S0010437X24007619

Definition 3.16. We set

$$\beta_{\text{crit}} := \inf\left\{\beta > 0;\ \sup_{\varphi\in\mathcal{T}_0(\Omega)}\left(\int_\Omega e^{-\varphi}\, d\mu_p / e^{-\beta E_0(\varphi)}\right) < +\infty\right\}.$$

Note that $\gamma_{\text{crit}}^n = 1/\beta_{\text{crit}}$, hence it follows from the previous analysis that F_γ is coercive if and only if $\gamma < \beta_{\text{crit}}^{-1/n}$. When $p\in X$ is smooth, it has been shown in [GKY13, Theorem 9] and independently [BB22, Theorem 1.5] that

$$\beta_{\text{crit}}(\Omega) = \frac{1}{(n+1)^n},$$

or, equivalently, that $\gamma_{\text{crit}}(\Omega) = n+1$. In particular, it does not depend on Ω.

We extend this independence to the case when p is the vertex of a cone over a Fano manifold.

Proposition 3.17. *Assume that (X,p) is the affine cone over a Fano manifold Z embedded in a projective space by the linear system $|-rK_Z|$ for $r\in\mathbb{N}$ such that $L = rK_Z^*$ is very ample, and fix $\lambda\in\mathbb{C}^*$. The Moser–Trudinger inequality holds for $(\gamma,\Omega,0)$ if and only if it does so for $(\gamma,\lambda\Omega,0)$.*

Proof. Let $L = rK_Z^*$, let D_λ denote the dilatation $z\mapsto\lambda z$ and set $\Omega_\lambda = D_\lambda(\Omega)$. We blow up p to obtain a resolution $f: Y\to X$, where Y is the total space of L^* and the exceptional divisor E is the zero section of L^*.

Recall that $K_Y = f^*K_X + aE$, where a is the discrepancy of Y along E. The adjunction formula yields $(K_Y+E)_{|E} = K_E$, hence $K_Z^* = (a+1)L$. In particular, $a = -1+1/r$ and (X,p) is log terminal. The fibration $\pi: Y = L^*\to Z$ yields $K_Y = \pi^*(K_Z+L)$, hence $f^*K_X = \pi^*(K_Z+L) - aE$.

Since $\pi^*(K_Z+L)$ is $\mathbb{C}^*$-invariant, we can cook up an adapted volume form $\mu_p = \mu_1\cdot\mu_E$ with $D_\lambda^*\mu_1 = \mu_1$ while $D_\lambda^*\mu_E = |\lambda|^{2a}\mu_E$. For $\varphi\in\mathcal{T}_0(\Omega_\lambda)$ we set $\varphi_\lambda = \varphi\circ D_\lambda\in\mathcal{T}_0(\Omega)$ and observe that

$$|\lambda|^{2a}\int_\Omega e^{-\gamma\varphi_\lambda}\, d\mu_p = \int_{\Omega_\lambda} e^{-\gamma\varphi}\, d\mu_p,$$

while $E_{\Omega,0}(\varphi_\lambda) = E_{\Omega_\lambda,0}(\varphi)$. The conclusion follows. □

We conjecture in § 5 that $\gamma_{\text{crit}}(X,p) \stackrel{?}{=} ((n+1)/n)\widehat{\text{vol}}(X,p)^{1/n}$ and give partial results towards this equality, which again suggest that $\gamma_{\text{crit}}(X,p)$ should be independent of (Ω,ϕ). In the whole article, we therefore use the notation $\gamma_{\text{crit}}(X,p)$ instead of the more precise, and heavy, $\gamma_{\text{crit}}(X,p,\Omega,\phi)$.

4. Upper bound for the coercivity

The purpose of this section is to establish the following upper bound:

$$\gamma_{\text{crit}}(X,p) \le \frac{n+1}{n}\widehat{\text{vol}}(X,p)^{1/n}.$$

Adapting the proof of [BB17, Theorem 1.6], we will construct approximate Green's functions to test the thresholds in the Moser–Trudinger inequality.

4.1 Functions with algebraic singularities

Let $\mathcal{I}$ be a coherent ideal sheaf, and $f_1,\dots,f_N\in\mathcal{O}_{X,p}$ be local generators of $\mathcal{I}_p$. The psh function

$$\varphi_{\mathcal{I}} := \log\left(\sum_{i=1}^N |f_i|^2\right)$$

is well defined near p, with *algebraic singularities* encoded in $\mathcal{I}$.

Proposition 4.1. *Let $\mathcal{I}$ be a coherent ideal sheaf supported at p. Then*

$$e(X,\mathcal{I}) = \int_{\{p\}} (dd^c \varphi_{\mathcal{I}})^n$$

and

$$\mathrm{lct}(X,\mathcal{I}) = \sup\left\{\alpha > 0 : \int_\Omega e^{-\alpha\varphi_{\mathcal{I}}}\, d\mu_p < +\infty\right\},$$

where Ω is any (small) neighborhood of $p \in X$.

These algebraic quantities are thus independent of the choice of generators.

Proof. The equality $e(X,\mathcal{I}) = \int_{\{p\}}(dd^c\varphi_{\mathcal{I}})^n$ is classical when Ω is smooth (see, e.g., [Dem12, Lemma 2.1]), and the proof can be adapted to the singular context (see [Dem85, Chapter 4]).

Let $\pi : \tilde{\Omega} \to \Omega$ be a local log resolution of the ideal $(X,\mathcal{I})$, i.e. a composition of blow-ups such that $\pi^*\mu_p = \prod_{j=1}^N |s_{E_j}|^{2a_j} dV_{\tilde{\Omega}}$ and

$$\pi^{-1}\mathcal{I}\cdot\mathcal{O}_{\tilde{\Omega}} = \mathcal{O}_{\tilde{\Omega}}\bigg(-\sum_{j=1}^M b_j E_j\bigg),$$

where $b_j \in \mathbb{N}$, $a_j \in \mathbb{Q}_{>-1}$, and $E_1,\dots,E_M$ have simple normal crossings. Observe that

$$\int_\Omega e^{-\alpha\varphi_{\mathcal{I}}}\, d\mu_p \sim \int_{\tilde{\Omega}} \frac{\prod_{j=1}^M |s_{E_j}|^{2a_j}}{\prod_{j=1}^M |s_{E_j}|^{2b_j\alpha}}\, dV_{\tilde{\Omega}} = \int_{\tilde{\Omega}} \prod_{j=1}^M |s_{E_j}|^{2(a_j-\alpha b_j)}\, dV_{\tilde{\Omega}},$$

is finite if and only if $a_j - \alpha b_j > -1$ for any $j = 1,\dots,M$, i.e. if and only if

$$\alpha < \inf_{j=1,\dots,M} \frac{a_j+1}{b_j} = \mathrm{lct}(X,\mathcal{I}),$$

as recalled in Definition 2.12. □

4.2 Approximate Green functions

The functions $\lambda\varphi_{\mathcal{I}}$ play the role of Green functions adapted to the singularity (X,p). We show here how to approximate them from above by smooth functions with prescribed boundary values.

Lemma 4.2. *Let $\mathcal{I}$ be a coherent ideal sheaf supported at p, and let $f_1,\dots,f_m$ denote local generators of $\mathcal{I}$. Fix an open set $\Omega' \Subset \Omega$. There exists a family $\{\varphi_{\mathcal{I},\lambda,\epsilon}\}_{\lambda>0,\epsilon>0} \in PSH(\Omega)\cap\mathcal{C}^\infty(\bar{\Omega})$ such that:*

(i) $\varphi_{\mathcal{I},\lambda,\epsilon\,|\,\partial\Omega} = \phi$ *for any* $\lambda > 0$, $\epsilon \in [0,1]$;
(ii) $\varphi_{\mathcal{I},\lambda,\epsilon} = \lambda\log\big(\sum_{j=1}^m |f_j|^2 + \epsilon^2\big) + \phi_0$ *in* Ω';
(iii) $\varphi_{\mathcal{I},\lambda,\epsilon} \searrow \varphi_{\mathcal{I},\lambda,0} =: \varphi_{\mathcal{I},\lambda}$ *as* $\epsilon \searrow 0$ *for any* $\lambda > 0$ *fixed.*

Proof. Without loss of generality we can assume that $\sum_{j=1}^m |f_j|^2 \le 1/e - 1$ in Ω. Let ρ be a smooth psh exhaustion for Ω and fix $0 < r \ll 1$, $0 < \delta \ll 1$ small enough. There exists $A > 0$ big enough and relatively compact open sets $B_r(0) \Subset \Omega' \Subset \tilde{\Omega} \Subset \Omega$ such that

$$\log\bigg(\sum_{j=1}^m |f_j|^2 + 1\bigg) + \delta \le A\rho \text{ over } \Omega\setminus\tilde{\Omega},$$

https://doi.org/10.1112/S0010437X24007619

while $\log(\sum_{j=1}^m |f_j|^2) - \delta \geq A\rho$ over $\Omega' \setminus B_r(0)$. We infer that

$$u_{\mathcal{I},\epsilon} := \begin{cases} A\rho & \text{on } \Omega \setminus \tilde{\Omega}, \\ \max_\delta\left(\log\left(\sum_{j=1}^m |f_j|^2 + \epsilon^2\right), A\rho\right) & \text{on } \tilde{\Omega} \setminus \Omega', \\ \log\left(\sum_{j=1}^m |f_j|^2 + \epsilon^2\right) & \text{on } \Omega', \end{cases}$$

is a decreasing family (in $\epsilon \in [0,1]$) of psh functions which are smooth in $\bar{\Omega} \setminus \{p\}$ (smooth in $\bar{\Omega}$ for $\epsilon > 0$) and which are identically 0 on $\partial\Omega$. Here $\max_\delta(\cdot,\cdot)$ denotes the regularized maximum. The lemma follows by setting $\varphi_{\mathcal{I},\lambda,\epsilon} := \lambda u_{\mathcal{I},\epsilon} + \phi_0$. □

We now compute the asymptotic behavior, as ε decreases to 0, of the quantities involved in the expected Moser–Trudinger inequality.

Lemma 4.3. *Let $\mathcal{I}$ and $\{\varphi_{\mathcal{I},\lambda,\epsilon}\}_{\epsilon\in(0,1]} \subset \mathcal{T}_\phi(\Omega)$ be as in Lemma 4.2. Then, for any $\gamma > 0, \lambda > 0$ fixed there exists a constant $C_{\lambda,\gamma} \in \mathbb{R}$ (independent of ϵ) such that*

$$C_{\lambda,\gamma} + (\gamma\lambda - \operatorname{lct}(X,\mathcal{I})) \log \epsilon^{-2} \leq \log \int_\Omega e^{-\gamma\varphi_{\mathcal{I},\lambda,\epsilon}}\, d\mu_p \tag{4.1}$$

for all $0 < \epsilon < \epsilon_0$.

Proof. Taking a log resolution $\pi : Y \to X$ we obtain

$$\int_\Omega e^{-\gamma\varphi_{\mathcal{I},\lambda,\epsilon}}\, d\mu_p \geq \int_{\Omega'} \frac{1}{(\sum_{j=1}^m |f_j|^2 + \epsilon^2)^{\gamma\lambda}}\, d\mu_p$$
$$\geq C_1 \int_{\pi^{-1}(\Omega')} \frac{\prod_{j=1}^M |s_{E_j}|^{2a_j}}{(\prod_{j=1}^M |s_{E_j}|^{2b_j} + \epsilon^2)^{\gamma\lambda}}\, dV_{\pi^{-1}(\Omega')},$$

where C_1 is a uniform constant (independent on ϵ). We set

$$f := \frac{\prod_{j=1}^M |s_{E_j}|^{2a_j}}{(\prod_{j=1}^M |s_{E_j}|^{2b_j} + \epsilon^2)^{\gamma\lambda}}.$$

We can assume without loss of generality that $\operatorname{lct}(X,\mathcal{I}) = (a_1+1)/b_1$. Pick $x \in E_1, x \notin E_j, j = 2,\dots,M$. We can find $0 < r \ll 1$ so small that $B_r(x) \cap E_j = \emptyset$ for any $j = 2,\dots,M$. We choose holomorphic coordinates $(z_1,\dots,z_n)$ centered at x such that $E_1 = \{z_1 = 0\}$. Thus, setting $a := a_1, b := b_1$ and $c := \gamma\lambda$ we get

$$\int_{\pi^{-1}(\Omega')} f\, dV_{\pi^{-1}(\Omega')} \geq C_2 \int_{B_r(0)} \frac{|z_1|^{2a}}{(|z_1|^{2b} + \epsilon^2)^c}\, d\lambda(z) = C_3 \int_0^r \frac{u^{2a+1}}{(u^{2b} + \epsilon^2)^c}\, du,$$

where C_2, C_3 are uniform constants. If $c \leq (a+1)/b$ (i.e. $\gamma\lambda \leq \operatorname{lct}(X,\mathcal{I})$), then

$$\int_0^r \frac{u^{2a+1}}{(u^{2b} + \epsilon^2)^c}\, du \geq \int_0^1 \frac{u^{2a+1}}{(u^{2b} + 1)^c} =: C_4$$

and (4.1) trivially follows. If $c > (a+1)/b$, then the substitution $v := u/\epsilon^{1/b}$ yields

$$\int_0^r \frac{u^{2a+1}}{(u^{2b} + \epsilon^2)^c}\, du = \epsilon^{-2(c-((a+1)/b))} \int_0^{r/\epsilon^{1/b}} \frac{v^{2a+1}}{(v^{2b} + 1)^c}$$
$$\geq \epsilon^{-2(\gamma\lambda - \operatorname{lct}(X,\mathcal{I}))} \int_0^r \frac{v^{2a+1}}{(v^{2b} + 1)^c}\, dv.$$

https://doi.org/10.1112/S0010437X24007619

The lemma follows. ∎

Lemma 4.4. *Let $\mathcal{I}$ and $\varphi_{\mathcal{I},\lambda,\epsilon} \in \mathcal{T}_\phi(\Omega)$ be as in Lemma 4.2. There exist positive constants $\{C_{\ell,\lambda}\}_{\ell\in\mathbb{N},\lambda>0}$ and a family of functions $F_\ell : (0,1] \to \mathbb{R}_{>0}$ such that*

$$-E_\phi(\varphi_{\mathcal{I},\lambda,\epsilon}) \le C_{\ell,\lambda} + \frac{\lambda^{n+1}}{n+1} F_\ell(\epsilon) \log \epsilon^{-2},$$

for any $\epsilon \in (0,1]$, where:

- *$\{C_{\ell,\lambda}\}_{\ell\in\mathbb{N},\lambda>0}$ is independent of $\epsilon \in (0,1]$;*
- *$F_\ell(\epsilon) \to F_\ell(0) =: e_\ell > 0$ as $\epsilon \searrow 0$;*
- *$e_\ell \searrow e(X,\mathcal{I})$ as $\ell \to +\infty$.*

Proof. We take a sequence $\{\Omega_\ell\}_{\ell\in\mathbb{N}}$ of open sets such that $\Omega_{\ell+1} \Subset \Omega_\ell$ for any $\ell \in \mathbb{N}$ and such that $\bigcap_{\ell\in\mathbb{N}} \Omega_\ell = \{p\}$. Since $\Omega_\ell \subset \Omega'$ (same notation as Lemma 4.2) for $\ell \in \mathbb{N}$ big enough, we obtain

$$\begin{aligned}
-E_\phi(\varphi_{\mathcal{I},\lambda,\epsilon}) &= \frac{1}{n+1}\sum_{j=0}^{n}\int_\Omega (\phi_0 - \varphi_{\mathcal{I},\lambda,\epsilon})(dd^c\varphi_{\mathcal{I},\lambda,\epsilon})^j \wedge (dd^c\phi_0)^{n-j} \\
&= \frac{1}{n+1}\sum_{j=0}^{n}\int_{\Omega\setminus\Omega_\ell} (\phi_0 - \varphi_{\mathcal{I},\lambda,\epsilon})(dd^c\varphi_{\mathcal{I},\lambda,\epsilon})^j \wedge (dd^c\phi_0)^{n-j} \\
&\quad - \frac{1}{n+1}\sum_{j=0}^{n}\int_{\Omega_\ell} \lambda^{j+1} \log\Big(\sum_{k=1}^m |f_k|^2 + \epsilon^2\Big)\Big(dd^c \log\Big(\sum_{k=1}^m |f_k|^2 + \epsilon^2\Big)\Big)^j \wedge (dd^c\phi_0)^{n-j}.
\end{aligned} \tag{4.2}$$

The first term on the right-hand side of (4.2) is uniformly bounded in $\epsilon \in [0,1]$, for $\lambda > 0$, $\ell \in \mathbb{N}$ fixed, since $\{\varphi_{\mathcal{I},\lambda,\epsilon}\}_{\epsilon\in[0,1]}$ is a continuous family of smooth functions on $\Omega \setminus \Omega_\ell$. We let $C_{\ell,\lambda}$ denote a uniform upper bound for this quantity.

The second term on the right-hand side of (4.2) is bounded from above by

$$\begin{aligned}
&- \frac{1}{n+1}\sum_{j=0}^{n}\int_{\Omega_\ell} \lambda^{j+1} \log\Big(\sum_{k=1}^m |f_k|^2 + \epsilon^2\Big)\Big(dd^c \log\Big(\sum_{k=1}^m |f_k|^2 + \epsilon^2\Big)\Big)^j \wedge (dd^c\phi_0)^{n-j} \\
&\le \frac{\lambda^{n+1}}{n+1} \log \epsilon^{-2} \sum_{j=0}^{n}\int_{\Omega_\ell} \Big(dd^c \log\Big(\sum_{k=1}^m |f_k|^2 + \epsilon^2\Big)\Big)^j \wedge (dd^c\phi_0)^{n-j}.
\end{aligned}$$

We set

$$F_\ell(\epsilon) := \sum_{j=0}^{n}\int_{\Omega_\ell} \Big(dd^c \log\Big(\sum_{k=1}^m |f_k|^2 + \epsilon^2\Big)\Big)^j \wedge (dd^c\phi_0)^{n-j}.$$

Observe that, for $j = 0, \ldots, n-1$,

$$\lim_{\ell\nearrow+\infty}\lim_{\epsilon\searrow 0}\int_{\Omega_\ell} \Big(dd^c \log\Big(\sum_{k=1}^m |f_k|^2 + \epsilon^2\Big)\Big)^j \wedge (dd^c\phi_0)^{n-j} = 0,$$

https://doi.org/10.1112/S0010437X24007619

since the ideal sheaf $\mathcal{I}$ generated by $f_1, \ldots, f_m$ is supported at one point, while

$$\int_{\Omega_\ell} \left(dd^c \log \left(\sum_{k=1}^m |f_k|^2 + \epsilon^2 \right) \right)^n \to e_\ell$$

as $\epsilon \searrow 0$, where $e_\ell \geq e(X, \mathcal{I})$ and $e_\ell \searrow \int_{\{p\}} (dd^c \varphi_{\mathcal{I},1})^n$ as $\ell \nearrow +\infty$.

Proposition 4.1 yields $\int_{\{p\}} (dd^c \varphi_{\mathcal{I},1})^n = e(X, \mathcal{I})$, ending the proof. □

4.3 The upper bound

We are now ready for the proof of the following result.

Theorem 4.5. *Let (X, p) be a an isolated log terminal singularity. Then*

$$\gamma_{\text{crit}} \leq \frac{n+1}{n} \widehat{\text{vol}}(X, p)^{1/n}.$$

Proof. Fix $\gamma < \gamma_{\text{crit}}$ and $C_1 > 0$ such that

$$\frac{1}{\gamma} \log \int_\Omega e^{-\gamma \varphi} \, d\mu_p \leq C_1 - E_\phi(\varphi) \tag{4.3}$$

for any $\varphi \in \mathcal{T}_\phi(\Omega)$.

Fix $\mathcal{I}$ coherent ideal sheaf supported at p, and let $\{\varphi_{\mathcal{I},\lambda,\epsilon}\}_{\lambda>0,\epsilon\in(0,1]} \in \mathcal{T}_\phi(\Omega)$ as defined in Lemma 4.2. Evaluating (4.3) at $\{\varphi_{\mathcal{I},\lambda,\epsilon}\}_{\epsilon\in(0,1]}$ yields

$$\begin{aligned} C_{\gamma,\lambda} + \left(\lambda - \frac{\text{lct}(X, \mathcal{I})}{\gamma} \right) \log \epsilon^{-2} &\leq \frac{1}{\gamma} \log \int_\Omega e^{-\gamma \varphi_{\mathcal{I},\lambda,\epsilon}} \, d\mu_p \\ &\leq C_1 - E_\phi(\varphi_{\mathcal{I},\lambda,\epsilon}) \\ &\leq C_1 + C_{N,\lambda} + \frac{\lambda^{n+1}}{n+1} F_N(\epsilon) \log \epsilon^{-2} \end{aligned}$$

for any $N \in \mathbb{N}, \epsilon \in (0, 1]$ thanks to Lemmas 4.3 and 4.4. We infer

$$\left(\lambda - \frac{\text{lct}(X, \mathcal{I})}{\gamma} - \frac{\lambda^{n+1}}{n+1} F_N(\epsilon) \right) \log \epsilon^{-2} \leq C_1 + C_{N,\lambda} - C_{\gamma,\lambda},$$

hence

$$\lambda - \frac{\lambda^{n+1}}{n+1} e_N \leq \frac{\text{lct}(X, \mathcal{I})}{\gamma} \tag{4.4}$$

for any $N \in \mathbb{N}, \lambda > 0$ since $F_N(\epsilon) \to e_N$ as $\epsilon \searrow 0$ (Lemma 4.4).

The function $g_N : \lambda \in (0, +\infty) \mapsto \lambda - (\lambda^{n+1}/(n+1)) e_N \in \mathbb{R}$ reaches its maximum at $\lambda_{N,M} := 1/e_N^{1/n}$. It follows therefore from (4.4) that

$$\gamma \leq \frac{\text{lct}(X, \mathcal{I})}{g_N(\lambda_{N,M})} = \frac{n+1}{n} \text{lct}(X, \mathcal{I}) e_N^{1/n}.$$

Now $e_N \searrow e(X, \mathcal{I})$ as $N \to +\infty$ by Lemma 4.4, hence

$$\gamma \leq \frac{n+1}{n} \text{lct}(X, \mathcal{I}) e(X, \mathcal{I})^{1/n}.$$

Since this holds for any coherent ideal sheaf $\mathcal{I}$ supported at p , we obtain

$$\gamma \leq \frac{n+1}{n} \inf_{\mathcal{I}} \text{lct}(X, \mathcal{I}) e(X, \mathcal{I})^{1/n} = \frac{n+1}{n} \widehat{\text{vol}}(X, p)^{1/n},$$

where the equality follows from Theorem 2.16. □

5. Moser–Trudinger inequality

5.1 Uniform integrability versus Moser–Trudinger inequality

Recall that

$$\alpha(X,\mu_p) := \sup\left\{\alpha>0,\ \sup_{\varphi\in\mathcal{F}_1(\Omega)}\int_\Omega e^{-\alpha\varphi}\,d\mu_p<+\infty\right\}.$$

This uniform integrability index is a local counterpart to Tian's celebrated α-invariant, introduced in [Tia87] in the quest for Kähler–Einstein metrics on Fano manifolds. We refer to [DK01, Dem09, Zer09, ACKPZ09, DP14, GZ15, Pha18] for some contributions to the local study of analogous invariants.

In this section we prove Theorem A, which can be seen as a local analogue of [BBEGZ19, Proposition 4.13].

Theorem 5.1. *One has* $\gamma_{\text{crit}}(X,p)\ge((n+1)/n)\alpha(X,\mu_p)$.

When (X,p) is smooth then $\alpha(X,\mu_p)=n$ and this statement is equivalent (after an appropriate rescaling) to [BB22, Theorem 1.5] and [GKY13, Theorem 9].

Together with Theorem 4.5, we would obtain the precise value

$$\gamma_{\text{crit}}(X,p)\overset{?}{=}\frac{n+1}{n}\widehat{\text{vol}}(X,p)^{1/n}$$

if we knew that $\alpha(X,\mu_p)=\widehat{\text{vol}}(X,p)^{1/n}$. We establish in § 5.2 the bound $\alpha(X,\mu_p)\le\widehat{\text{vol}}(X,p)^{1/n}$ and analyze the reverse inequality in § 5.3.

5.1.1 *Entropy.* We let $\mathcal{P}(\Omega)$ denote the set of probability measures on Ω. Given two measures $\mu,\nu\in\mathcal{P}(\Omega)$, the *relative entropy of* ν *with respect to* μ is

$$H_\mu(\nu):=\int_\Omega\frac{d\nu}{d\mu}\log\frac{d\nu}{d\mu}\,d\mu=\int_\Omega\log\frac{d\nu}{d\mu}\,d\nu$$

if ν is absolutely continuous with respect to μ, and as $H_\mu(\nu):=+\infty$ otherwise.

Given $\mu\in\mathcal{P}(X)$, the relative entropy $H_\mu(\cdot)$ is the Legendre transform of the convex functional $g\in\mathcal{C}^0(\Omega)\cap L^\infty(\Omega)\mapsto\log\int_\Omega e^g\,d\mu\in\mathbb{R}$, i.e.

$$H_\mu(\nu)=\sup_{g\in\mathcal{C}^0(\Omega)\cap L^\infty(\Omega)}\left(\int_\Omega g\,d\nu-\log\int_\Omega e^g\,d\mu\right).$$

We shall need the following duality result.

Lemma 5.2 [BBEGZ19, Lemma 2.11]. *Fix* $\mu\in\mathcal{P}(\Omega)$. *Then*

$$\log\int_\Omega e^g\,d\mu=\sup_{\nu\in\mathcal{P}(\Omega)}\left(\int_\Omega g\,d\nu-H_\mu(\nu)\right)$$

for each lower semi-continuous function $g:\Omega\to\mathbb{R}\cup\{+\infty\}$.

Recall that we have normalized the adapted volume form so that $\mu_p\in\mathcal{P}(\Omega)$.

Corollary 5.3. *Fix* $0<\alpha<\alpha(X,\mu_p)$. *Then there exists* $C_\alpha>0$ *such that*

$$H_{\mu_p}(\nu)\ge-\alpha\int_\Omega\varphi\,d\nu-C_\alpha$$

for all $\varphi\in\mathcal{F}_1(\Omega)$ *and for all* $\nu\in\mathcal{P}(\Omega)$ *such that* $H_{\mu_p}(\nu)<+\infty$.

Proof. This follows from Lemma 5.2 applied to $g=-\alpha\varphi$ and $\mu=\mu_p$. By the definition of $\alpha(X,\mu_p)$, we obtain $-\log\int_\Omega e^{-\alpha\varphi}\,d\mu_p\geq -C_\alpha$. □

This corollary shows, in particular, that $\mathcal{F}_1(\Omega)\subset L^1(\nu)$ for any probability measure $\nu\in\mathcal{P}(X)$ with finite μ_p-entropy. Since the measure ν is moreover *non-pluripolar*, the following result is a consequence of Theorem 3.7.

Proposition 5.4. *Fix $\nu\in\mathcal{P}(\Omega)$ such that $H_{\mu_p}(\nu)<+\infty$. Then there exists a unique $v\in\mathcal{F}_1(\Omega)\cap\mathcal{E}^1(\Omega)$ such that*

$$\nu=(dd^cv)^n.$$

5.1.2 *Proof of Theorem 5.1.* The proof is similar to the derivation of the Moser–Trudinger inequality from Brezis–Merle inequality by Berman and Berndtsson, see [BB22, § 4.2]. Fix $\varphi\in\mathcal{T}_\phi(\Omega)$ and $0<\alpha<\alpha(X,\mu_p)$. By Lemma 5.2 for any $\epsilon>0$ there exists $\nu_\epsilon\in\mathcal{P}(\Omega)$ such that $H_{\mu_p}(\nu_\epsilon)<+\infty$ and

$$\log\int_\Omega e^{-((n+1)/n)\alpha\varphi}\,d\mu_p\leq\epsilon-\frac{n+1}{n}\alpha\int_\Omega\varphi\,d\nu_\epsilon-H_{\mu_p}(\nu_\epsilon). \tag{5.1}$$

Proposition 5.4 ensures the existence of $v_\epsilon\in\mathcal{F}_1(\Omega)\cap\mathcal{E}^1(\Omega)$ such that $\nu_\epsilon=(dd^cv_\epsilon)^n$. It follows, moreover, from Corollary 5.3 thatd

$$H_{\mu_p}(\nu_\epsilon)\geq-\alpha\int_\Omega v_\epsilon\,d\nu_\epsilon-C_\alpha. \tag{5.2}$$

Combining (5.1) and (5.2) we obtain

$$\log\int_\Omega e^{-((n+1)/n)\alpha\varphi}\,d\mu_p\leq\epsilon+C_\alpha-\frac{n+1}{n}\alpha\int_\Omega\varphi\,d\nu_\epsilon+\alpha\int_\Omega v_\epsilon\,d\nu_\epsilon. \tag{5.3}$$

We observe that

$$\begin{aligned}-\frac{n+1}{n}\alpha\int_\Omega\varphi\,d\nu_\epsilon+\alpha\int_\Omega v_\epsilon\,d\nu_\epsilon&=\frac{n+1}{n}\alpha\int_\Omega(v_\epsilon-\varphi)(dd^cv_\epsilon)^n-\frac{\alpha}{n}\int_\Omega v_\epsilon(dd^cv_\epsilon)^n\\&\leq-\frac{n+1}{n}\alpha E_\phi(\varphi)+\frac{\alpha}{n}\left\{(n+1)E_\phi(v_\varepsilon)-\int_\Omega v_\epsilon(dd^cv_\epsilon)^n\right\}\end{aligned}$$

by using Lemma 3.5 (the latter has been stated for functions in $\mathcal{T}_\phi(\Omega)$, it easily extends to the class $\mathcal{F}_1(\Omega)\cap\mathcal{E}^1(\Omega)$ by approximation). Since $v_\varepsilon\leq\phi_0$ and $E_\phi(\phi_0)=0$, the same lemma ensures

$$(n+1)E_\phi(v_\varepsilon)-\int_\Omega v_\epsilon(dd^cv_\epsilon)^n\leq-\int_\Omega\phi_0(dd^cv_\varepsilon)^n\leq-\inf_\Omega\phi_0,$$

using that $\nu_\epsilon=(dd^cv_\epsilon)^n$ is a probability measure. Altogether this yields

$$\log\int_\Omega e^{-((n+1)/n)\alpha\varphi}\,d\mu_p\leq\epsilon+C_\alpha-\frac{\alpha}{n}\inf_\Omega\phi_0-\frac{n+1}{n}\alpha E_\phi(\varphi).$$

Letting $\epsilon\searrow 0$ we conclude that

$$\left(\int_\Omega e^{-((n+1)/n)\alpha\varphi}\,d\mu_p\right)^{n/((n+1)\alpha)}\leq C'_\alpha e^{-E_\phi(\varphi)}$$

for any function $\varphi\in\mathcal{T}_\phi(\Omega)$. Thus, $\gamma_{\text{crit}}(X,p)\geq((n+1)/n)\alpha(X,\mu_p)$.

5.2 Upper bound on the α-invariant

Definition 5.5. We set

$$\tilde{\alpha}(X,\mu_p):=\inf\{c(\varphi),\ \varphi\in\mathcal{F}_1(\Omega)\},$$

where $c(\varphi):=\sup\{c>0;\ \int_\Omega e^{-c\varphi}\,d\mu_p<+\infty\}$.

5.2.1 *Bounding the α-invariant by the normalized volume.*

Proposition 5.6. *One has $\alpha(X,\mu_p) \le \tilde{\alpha}(X,\mu_p) \le \widehat{\mathrm{vol}}(X,p)^{1/n}$.*

Proof. It follows from the definition that $\alpha(X,\mu_p) \le \tilde{\alpha}(X,\mu_p)$.

For any $\epsilon > 0$ and $\mathcal{I}$ coherent ideal sheaf supported at 0, the function

$$\psi_{\mathcal{I},\epsilon} := \psi_{\mathcal{I},\lambda,\epsilon}, \quad \text{with } \lambda = \left(\frac{1-\epsilon}{e(X,\mathcal{I})}\right)^{1/n}$$

given by Lemma 5.7, belongs to $\mathcal{F}_1(\Omega)$ and yields

$$\tilde{\alpha}(X,\mu_p) \le c(\psi_{\mathcal{I},\epsilon}) = \frac{1}{(1-\epsilon)^{1/n}} \mathrm{lct}(X,\mathcal{I}) e(X,\mathcal{I})^{1/n}.$$

The latter equality is a consequence of Proposition 4.1. We conclude the proof by taking the infimum over all $\mathcal{I}$ and letting $\epsilon \searrow 0$. □

Lemma 5.7. *Let $\mathcal{I}$ be a coherent ideal sheaf supported at p. Then, for any $\lambda, \epsilon > 0$ there exists a function $\psi_{\mathcal{I},\lambda,\epsilon} \in \mathcal{F}(\Omega)$ such that:*

(i) *$\psi_{\mathcal{I},\lambda,\epsilon} = \lambda \log(\sum_{j=1}^m |f_j|^2)$ near 0 for local generators $f_1, \dots, f_m$ of $\mathcal{I}$;*

(ii) *$\lambda^n e(X,\mathcal{I}) \le \int_\Omega (dd^c \psi_{\mathcal{I},\lambda,\epsilon})^n \le \lambda^n e(X,\mathcal{I}) + \epsilon$.*

Proof. Assume that ϕ_0 is the maximal psh extension of ϕ to Ω, i.e. the largest psh function in Ω which lies below ϕ on $\partial\Omega$. It satisfies $(dd^c\phi_0)^n = 0$ in Ω.

Fix $f_1, \dots, f_m$ local generators of the ideal $\mathcal{I}$ and set $\psi_\lambda := \lambda \log(\sum_{j=1}^m |f_j|^2)$. We can assume without loss of generality that the f_j are well defined in Ω and normalized so that $\psi_\lambda \le \phi_0 - 1$ in Ω. For $r > 0$, we consider

$$\varphi_r := \sup\{u \in PSH(\Omega),\ u \le \psi_\lambda \text{ in } B(r) \text{ and } u \le \phi_0 \text{ in } \Omega\}.$$

The corresponding family of psh functions is non-empty as it contains ψ_λ. For $A > 1$ large enough, the function

$$w_r = \begin{cases} \psi_\lambda & \text{in } B(r), \\ \max(\psi_\lambda, A\rho + \phi_0) & \text{in } \Omega \setminus B(r), \end{cases}$$

is psh and coincides with $A\rho + \phi_0$ near $\partial\Omega$. It follows that:

- $\varphi_r \in PSH(\Omega)$ with $\varphi_r = \phi$ on $\partial\Omega$;
- $\varphi_r \equiv \psi_\lambda$ in $B(r)$, hence $\lambda^n e(X,\mathcal{I}) \le \int_\Omega (dd^c\varphi_r)^n$;
- $(dd^c\varphi_r)^n = 0$ in $\Omega \setminus \overline{B}(r)$ (balayage argument).

The family $r \mapsto \varphi_r$ increases, as $r > 0$ decreases to 0, to some psh limit φ whose Monge–Ampère measure $(dd^c\varphi)^n$ is concentrated at the origin. It follows from Bedford–Taylor continuity theorem that $(dd^c\varphi)^n$ is the weak limit of $(dd^c\varphi_r)^n \ge \lambda^n e(X,\mathcal{I})\delta_0$, hence $(dd^c\varphi)^n \ge \lambda^n e(X,\mathcal{I})\delta_0$. Conversely, $\psi_\lambda \le \varphi$ near 0, hence Demailly's comparison theorem ensures that

$$(dd^c\varphi)^n(0) \le (dd^c\psi_\lambda)^n(0) \le \lambda^n e(X,\mathcal{I}),$$

whence equality. Thus, $\phi_{\mathcal{I},\lambda,\epsilon} := \varphi_{r_\epsilon}$ satisfies the required properties. □

5.2.2 *Normalized volume versus uniform integrability.*

Proposition 5.8. *One has $\tilde{\alpha}(X,\mu_p) = \widehat{\mathrm{vol}}(X,p)^{1/n}$.*

We refer the reader to Appendix A for a more algebraic approach based on [BdFF12], which moreover provides a slightly stronger result.

https://doi.org/10.1112/S0010437X24007619

When (X,p) is smooth, it follows from [DK01] that $\tilde{\alpha}(X,\mu_p) = \alpha(X,\mu_p)$. The situation is, however, more subtle in the singular context (see § 5.3.2).

Proof. By Proposition 5.6 it suffices to show that $\tilde{\alpha}(X,\mu_p) \geq \widehat{\mathrm{vol}}(X,p)^{1/n}$, i.e. $\int_\Omega e^{-\alpha\varphi}\, d\mu_p < +\infty$ for all $\varphi \in \mathcal{F}_1(\Omega)$ and $\alpha < \widehat{\mathrm{vol}}(X,p)^{1/n} = \inf_{\mathcal{I}} \mathrm{lct}(X,\mathcal{I})^n e_p(\mathcal{I})$.

In a log resolution $\pi : \tilde{\Omega} \to \Omega$, this boils down to $\int_{\tilde{\Omega}} e^{-\alpha\varphi\circ\pi} \prod_{i=1}^M |s_i|_{h_i}^{2a_i}\, dV < +\infty$, where s_i are holomorphic sections defining simple normal crossing exceptional divisors $E_1, \ldots, E_M$, $K_{\tilde{\Omega}/\Omega} = \sum_{j=1}^M a_i E_i$ and where dV is a smooth volume form. The log terminal condition ensures that $a_i > -1$ for all $i = 1, \ldots, M$.

As $\alpha < \widehat{\mathrm{vol}}(X,p)^{1/n} \leq n$, the integrability condition is equivalent to show that for any point $x \in \bigcup_{i=1}^M E_i$ there exists a small ball $B(x,r)$ such that

$$\int_{B(x,r)} e^{-\alpha\varphi\circ\pi} \prod_{i=1}^M |s_i|_{h_i}^{2a_i}\, dV < +\infty. \tag{5.4}$$

Set $U := \sum_{i: a_i \geq 0} a_i \log|s_i|_{h_i}^2$, $V := \alpha\varphi\circ\pi$ and $W := -\sum_{i: a_i < 0} a_i \log|s_i|_{h_i}^2$. By [BBJ21, Theorem B.5] the condition (5.4) holds if and only if there exists $\epsilon > 0$ such that

$$\nu(U\circ g, F) + A_{\tilde{\Omega}}(F) \geq (1+\epsilon)\nu(V\circ g, F) + (1+\epsilon)\nu(W\circ g, F) \tag{5.5}$$

for any F prime divisor *over* $\tilde{\Omega}$ with center in a small ball $\overline{B(x,r')} \subset B(x,r)$, i.e. $F \subset \Omega'$ for $g : \Omega' \to \tilde{\Omega}$ modification. Observe that

$$\begin{aligned} \nu(U\circ g, F) + A_{\tilde{\Omega}}(F) - \nu(W\circ g, F) &= \mathrm{ord}_F(g^* K_{\tilde{\Omega}/\Omega}) + 1 + \mathrm{ord}_F(K_{\Omega'/\tilde{\Omega}}) \\ &= 1 + \mathrm{ord}_F(K_{\Omega'/\Omega}) = A_\Omega(F). \end{aligned}$$

Thus, (5.5) becomes

$$\alpha(1+\epsilon) \leq \frac{A_\Omega(F) - \epsilon\nu(W\circ g, F)}{\nu(\varphi\circ\pi\circ g, F)}. \tag{5.6}$$

As $a_i > -1$ for all i, [BBJ21, Theorem B.5] ensures the existence of $a > 0$ such that $A_{\tilde{\Omega}}(F) \geq (1+a)\nu(W\circ g, F)$ for any prime divisor F over $\tilde{\Omega}$ as above. Thus,

$$A_{\tilde{\Omega}}(F) \leq A_\Omega(F) + \nu(W\circ g, F) \leq \frac{1}{1+a} A_{\tilde{\Omega}}(F) + A_\Omega(F),$$

and $\nu(W\circ g, F) \leq (1/(1+a)) A_{\tilde{\Omega}}(F) \leq (1/a) A_\Omega(F)$. Therefore, (5.6) holds if

$$\alpha(1+\epsilon) \leq \frac{a-\epsilon}{a} \frac{A_\Omega(F)}{\nu(\varphi\circ\pi\circ g, F)}. \tag{5.7}$$

Since $\varphi \in \mathcal{F}_1(\Omega)$, it follows from the comparison theorem of Demailly [Dem85, Theorem 4.2] that for a coherent ideal sheaf $\mathcal{I}$ supported at $p \in \Omega$,

$$1 \geq \int_\Omega (dd^c\varphi)^n \geq \nu_{\mathcal{I}}(\varphi, p)^n \int_{\mathbb{C}^N} (dd^c f_{\mathcal{I}})^n \wedge [X] = \nu_{\mathcal{I}}(\varphi,p)^n e_p(\mathcal{I}), \tag{5.8}$$

where $f_{\mathcal{I}} = \log(\sum_i |f_i|^2)$ for generators $\{f_i\}_i$ of $\mathcal{I}$ and

$$\nu_{\mathcal{I}}(\varphi, p) := \sup\{s > 0 : \varphi \leq s f_{\mathcal{I}} + O(1)\} = \min_G \frac{\nu(\varphi\circ\pi\circ g, G)}{\mathrm{ord}_G \mathcal{I}},$$

where $\pi\circ g : \Omega' \to \Omega$ is a log resolution for $\mathcal{I}$ and the minimum is over all exceptional divisors of $\Omega' \to \Omega$. Lemma 5.9 ensures that for any prime divisor F and $\delta > 0$ there exists an ideal $\mathcal{I}$ such

https://doi.org/10.1112/S0010437X24007619

that

$$\frac{A_\Omega(F)}{\nu(\varphi\circ\pi\circ p,F)}\geq(1-\delta)\frac{A_\Omega(F)}{\mathrm{ord}_F\mathcal{I}}(\nu_{\mathcal{I}}(\varphi,p))^{-1}\geq(1-\delta)\frac{A_\Omega(F)}{\mathrm{ord}_F\mathcal{I}}e_p(\mathcal{I})^{1/n}$$
$$\geq(1-\delta)\mathrm{lct}(X,\mathcal{I})e_p(\mathcal{I})^{1/n}\geq(1-\delta)\widehat{\mathrm{vol}}(X,p)^{1/n}.$$

Thus, (5.7) holds if $\alpha(1+\epsilon)\leq(((a-\epsilon)(1-\delta))/a)\widehat{\mathrm{vol}}(X,p)^{1/n}$, concluding the proof. □

Lemma 5.9. *Fix $\varphi\in\mathcal{F}_1(\Omega)$ and $F\subset\Omega'$ prime divisor such that $\pi\circ g(F)=p$. For any $\epsilon>0$, there exists a coherent ideal sheaf $\mathcal{I}$ supported at p such that*

$$\nu_{\mathcal{I}}(\varphi,p)\geq(1-\epsilon)\frac{\nu(\varphi\circ\pi\circ g,F)}{\mathrm{ord}_F(\mathcal{I})}.$$

Proof. Let $c:=\nu(\varphi\circ\pi\circ g,F)$ and for $c'\in\mathbb{Q}$, $c'\leq c$, set

$$\mathcal{A}_{mc'}(F):=\{f\in\mathcal{O}_{X,p}:\mathrm{ord}_F(f\circ\pi\circ g)\geq mc'\}$$

for $m\in\mathbb{N}$ divisible enough. Then $\mathcal{A}_{mc'}(F)$ is an ideal sheaf and

$$\limsup_{m\to+\infty}\frac{\mathrm{ord}_F(\mathcal{A}_{mc'}(F))}{m}=c'. \tag{5.9}$$

In particular, if $\varphi_{mc'}\in\mathrm{PSH}(B(p,r))$ has algebraic singularities along $\mathcal{A}_{mc'}(F)$, then for any $\epsilon>0$, $\varphi_{mc'}$ is less singular than $(mc'/(c-\epsilon))\varphi$ around p if $m\geq m_1(\epsilon)\gg1$. For any G exceptional divisor on Ω' and $m\geq m_1(\epsilon)$ we infer

$$\frac{\nu(\varphi\circ\pi\circ g,G)}{\mathrm{ord}_G(\mathcal{A}_{mc'}(F))/m}=\frac{\nu(\varphi\circ\pi\circ g,G)}{\nu(\varphi_{mc'}\circ\pi\circ g,G)/m}\geq\frac{c-\epsilon}{c'}. \tag{5.10}$$

On the other hand, (5.9) implies that there exists $m_0(\epsilon)\geq m_1(\epsilon)\gg1$ with

$$\frac{\nu(\varphi\circ\pi\circ g,F)}{\mathrm{ord}_F(\mathcal{A}_{m_0c'}(F))/m_0}\leq\frac{c}{c'-\epsilon}. \tag{5.11}$$

Combining (5.10) and (5.11) we obtain

$$\min_G\frac{\nu(\varphi\circ\pi\circ g,G)}{\mathrm{ord}_G(\mathcal{A}_{m_0c'}(F))/m_0}\geq\frac{c-\epsilon}{c'}\geq\left(1-\epsilon\frac{c+c'}{cc'}\right)\frac{c}{c'-\epsilon}$$
$$\geq\left(1-\epsilon\frac{c+c'}{cc'}\right)\frac{\nu(\varphi\circ\pi\circ g,F)}{\mathrm{ord}_F(\mathcal{A}_{m_0c'}(F))/m_0}.$$

Since c' and ϵ are arbitrary, and $x\mapsto f(x)=(c+x)/cx$ is decreasing, we deduce that for any $\epsilon>0$ there exists $c'\in\mathbb{Q}$ and $m_0=m_0(c,c',\epsilon)$ such that

$$\nu_{\mathcal{A}_{m_0c'}(F)}(\varphi,p)\geq(1-\epsilon)\frac{\nu(\varphi\circ\pi\circ g,F)}{\mathrm{ord}_F(\mathcal{A}_{m_0c'}(F))}.$$

Setting $\mathcal{A}:=\mathcal{A}_{m_0c'}(F)$ concludes the proof. □

5.3 Lower bounds on the α-invariant

We provide in this section two effective (but not sharp) lower bounds on $\alpha(X,\mu_p)$.

5.3.1 *Using projections on n-planes.* A result of Skoda ensures that $e^{-\varphi}$ is integrable if the Lelong numbers of φ are small enough (see [GZ17, Theorem 2.50]). This has been largely extended by Demailly and Zeriahi who provided uniform integrability results for functions $\varphi\in\mathcal{F}_1(\Omega)$ (see [Dem09, ACKPZ09]). In this section, we extend these results to our singular setting.

Theorem 5.10. *One has*

$$\alpha(X,\mu_p) \geq \frac{n}{\mathrm{mult}(X,p)^{1-1/n}} \frac{\mathrm{lct}(X,p)}{1+\mathrm{lct}(X,p)}.$$

Proof. Recall that $\mu_p = f dV_X$ with $f \in L^r(dV_X)$. The exponent $r>1$ has been estimated in Lemma 2.14. Using Hölder inequality, we thus obtain

$$\alpha(X,\mu_p) \geq \frac{\mathrm{lct}(X,p)}{1+\mathrm{lct}(X,p)} \alpha(\Omega, dV_X).$$

The remainder of the proof consists of establishing the lower bound

$$\alpha(\Omega, dV_X) \geq \frac{n}{\mathrm{mult}(X,p)^{1-1/n}}.$$

Recall that $dV_X = \omega_{\mathrm{eucl}}^n \wedge [X]$, where ω_{eucl} denotes the euclidean Kähler form. Thus, $dV_X = \sum_I (\pi_I)^*(dV_I)$, where $I=(i_1,\dots,i_n)$ is a n-tuple, $\pi_I : \mathbb{C}^N \to \mathbb{C}^n_I$ denotes the linear projection on $\mathbb{C}^n_I$, and dV_I is the euclidean volume form on $\mathbb{C}^n_I$. We choose coordinates in $\mathbb{C}^N$ so that each projection map $\pi_I : \Omega \to \Omega_I \subset \mathbb{C}^n$ is proper. For $\varphi \in \mathcal{F}_1(\Omega)$, we obtain

$$\int_\Omega e^{-\alpha\varphi}\, dV_X = \sum_I \int_{\Omega_I} (\pi_I)_*(e^{-\alpha\varphi})\, dV_I \leq \mathrm{mult}(X,p) \sum_I \int_{\Omega_I} e^{-\alpha(\pi_I)_*\varphi}\, dV_I.$$

We assume here, without loss of generality, that $\varphi \leq 0$, and use the (sub-optimal) inequality $(\pi_I)_*(e^{-\alpha\varphi}) \leq \mathrm{mult}(X,p) e^{-\alpha(\pi_I)_*\varphi}$. The function $\varphi_I := (\pi_I)_*\varphi$ is psh in $\Omega_I = \pi_I(\Omega)$, with boundary values $(\pi_I)_*(\phi)$. We claim that

$$\int_{\Omega_I} (dd^c\varphi_I)^n \leq \mathrm{mult}(X,p)^{n-1}. \tag{5.12}$$

Once this is established, it follows from the main result of [ACKPZ09] that for all $0<\varepsilon$ small enough, there exists $C_\varepsilon > 0$ independent of φ such that

$$\int_{\Omega_I} e^{-((n-\varepsilon)/\mathrm{mult}(X,p)^{1-1/n})\varphi_I}\, dV_I \leq C_\varepsilon,$$

which yields the desired lower bound $\alpha(\Omega, dV_X) \geq (n/(\mathrm{mult}(X,p)^{1-1/n}))$.

It remains to check (5.12). We decompose $\varphi_I(z) = \sum_{i=1}^m \varphi(x_i)$, where $m = \mathrm{mult}(X,p)$ and $x_1,\dots,x_m$ denote the preimages of z counted with multiplicities. The assumption on the Monge–Ampère mass of φ reads

$$\sum_{i=1}^m \int (dd^c\varphi)^n(x_i) \leq 1.$$

We set $a_i^n := \int (dd^c\varphi)^n(x_i)$ and use [Ceg04, Corollary 5.6] to estimate

$$\int (dd^c\varphi_I)^n = \sum_{i_1,\dots,i_n=1}^m \int dd^c\varphi(x_{i_1}) \wedge \dots \wedge dd^c\varphi(x_{i_n}) \leq \sum_{i_1,\dots,i_n=1}^m a_{i_1}\cdots a_{i_n} = \left(\sum_i^m a_i\right)^n.$$

The latter sum is maximized when $a_1 = \dots = a_m = m^{-1/n}$, yielding (5.12). □

Example 5.11. Let $X = \{z \in \mathbb{C}^{n+1},\ F(z)=0\}$ be the A_k-singularity, where $F(z) = z_0^{k+1} + z_1^2 + \cdots z_n^2$. Arguing as we have done for the ODP ($k=1$), one can check that $\mu_p \sim dV_X/\|F'\|^2$ so that $\mathrm{mult}(X,p)=2$ and $\mathrm{lct}(X,p)=n-2+2/(k+1)$. Now

$$\widehat{\mathrm{vol}}(A_k,p)^{1/n}=\begin{cases}2^{1/n}\left(\dfrac{n-2}{n-1}\right)^{1-1/n}n & \text{if } \dfrac{k+1}{2}\geq\dfrac{n-1}{n-2},\\ (k+1)^{1/n}\left(\dfrac{(n-2)(k+1)+2}{k+1}\right) & \text{if } \dfrac{k+1}{2}<\dfrac{n-1}{n-2},\end{cases}$$

as computed by Li in [Li18, Example 5.3]. For $n\gg 1$, the lower bound provided by Theorem 5.10 is thus short of a factor $2=\mathrm{mult}(X,p)$ by comparison with the expected lower bound $\widehat{\mathrm{vol}}(A_k,p)^{1/n}$.

5.3.2 *Using resolutions.*

Proposition 5.12. *Let $\pi:\tilde{\Omega}\to\Omega$ be a resolution of singularities with simple normal crossing, and let $\{a_i\}_{i=1,\ldots,M}$ be the discrepancies. Then*

$$\alpha(X,\mu_p)\geq\frac{\widehat{\mathrm{vol}}(X,p)^{1/n}}{1+(\max_i a_i)_+}.$$

In particular, if the singularity is 'admissible', then $\alpha(X,\mu_p)=\widehat{\mathrm{vol}}(X,p)^{1/n}$.

Following [LTW21, Definition 1.1] we say here that (X,p) is an admissible singularity if there exists a resolution $\pi:\tilde{X}\to X$ (with snc exceptional divisor $E=\sum_j E_j$ and π-ample divisor $-\sum b_jE_j,\ b_j\in\mathbb{Q}^+$) such that the discrepancies $a_i\in(-1,0]$ are all non-positive. Recall that:

- any two-dimensional log terminal singularity is admissible;
- the vertex of the affine cone over a Fano manifold embedded in a projective space by the linear system associated to a multiple of the anticanonical bundle is admissible (cf. the proof of Proposition 3.17);
- (X,p) is admissible if it is $\mathbb{Q}$-factorial and admits a crepant resolution.

Theorem B from the introduction follows from the combination of Proposition 5.6, Theorem 5.10 and Proposition 5.12.

Proof. We seek $\alpha>0$ such that

$$\sup_{\varphi\in\mathcal{F}_1(\Omega)}\int_{\tilde{\Omega}}e^{-\alpha\varphi\circ\pi}\prod_{i=1}^{M}|s_i|_{h_i}^{2a_i}\,dV<+\infty. \tag{5.13}$$

If all the a_i are non-positive we can use [DK01, Main Theorem] to show that $\alpha(X,\mu_p)=\tilde{\alpha}(X,\mu_p)$, hence $\alpha(X,\mu_p)=\widehat{\mathrm{vol}}(X,p)^{1/n}$ by Proposition 5.8. Indeed, assume that there exists $\gamma>0$ such that $\alpha(X,\mu_p)<\gamma<\tilde{\alpha}(X,\mu_p)$. By definition, we can find $\psi_j\in\mathcal{F}_1(\Omega)$ such that $\int_\Omega e^{-\gamma\psi_j}\,d\mu_p\to+\infty$. Extracting and relabelling, we can assume that $\psi_j\to\psi$ in L^1 with $c(\psi)>\gamma$. The psh functions $\varphi_j=\psi_j+\gamma^{-1}\sum_{i=1}^M(-a_i)\log|s_i|_{h_i}^2$ converge to $\varphi=\psi+\gamma^{-1}\sum_{i=1}^M(-a_i)\log|s_i|_{h_i}^2$ in $L^1(\tilde{\Omega})$ and $c(\varphi)>\gamma$. It follows therefore from [DK01, Theorem 0.2.2] that

$$\int_\Omega e^{-\gamma\psi_j}\,d\mu_p=\int_{\tilde{\Omega}}e^{-\gamma\varphi_j}\,dV\longrightarrow\int_{\tilde{\Omega}}e^{-\gamma\varphi}\,dV<+\infty,$$

contradicting the assumption $\int_\Omega e^{-\gamma\psi_j}\,d\mu_p\to+\infty$.

In general, we set $U := \sum_{i:a_i>0} a_i \log|s_i|^2_{h_i}$ and $W := -\sum_{i:a_i\le 0} a_i \log|s_i|^2_{h_i}$. Using [DK01, Main Theorem], we obtain

$$\alpha(X,\mu_p) \ge \inf_{\varphi\in\mathcal{F}_1(\Omega)} c_W(\varphi\circ\pi), \tag{5.14}$$

where

$$c_W(\varphi\circ\pi) := \sup\left\{\alpha>0 : \int_{\tilde\Omega} e^{-\alpha\varphi\circ\pi - W}\, dV < +\infty\right\}$$

is the twisted complex singularity exponent. It then remains to estimate $c_W(\varphi\circ\pi)$ for a fixed $\varphi\in\mathcal{F}_1(\Omega)$. As $\pi^* d\mu_p = e^{U-W} dV$, Hölder inequality yields

$$\int_{\tilde\Omega} e^{-\alpha\varphi\circ\pi - W}\, dV \le \left(\int_{\tilde\Omega} e^{-p'\alpha\varphi\circ\pi}\pi^*\, d\mu_p\right)^{1/p'} \left(\int_{\tilde\Omega} e^{(1-q')U-W}\, dV\right)^{1/q'}. \tag{5.15}$$

Set $A := (\max_i a_i)_+ > 0$. The second factor on the right-hand side of (5.15) is finite for any $q' < ((A+1)/A)$, while the first factor on the right-hand side gives the condition $p'\alpha < \tilde\alpha = \widehat{\mathrm{vol}}(X,p)^{1/n}$. We infer $c_W(\varphi\circ\pi) \ge (\widehat{\mathrm{vol}}(X,p)^{1/n})/(1+A)$, which concludes the proof. □

As the proof shows, the main obstruction to proving the equality $\alpha(X,\mu_p) = \tilde\alpha(X,\mu_p) = \widehat{\mathrm{vol}}(X,p)^{1/n}$ is the lack of a Demailly–Kollàr result on complex spaces. Resolving the singularities, one ends up with a twisted version of Demailly and Kollàr's problem on a smooth manifold. It is known that the general form of such a problem has a negative answer [Pha14, Remark 1.3].

6. Ricci inverse iteration

In this section, we prove Theorem C from the introduction. The strategy is similar to that of [GKY13, Theorem 1], with a singular twist.

We fix $\gamma < \gamma_{\mathrm{crit}}(X,p)$ and consider, for $j\in\mathbb{N}$, the sequence of functions $\varphi_j \in PSH(\Omega)$ defined by induction as follows: pick $\varphi_0 \in \mathcal{T}^\infty_\phi(\Omega)$ a smooth initial data, and let $\varphi_{j+1} \in PSH(\Omega)\cap\mathcal{C}^0(\overline\Omega)\cap\mathcal{C}^\infty(\overline\Omega\setminus\{p\})$ be the unique solution to

$$(dd^c\varphi_{j+1})^n = \frac{e^{-\gamma\varphi_j}\mu_p}{\int_\Omega e^{-\gamma\varphi_j}\mu_p}$$

with boundary values $\varphi_{j+1|\partial\Omega} = \phi$. The existence and regularity of φ_j off the singular locus follows from [Fu23, Theorem 1.4], while the continuity of φ_j near p is a consequence of [GGZ23, Theorem A].

We are going to establish uniform *a priori* estimates on arbitrary derivatives of the φ_j in $\overline\Omega\setminus\{p\}$, thus (φ_j) admits 'smooth' cluster values. We show that the functional F_γ is constant on the set $\mathcal{K}$ of these cluster points, so that any such ψ is a solution of the Monge–Ampère equation

$$(dd^c\psi)^n = \frac{e^{-\gamma\psi}\mu_p}{\int_\Omega e^{-\gamma\psi}\mu_p}$$

with boundary values $\psi_{|\partial\Omega} = \phi$.

6.1 Uniform estimates

Proposition 6.1. *There exists $C_0>0$ such that $\|\varphi_j\|_{L^\infty(\Omega)} \le C_0$ for all $j\in\mathbb{N}$.*

This uniform estimate relies crucially on a technique introduced by Kolodziej in [Kol98], which has been extended to this singular setting in [GGZ23].

Proof. We assume without loss of generality that ϕ_0 is the maximal psh extension of ϕ in Ω. In particular, $\varphi_j \leq \phi_0$ for all $j \in \mathbb{N}$, and $E_\phi(\varphi_j) \leq E_\phi(\phi_0) = 0$. Our task is to establish a uniform lower bound $\varphi_j \geq -C_0$.

The assumption $\gamma < \gamma_{\text{crit}}(X,p)$ ensures, by Lemma 3.13, that the functional F_γ is coercive, and in particular there exist $0 < a < 1$ and $0 < b$ such that

$$F_\gamma(\varphi_j) \leq a E_\phi(\varphi_j) + b$$

for all $j \in \mathbb{N}$. It follows from [GKY13, Proposition 12] (exactly the same proof applies here) that $j \mapsto F_\gamma(\varphi_j)$ is increasing, hence

$$F_\gamma(\varphi_0) \leq F_\gamma(\varphi_j) \leq a E_\phi(\varphi_j) + b \leq b,$$

showing that the energies $(E_\phi(\varphi_j))$ are uniformly bounded, $-b' \leq E_\phi(\varphi_j) \leq 0$.

The corresponding family $\mathcal{G}_{b'}$ of psh functions with ϕ-boundary values and energy bounded by b' is compact, and all its members have zero Lelong number at all points in Ω (see Theorem 3.7). Passing through a resolution, one can thus invoke Skoda's uniform integrability theorem [GZ17, Theorem 2.50] to conclude that the densities $e^{-\gamma\varphi_j}$ are uniformly in $L^r(dV_X)$ for any $r > 1$.

Now $\mu_p = f dV_X$ with $f \in L^{1+\varepsilon}$ for some $\varepsilon > 0$ since (X,p) is log terminal. Hölder inequality thus ensures that the densities $g_j := e^{-\gamma\varphi_j} f / \int_\Omega e^{-\gamma\varphi_j}\, d\mu_p$ are uniformly in $L^{1+\varepsilon'}(dV_X)$ for some $0 < \varepsilon' < \varepsilon$.

It therefore follows from [GGZ23, Proposition 1.8] (an extension of the main result of [Kol98] to the setting of pseudoconvex subsets of a singular complex space) that the φ_j are uniformly bounded. □

6.2 $\mathcal{C}^2$-estimates

In this section we establish the following *a priori* estimates.

Proposition 6.2. *For any compact subset K of $\overline{\Omega} \setminus \{p\}$, there exists a constant $C_2(K) > 0$ such that for all $j \in \mathbb{N}$,*

$$0 \leq \sup_K \Delta_{\omega_X} \varphi_j \leq C_2(K).$$

Here $\Delta_{\omega_X} h := n((dd^c h \wedge \omega_X^{n-1})/\omega_X^n)$ denotes the Laplace operator with respect to the Kähler form ω_X. Such an estimate goes back to the regularity theory developed in [CKNS85]. The strategy of the proof is similar to that of [GKY13, Theorem 15], with a twist due to the presence of the singular point p.

Proof. To obtain these estimates, one considers a resolution of the singularity $\pi : \tilde{\Omega} \to \Omega$. We let $E = \bigcup_{\ell=1}^m E_\ell$ denote the exceptional divisor and let:

- s_ℓ denote a holomorphic section of $\mathcal{O}(E_\ell)$ such that $E_\ell = (s_\ell = 0)$;
- b_ℓ be positive rational numbers such that $-\sum_\ell b_\ell E_\ell$ is π-ample;
- h_ℓ denote a smooth hermitian metric of $\mathcal{O}(E_\ell)$ and $K \gg 1$ such that

$$\beta := K dd^c \rho \circ \pi - \sum_{\ell=1}^m b_\ell \Theta_{h_\ell} \text{ is a Kähler form on } \tilde{\Omega}.$$

Observe that the function $\rho' := K\rho \circ \pi + \sum_{\ell=1}^m b_\ell \log |s_\ell|^2_{h_\ell}$ is strictly psh in $\tilde{\Omega}$, with $dd^c \rho' \geq \beta$ and $\rho'(z) \to -\infty$ as $z \to E$.

https://doi.org/10.1112/S0010437X24007619

Recall that $\pi^*\mu_p = \Pi_{\ell=1}^m |s_\ell|^{2a_\ell} dV_{\tilde{\Omega}}$ with $a_\ell > -1$, and set $|s|^2 = \Pi_{\ell=1}^m |s_\ell|^{2b_\ell}$. We are going to show that there exist uniform constants $C_2 > 0, m \in \mathbb{N}$ such that

$$0 \le |s|^{2m} |\Delta_\beta \varphi_j|(z) \le C_2 \tag{6.1}$$

for all $j \in \mathbb{N},\ z \in \overline{\Omega}$, from which Proposition 6.2 follows. Slightly abusing notation, we still denote here by φ_j the function $\varphi_j \circ \pi$.

We approximate φ_j by the smooth solutions $\varphi_{j,\varepsilon}$ of the Dirichlet problem

$$\begin{cases} (\varepsilon\beta + dd^c \varphi_{j+1,\varepsilon})^n = \dfrac{e^{-\gamma\varphi_{j,\varepsilon}} \prod_{l=1}^m (|s_l|^2_{h_l} + \varepsilon^2)^{a_l}}{c_j} \, dV_{\tilde{\Omega}}, \\ \varphi_{j+1,\varepsilon|\partial\tilde{\Omega}} = \phi, \end{cases} \tag{6.2}$$

with $\varphi_{0,\varepsilon} = \varphi_0$ and $c_j = \int_\Omega e^{-\gamma\varphi_j} \, d\mu_p$. We are going to establish *a priori* estimates on these smooth approximants, whose existence is guaranteed by [GL10, Theorem 1.1]. We then show that $\varphi_{j,\varepsilon}$ converges to φ_j as ε decreases to zero.

Step 1. We first claim that for all j, ε,

$$\sup_{\partial\tilde{\Omega}} |\nabla \varphi_{j+1,\varepsilon}| \le A_{1,j,\varepsilon}, \tag{6.3}$$

where $A_{1,j,\varepsilon} > 0$ only depends on an upper-bound on $\|\varphi_{j,\varepsilon}\|_{L^\infty(\tilde{\Omega})}$.

Let Φ^- be a smooth psh extension of $-\phi$ to a neighborhood of $\overline{\Omega}$. Observe that $\varphi_{j+1,\varepsilon} + \Phi^- \circ \pi$ is β-psh in $\tilde{\Omega}$, with zero boundary values. Thus, $\varphi_{j+1,\varepsilon} + \Phi^- \circ \pi \le u$, where u is the smooth solution in $\tilde{\Omega}$ to the Laplace equation $\Delta_\beta u = -n$ with zero boundary values. We infer $\varphi_{j+1,\varepsilon} \le \psi_1 := u - \Phi^- \circ \pi$ in $\tilde{\Omega}$.

We now construct a psh function $\psi_2 \le \varphi_{j+1,\varepsilon}$ with ϕ-boundary values and such that $\sup_{\partial\tilde{\Omega}} |\psi_2|$ is controlled from above by $\|\varphi_{j,\varepsilon}\|_{L^\infty(\tilde{\Omega})}$. The upper bound on $\sup_{\partial\tilde{\Omega}} |\nabla \varphi_{j+1,\varepsilon}|$ thus follows from the inequalities $\psi_2 \le \varphi_{j,\varepsilon} \le \psi_1$.

Recall that $\pi^*\mu_p = \Pi_{\ell=1}^m |s_\ell|^{2a_\ell} dV_{\tilde{\Omega}}$. We let $P \subset [1, m]$ denote the subset of indices such that $-1 < a_\ell < 0$. For $\delta > 0$ small enough, we observe that $v := \rho' + \delta \sum_{\ell \in P} |s_\ell|^{2\delta}$ is strictly psh in $\tilde{\Omega}$ and satisfies, in $\tilde{\Omega} \setminus E$,

$$dd^c v \ge c \left\{ \beta + \sum_{\ell \in P} \frac{i \, ds_\ell \wedge d\overline{s_\ell}}{|s_\ell|^{2(1-\delta)}} \right\}$$

for some $c > 0$, hence $(dd^c v)^n \ge c' \pi^* \mu_p$. Replacing v by $\lambda_{j,\varepsilon} v$, we obtain

$$(\varepsilon\beta + dd^c \lambda_{j,\varepsilon} v)^n \ge \lambda_{j,\varepsilon}^n (dd^c v)^n \ge \frac{e^{-\gamma\varphi_{j,\varepsilon}} \prod_{l=1}^m (|s_l|^2_{h_l} + \varepsilon^2)^{a_l}}{c_j} \, dV_{\tilde{\Omega}},$$

for some $\lambda_{j,\varepsilon} > 0$ which only depends on an upper bound on $\|\varphi_{j,\varepsilon}\|_{L^\infty(\tilde{\Omega})}$. In other words, $\lambda_{j,\varepsilon} v$ is a subsolution to the Monge–Ampère equation in $\tilde{\Omega} \setminus E$.

We modify $\lambda_{j,\varepsilon} v$ near $\partial\tilde{\Omega}$ to produce a subsolution with the right boundary values. Let χ be a cut-off function which is 1 near E and has compact support in $\tilde{\Omega}$. The function $\psi_2 = \chi \lambda_{j,\varepsilon} v + (1 - \chi)\phi_0 + A\rho \circ \pi$ satisfies all our requirements for $A > 0$ large enough. Note, however, that it is only locally bounded in $\tilde{\Omega} \setminus E$.

Finally, consider $\max(\psi_2, \varphi_{j,\varepsilon})$. This is a subsolution of the Dirichlet problem which is globally bounded in $\tilde{\Omega}$. It follows from the maximum principle that $\max(\psi_2, \varphi_{j,\varepsilon}) \le \varphi_{j,\varepsilon}$, hence $\psi_2 \le \max(\psi_2, \varphi_{j,\varepsilon}) \le \varphi_{j,\varepsilon}$.

Step 2. We next claim that there exist constants $A_2, A_{3,j+1,\varepsilon} > 0$ such that

$$\sup_{\tilde{\Omega}}[|s|_h^{2A_2}|\nabla\varphi_{j+1,\varepsilon}|_\beta^2] \leq A_{3,j+1,\varepsilon}, \tag{6.4}$$

where $A_{3,j+1,\varepsilon}$ only depends on an upper bound on $\|\varphi_{k,\varepsilon}\|_{L^\infty(\tilde{\Omega})}$ for $k \leq j+1$.

Proof. The proof is a variant of [DFS23, Proposition 2.2], which itself relies on previous estimates due to Blocki and Phong-Sturm.

As we work in $\tilde{\Omega} \setminus \mathrm{Supp}(E)$, we identify β with $dd^c(K\rho\circ\pi + \log|s|_h^2)$. Replacing $\varphi_{j+1,\epsilon}$ by $\tilde{\varphi}_{j+1,\epsilon} := \varphi_{j+1,\epsilon} - (K\rho\circ\pi + \log|s|_h^2)$, (6.2) becomes

$$\begin{cases} ((1+\epsilon)\beta + dd^c\tilde{\varphi}_{j+1,\epsilon})^n = c_j^{-1}e^{-\gamma\varphi_{j,\epsilon}}\displaystyle\prod_{l=1}^m(|s_l|_{h_l}^2+\epsilon^2)^{a_l}\,dV_{\tilde{\Omega}}, \\ \tilde{\varphi}_{j+1,\epsilon|\partial\tilde{\Omega}} = \phi - \log|s|_h^2. \end{cases} \tag{6.5}$$

As

$$||\nabla\varphi_{j,\epsilon}|_\beta - |\nabla\tilde{\varphi}_{j,\epsilon}|_\beta| \leq \frac{|\nabla|s|_h^2|_\beta}{|s|_h} + C,$$

to get the estimate (6.4) for $\tilde{\varphi}_{j,\epsilon}$ it is enough to prove by induction that there exists positive constants $B_2, B_{3,j+1,\varepsilon}$ such that

$$\sup_{\tilde{\Omega}}[|s|_h^{2B_2}|\nabla\tilde{\varphi}_{j+1,\varepsilon}|_\beta^2] \leq \max\left\{\sup_{\tilde{\Omega}}[|s|_h^{2B_2}|\nabla\tilde{\varphi}_{j,\varepsilon}|_\beta^2], B_{3,j+1,\varepsilon}\right\}, \tag{6.6}$$

where B_2 is uniform in j,ε while $B_{3,j+1,\varepsilon}$ only depends on upper bounds on $\|\varphi_{j+1,\varepsilon}\|_{L^\infty(\tilde{\Omega})}, \|\varphi_{j,\varepsilon}\|_{L^\infty(\tilde{\Omega})}$, and where $\tilde{\varphi}_{0,\varepsilon} := -(K\rho\circ\pi + \log|s|_h^2)$. To lighten notation we rewrite the equation

$$\begin{cases} (\beta_\epsilon + dd^c u)^n = e^{-v}\displaystyle\prod_{l=1}^m(|s_l|_{h_l}^2+\epsilon^2)^{a_l}\beta_\epsilon^n, \\ u_{|\partial\tilde{\Omega}} = \tilde{\phi}, \end{cases} \tag{6.7}$$

where $\beta_\epsilon := (1+\epsilon)\beta$ is a non-degenerate smooth family of Kähler forms. Note that (6.6) becomes

$$\sup_{\tilde{\Omega}}[|s|_h^{2B_2}|\nabla u|_\beta^2] \leq \max\left\{\sup_{\tilde{\Omega}}[|s|_h^{2B_2}|\nabla(v/\gamma - \log|s|_h^2 - f_\epsilon)|_\beta^2], B_{3,j+1,\varepsilon}\right\}, \tag{6.8}$$

where $\{f_\epsilon\}_{\epsilon>0}$ is a non-degenerate smooth family. In the estimates that follow we indicate with C_i all the constants *under control*, i.e. that depend on a upper bound on $\|\varphi_{j+1,\epsilon}\|_{L^\infty(\tilde{\Omega})}, \|\varphi_{j,\epsilon}\|_{L^\infty(\Omega)}$. Observe that $\|u+\log|s|_h^2\|_{L^\infty(\tilde{\Omega})}, \|v\|_{L^\infty(\tilde{\Omega})}$ and $\sup_{\partial\tilde{\Omega}}|\nabla u|$ are under control. The constant $B_{3,j+1,\varepsilon}$ in (6.8) will clearly depend on the C_i. We indicate with D_i all the constants uniform in j,ε, which will be used to determine the uniform constant B_2 in (6.8).

We denote by Δ_ϵ and Δ'_ϵ the Laplacian operators with respect to β_ϵ and to $\eta_\epsilon := \beta_\epsilon + dd^c u$, respectively. Consider

$$H := \log|\nabla u|_{\beta_\epsilon}^2 + \log|s|_h^{2k} - G(u),$$

where $G(x) = Ax - B/(x+C+1)$ for C chosen so that $u \geq -C$, while $A>0, B>0$ to be determined later. The constants A,k are chosen to be uniform in j,ε while B is under control. If H reaches its maximum at x_M, then

$$|\nabla u|_{\beta_\epsilon}^2|s|_h^{2(k+A)} \leq C_1(|\nabla u|_{\beta_\epsilon}^2|s|_h^{2(k+A)})(x_M) \tag{6.9}$$

for a constant C_1 under control.

As $u+\log|s|_h^2$ is smooth on $\tilde{\Omega}$, we ensure that $H(x)\simeq(k+A-1)\log|s|_h^2\to-\infty$ as $x\to\mathrm{Supp}(E_j)$ by imposing $k\geq 1$. If H reaches its maximum on $\partial\tilde{\Omega}$, then we are done since $\sup_{\partial\tilde{\Omega}}|\nabla u|$ is under control. From now on we thus suppose that H reaches its maximum in $\tilde{\Omega}\setminus\{s=0\}$. A direct computation [PSS12, (5.11) and (5.20)] yields

$$\begin{aligned}\Delta_\epsilon'\log|\nabla u|^2_{\beta_\epsilon}\geq&\frac{2\mathrm{Re}\langle\nabla v+\sum_{l=1}^m a_l\nabla\log(|s_l|^2_{h_l}+\epsilon^2),\nabla u\rangle_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}}-\Lambda\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon\\&+2\mathrm{Re}\left\langle\frac{\nabla|\nabla u|^2_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}},\frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\right\rangle_{\eta_\epsilon}-2\mathrm{Re}\left\langle\frac{\nabla|\nabla u|^2_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}},\frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\right\rangle_{\beta_\epsilon},\end{aligned}\tag{6.10}$$

where Λ denotes a (uniform in ε) lower bound on the holomorphic bisectional curvature of β_ϵ. At the point where H reaches its maximum we obtain

$$\frac{\nabla|\nabla u|^2_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}}=\nabla\log|\nabla u|^2_{\beta_\epsilon}=-\nabla(\log|s|_h^{2k}-G(u))=-\frac{k\nabla|s|^2_h}{|s|^2_h}+G'(u)\nabla u,$$

hence

$$\begin{aligned}&2\mathrm{Re}\left\langle\frac{\nabla|\nabla u|^2_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}},\frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\right\rangle_{\eta_\epsilon}-2\mathrm{Re}\left\langle\frac{\nabla|\nabla u|^2_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}},\frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\right\rangle_{\beta_\epsilon}\\&\quad=2k\mathrm{Re}\left\langle\frac{\nabla|s|^2_h}{|s|^2_h},\frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\right\rangle_{\beta_\epsilon}-2k\mathrm{Re}\left\langle\frac{\nabla|s|^2_h}{|s|^2_h},\frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\right\rangle_{\eta_\epsilon}+2G'(u)\frac{|\nabla u|^2_{\eta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}}-2G'(u)\\&\quad\geq 2k\mathrm{Re}\left\langle\frac{\nabla|s|^2_h}{|s|^2_h},\frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\right\rangle_{\beta_\epsilon}-2k\mathrm{Re}\left\langle\frac{\nabla|s|^2_h}{|s|^2_h},\frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\right\rangle_{\eta_\epsilon}-2G'(u),\end{aligned}$$

using the monotonicity of $G(x)$ in the last inequality. By (6.9) and asking $k\geq 2$, we can assume that $|s|_h^2|\nabla u|_{\beta_\epsilon}\geq 1$ at x_M. Thus,

$$\left|2\mathrm{Re}\left\langle\frac{\nabla|s|^2_h}{|s|^2_h},\frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\right\rangle_{\beta_\epsilon}\right|\leq 2\left|\mathrm{Re}\left\langle\nabla|s|^2_h,\frac{\nabla u}{|\nabla u|_{\beta_\epsilon}}\right\rangle_{\beta_\epsilon}\right|\leq D_1$$

and

$$\begin{aligned}\left|2\mathrm{Re}\left\langle\frac{\nabla|s|^2_h}{|s|^2_h},\frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\right\rangle_{\eta_\epsilon}\right|\leq 2\left|\mathrm{Re}\left\langle\nabla|s|^2_h,\frac{\nabla u}{|\nabla u|_{\beta_\epsilon}}\right\rangle_{\eta_\epsilon}\right|&\leq|\nabla|s|^2_h|^2_{\eta_\epsilon}+\frac{|s|^4_h|\nabla u|^2_{\eta_\epsilon}}{|s|^4_h|\nabla u|^2_{\beta_\epsilon}}\\&\leq|\nabla|s|^2_h|^2_{\beta_\epsilon}\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon+|s|^4_h|\nabla u|^2_{\eta_\epsilon}.\end{aligned}$$

We infer that at $x=x_M$,

$$\begin{aligned}&2\mathrm{Re}\left\langle\frac{\nabla|\nabla u|^2_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}},\frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\right\rangle_{\eta_\epsilon}-2\mathrm{Re}\left\langle\frac{\nabla|\nabla u|^2_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}},\frac{\nabla u}{|\nabla u|^2_{\beta_\epsilon}}\right\rangle_{\beta_\epsilon}\\&\quad\geq-kD_1-k|\nabla|s|^2_h|^2_{\beta_\epsilon}\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon-k|s|^4_h|\nabla u|^2_{\eta_\epsilon}-2G'(u)\\&\quad\geq-kD_1-kD_2\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon-k|s|^4_h|\nabla u|^2_{\eta_\epsilon}-2G'(u),\end{aligned}$$

which is the first estimate of the right-hand side in (6.10).

Next, as we want to prove (6.8), as a consequence of (6.9) and of $|\nabla u|^2_\beta\leq D_3|\nabla u|^2_{\beta_\varepsilon}$ in the estimate that follows we can assume that

$$D_3C_1|\nabla u|^2_{\beta_\varepsilon}\geq\max\{|\nabla(v/\gamma-\log|s|^2_h-f_\varepsilon)|^2_\beta,1\}$$

at the point x_M. We deduce

$$\left|\frac{2\mathrm{Re}\langle \nabla v + \sum_{l=1}^m \nabla a_l \log(|s_l|^2_{h_l}+\epsilon^2), \nabla u\rangle_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}}\right|$$
$$\leq D_4 + \frac{|\nabla(v-\gamma\log|s|^2_h - \gamma f_\varepsilon)|^2_{\beta_\epsilon}}{|\nabla u|^2_{\beta_\epsilon}} + \frac{D_5}{|\nabla u|^2_{\beta_\varepsilon}}|s|_h^{-2} + \frac{D_6}{|\nabla u|^2_{\beta_\epsilon}}\sum_{l=1}^m |s_l|_{h_l}^{-2} \leq C_2 + C_3|s|_h^{-2M}$$

for $M := 1/\min_l b_l$ so that $Mb_l \geq 1$ for any l. The previous inequalities yield

$$\Delta'_\epsilon \log|\nabla u|^2_{\beta_\epsilon} \geq -kD_1 - C_2 - C_3|s|_h^{-2M} - (kD_2+\Lambda)\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon - k|s|^4_h|\nabla u|^2_{\eta_\epsilon} - 2G'(u). \tag{6.11}$$

Moreover,

$$-\Delta'_\epsilon G(u) = -G'(u)\Delta'_\epsilon u - G''(u)|\nabla u|^2_{\eta_\epsilon} = G'(u)\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon - nG'(u) - G''(u)|\nabla u|^2_{\eta_\epsilon}$$

and $\Delta'_\epsilon \log|s|_h^{2k} \geq -kD_7\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon$. Together with (6.11) we obtain

$$\Delta'_\epsilon H \geq (G' - kD_2 - \Lambda - kD_7)\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon - (n+2)G' - (G'' + k|s|^4_h)|\nabla u|^2_{\eta_\epsilon} - kD_1 - C_2 - C_3|s|_h^{-2M}.$$

Taking $k = M(n+1)+1$, this can be rewritten

$$\Delta'_\epsilon H \geq (G' - D_8)\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon - (n+2)G' - (G'' + D_9|s|^4_h)|\nabla u|^2_{\eta_\epsilon} - C_4|s|_h^{-2M}. \tag{6.12}$$

We now define $G(x) := (D_8+1)x - B/(x+C+1)$, where $B>0$ is so large that

$$\frac{2B}{(u+C+1)^3} - D_9|s|^4_h \geq |s|^2_h$$

at x_M. Note that B can be chosen such that it only depends on C, D_9 and on $\|u + \log|s|^2_h\|_{L^\infty(\tilde\Omega)}$, i.e. it is under control. From (6.12) we deduce at x_M

$$0 \geq \Delta'_\epsilon H \geq \mathrm{tr}_{\eta_\epsilon}\beta_\epsilon + |s|^2_h|\nabla u|^2_{\eta_\epsilon} - C_5|s|_h^{-2M}.$$

This yields $\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon \leq C_5|s|_h^{-2M}$ and $|\nabla u|^2_{\eta_\epsilon} \leq C_5|s|_h^{-2M-2}$, hence

$$|\nabla u|^2_{\beta_\epsilon} \leq |\nabla u|^2_{\eta_\epsilon}\mathrm{tr}_{\beta_\epsilon}\eta_\epsilon \leq |\nabla u|^2_{\eta_\varepsilon}(\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon)^{n-1}\left(\frac{\eta_\varepsilon^n}{\beta_\varepsilon^n}\right) \leq C_6|s|_h^{-2M}|\nabla u|^2_{\eta_\varepsilon}(\mathrm{tr}_{\eta_\epsilon}\beta_\epsilon)^{n-1} \leq C_7|s|_h^{-2k},$$

where we also used [GZ17, Lemma 14.4], the Monge–Ampère equation (6.7) and the fact that $\prod_{l=1}^m(|s_l|^2_{h_l}+\varepsilon^2)^{a_l} \leq D_{10}|s|_h^{-2M}$ as $a_l > -1$. From (6.9) we deduce $|\nabla u|^2_{\beta_\epsilon}|s|_h^{2(k+D_8+1)} \leq C_8$. As $\{\beta_\epsilon\}_{\epsilon>0}$ is a non-degenerate continuous family of Kähler forms converging to β as $\epsilon\to 0$, we get

$$|s|_h^{2(k+D_8+1)}|\nabla u|^2_\beta \leq \max\left\{\sup_{\tilde\Omega}[|s|_h^{2(k+D_8+1)}|\nabla(v/\gamma - \log|s|^2 - f_\varepsilon)|^2_\beta], C_9\right\},$$

i.e. (6.8), which concludes the proof by setting $B_2 := k + D_8 + 1$, $B_{3,j+1,\varepsilon} := C_9$. □

Step 3. Fix V a small neighborhood of $\partial\tilde\Omega$ (intersected with $\overline{\tilde\Omega}$). We claim that

$$\sup_{\partial\tilde\Omega}|\Delta_\beta\varphi_{j,\varepsilon}| \leq C_V[1 + \sup_V|\nabla\varphi_{j,\varepsilon}|^2], \tag{6.13}$$

for some uniform constant C_V independent of j,ε. This follows from a long series of estimates established in [GKY13, Lemma 18] (which itself was adapting the technique developed by [CKNS85]) when μ_p and Ω are smooth. The statement of [GKY13, Lemma 18] mentions $\sup_{\tilde\Omega}|\nabla\varphi_j|^2$; however, the arguments only involve:

– local reasonings in a small fixed neighborhood of the boundary;
– smoothness of μ_p in this neighborhood and pseudoconvexity of $\partial\tilde\Omega$.

Step 4. We now show that there exist constants $m, B_{3,j,\varepsilon} > 0$ such that

$$\sup_{\tilde{\Omega}} |s|^{2m} |\Delta_\beta \varphi_{j,\varepsilon}| \leq B_{3,j,\varepsilon} \left[1 + \sup_{\partial\tilde{\Omega}} |\Delta_\beta \varphi_{j,\varepsilon}| \right], \tag{6.14}$$

where $B_{3,j,\varepsilon}$ only depends on an upper bound on $\|\varphi_{k,\varepsilon}\|_{L^\infty(\tilde{\Omega})}$, for $k \leq j$. This is a variant of [GKY13, Lemma 17], for which we provide a detailed proof.

We set $\omega_j := \varepsilon\beta + dd^c\varphi_{j,\varepsilon}$ and observe that

$$\omega_j^n = e^{\psi_\varepsilon - \varphi_{j-1,\varepsilon} - c'_{j-1}} \beta^n,$$

where ψ_ε is a difference of quasi-psh functions in $\tilde{\Omega}$ such that $e^{\psi_\varepsilon} \leq c_1 |s|^{-2a}$ and $dd^c\psi_\varepsilon \geq -c_1|s|^{-2}\beta$ in $\tilde{\Omega}$, for some uniform constants $a, c_1 > 0$. We consider

$$H_j := \log \mathrm{Tr}_\beta(\omega_j) + \varphi_{j-1,\varepsilon} - A\varphi_{j,\varepsilon} + A\rho',$$

where $A > 0$ is chosen below. We use here the classical notation

$$\mathrm{Tr}_\eta(\omega) := n\frac{\omega \wedge \eta^{n-1}}{\eta^n} \quad \text{and} \quad \Delta_\eta(h) := n\frac{dd^c h \wedge \eta^{n-1}}{\eta^n}.$$

Either H_j reaches its maximum on $\partial\tilde{\Omega}$ and we are done, or it reaches its maximum at some point $x_j \in \tilde{\Omega} \setminus E$ since $\rho \to -\infty$ along E. We are going to estimate $\Delta_{\omega_j} H_j$ from below and use the fact that $0 \geq \Delta_{\omega_j} H_j(x_j)$.

It follows from [Siu87] that

$$\Delta_{\omega_j} \log \mathrm{Tr}_\beta(\omega_j) \geq -\frac{\mathrm{Tr}_\beta(\mathrm{Ric}(\omega_j))}{\mathrm{Tr}_\beta(\omega_j)} - B\mathrm{Tr}_{\omega_j}(\beta),$$

where $-B$ is a lower bound on the holomorphic bisectional curvature of β. Now

$$-\mathrm{Ric}(\omega_j) = -\mathrm{Ric}(\beta) + dd^c(\psi_\varepsilon - \varphi_{j-1,\varepsilon}) \geq -\omega_{j-1} - \frac{A_1}{|s|^2}\beta$$

in $\tilde{\Omega} \setminus E$. Moreover, $\mathrm{Tr}_\beta(\omega_{j-1}) \leq \mathrm{Tr}_\beta(\omega_j)\mathrm{Tr}_{\omega_j}(\omega_{j-1})$, hence

$$\Delta_{\omega_j} \log \mathrm{Tr}_\beta(\omega_j) \geq -\mathrm{Tr}_{\omega_j}(\omega_{j-1}) - \frac{nA_1}{|s|^2 \mathrm{Tr}_\beta(\omega_j)} - B\mathrm{Tr}_{\omega_j}(\beta).$$

Using that $dd^c\rho' \geq \beta$, we obtain

$$\Delta_{\omega_j} H_j \geq -An + (A - B)\mathrm{Tr}_{\omega_j}(\beta) - \frac{nA_1}{|s|^2 \mathrm{Tr}_\beta(\omega_j)}.$$

Using the classical inequality $n[\mathrm{Tr}_{\omega_j}(\beta)]^{n-1} \geq (\beta^n/\omega_j^n)\mathrm{Tr}_\beta(\omega_j)$, we infer

$$\Delta_{\omega_j} H_j \geq -An + c(A - B)e^{-\psi_\varepsilon/(n-1)}[\mathrm{Tr}_\beta(\omega_j)]^{1/(n-1)} - \frac{nA_1}{|s|^2 \mathrm{Tr}_\beta(\omega_j)}. \tag{6.15}$$

Let us stress that the constant c depends here on an upper bound on $\|\varphi_{j-1,\varepsilon}\|_{L^\infty(\tilde{\Omega})}$.

We fix A so large that $A > B$ and $\psi_\varepsilon + A\rho' \leq c'_1 - a\log|s|^2 + A\rho'$ is bounded from above. At the point x_j we obtain $0 \geq \Delta_{\omega_j} H_j$, therefore:

– either $|s|^2 \mathrm{Tr}_\beta(\omega_j) \leq 1$, hence $H_j(x_j) \leq (\varphi_{j-1,\varepsilon} - A\varphi_{j,\varepsilon} + A\rho')(x_j) \leq C$;
– or $|s|^2 \mathrm{Tr}_\beta(\omega_j) \geq 1$ and (6.15) yields $\mathrm{Tr}_\beta(\omega_j) \leq C' e^{\psi_\varepsilon(x_j)}$, hence

$$H_j(x_j) \leq \psi_\varepsilon(x_j) + A\rho'(x_j) + C'' \leq C'''.$$

Thus, H_j is uniformly bounded from above in both cases, and (6.14) follows (we use here an upper bound on $\|\varphi_{j-1,\varepsilon}\|_{L^\infty(\tilde{\Omega})}$ and $\|\varphi_{j,\varepsilon}\|_{L^\infty(\tilde{\Omega})}$).

Step 5. We finally show by induction on j that $\varphi_{j,\varepsilon}$ uniformly converges towards φ_j as ε decreases to 0. There is nothing to prove for $j=0$ since $\varphi_{0,\varepsilon}=\varphi_0$.

For $j=1$, it follows from (a slight generalization of) [GGZ23, Proposition 1.8] that $\|\varphi_{1,\varepsilon}\|_{L^\infty(\tilde{\Omega})}\le C_1$ is bounded uniformly in $\varepsilon>0$. Proceeding by induction, we similarly obtain that for all $j\in\mathbb{N}$,

$$\|\varphi_{j,\varepsilon}\|_{L^\infty(\tilde{\Omega})}\le C_j$$

is bounded uniformly in $\varepsilon>0$. By previous steps, the family $(\varphi_{j,\varepsilon})_\varepsilon$ is relatively compact in $\mathcal{C}^{1,\alpha}$ for all $0<\alpha<1$. Any cluster point ψ_j, as $\varepsilon\to 0$, is a solution of

$$(dd^c\psi_{j+1})^n=\frac{e^{-\gamma\psi_j}\mu_p}{c_j}$$

with boundary values $\psi_{j+1|\partial\Omega}=\phi$, hence $\psi_j=\varphi_j$ by uniqueness. Thus, $\varphi_{j,\varepsilon}$ converges to φ_j as ε decreases to zero, and the convergence is moreover uniform on $\tilde{\Omega}$ by [GGZ23, Proposition 1.8].

We can thus let ε tend to zero in previous inequalities. Now $\|\varphi_{j,\varepsilon}\|_{L^\infty(\tilde{\Omega})}\to\|\varphi_j\|_{L^\infty(\Omega)}$, and the latter is uniformly bounded in j by Proposition 6.1. For $\varepsilon=0$, (6.3), (6.4), (6.13) and (6.14) thus provide uniform bounds in j, and conclude the proof of (6.1). The proof of Proposition 6.2 is thus complete. □

6.3 Higher-order estimates and convergence

Once the uniform $\mathcal{C}^2$-estimate is established (Proposition 6.2), one can then linearize the complex Monge–Ampère equation and apply standard elliptic theory (Evans–Krylov method and Schauder bootstrapping) to derive higher-order estimates.

PROPOSITION 6.3. *Given K a compact subset of $\Omega\setminus\{p\}$ and $\alpha>0,\ell\in\mathbb{N}$, there exists $C(K,\ell,\alpha)>0$ such that for all $j\in\mathbb{N}$, $\|\varphi_j\|_{\mathcal{C}^{\ell,\alpha}(K)}\le C(K,\ell,\alpha)$.*

It follows that the sequence (φ_j) is relatively compact in $\mathcal{C}^\infty(\Omega\setminus\{p\})$. We let $\mathcal{K}$ denote the set of cluster values of the sequence (φ_j). Any function $\psi\in\mathcal{K}$ is:

- psh in Ω and smooth in $\Omega\setminus\{p\}$, with $\psi_{|\partial\Omega}=\phi$;
- uniformly bounded in $\overline{\Omega}$ (Proposition 6.1);
- continuous on $\overline{\Omega}$, as the uniform limit of (φ_{j_k}) (see [GGZ23, Proposition 1.8]);

The set $\mathcal{K}$ is invariant under the action of $T_\gamma:\varphi\in\mathcal{T}_\phi(\Omega)\mapsto\psi\in\mathcal{T}_\phi(\Omega)$, which associates, to a given $\varphi\in\mathcal{T}_\phi(\Omega)$, the unique solution $\psi\in\mathcal{T}_\phi(\Omega)$ to the complex Monge–Ampère equation

$$(dd^c\psi)^n=\frac{e^{-\gamma\varphi}\mu_p}{\int_\Omega e^{-\gamma\varphi}\,d\mu_p}.$$

It follows from [GKY13, Proposition 12] that the functional F_γ is constant on $\mathcal{K}$ and that $\mathcal{K}$ is pointwise invariant under the action of T_γ. Thus, a cluster value of (φ_j) provides a desired solution to Theorem C.

Appendix A.

Sèbastien Boucksom

The purpose of this appendix is to provide an alternative approach to Proposition 5.8, emphasizing the role of b-divisors. We use [BdFF12] as a main reference for what follows.

https://doi.org/10.1112/S0010437X24007619

A.1 Nef b-divisors over a point

Consider for the moment any normal singularity (X,p), and set $n := \dim X$.

In what follows, a *birational model* means a projective birational morphism $\pi\colon X_\pi \to X$ with X_π normal. A *b-divisor over p* is defined as a collection $B = (B_\pi)_\pi$ of $\mathbb{R}$-divisors B_π on X_π for all birational models π, compatible under push-forward, and such that each B_π has support in $\pi^{-1}(p)$. The $\mathbb{R}$-vector space of b-divisors over p can thus be written as the projective limit

$$\mathrm{Div}_{\mathrm{b}}(X,p) := \varprojlim_{\pi} \mathrm{Div}_p(X_\pi),$$

where $\mathrm{Div}_p(X_\pi)$ denotes the (finite-dimensional) $\mathbb{R}$-vector space of divisors on X_π with support in $\pi^{-1}(p)$, and we endow $\mathrm{Div}_{\mathrm{b}}(X,p)$ with the projective limit topology.

A b-divisor $B \in \mathrm{Div}_{\mathrm{b}}(X,p)$ is said to be *Cartier* if it is determined by some birational model π, in the sense that $B_{\pi'}$ is the pullback of B_π for any higher birational model π'. There is a symmetric, multilinear *intersection pairing*

$$(B_1,\dots,B_n) \mapsto (B_1 \cdot \dots \cdot B_n) \in \mathbb{R} \tag{A.1}$$

for Cartier b-divisors B_i, defined as the intersection number $(B_{1,\pi} \cdot \dots \cdot B_{n,\pi})$ computed on X_π for any choice of common determination π of the B_i (the result being independent of the choice of π, by the projection formula).

A *valuation centered at p* is a valuation $v\colon \mathcal{O}_{X,p} \to \mathbb{R}_{\geq 0}$ such that $v(\mathfrak{m}_p) > 0$ on the maximal ideal $\mathfrak{m}_p \subset \mathcal{O}_{X,p}$. It is further *divisorial* if it can be written as $v = c\,\mathrm{ord}_E$ for a prime divisor $E \subset \pi^{-1}(p)$ on some birational model X_π and $c \in \mathbb{Q}_{>0}$. Given a b-divisor B over $p \in X$, we then set $v(B) := c\,\mathrm{ord}_E(B_\pi)$. The function $v \mapsto v(B)$ so defined on the space $\mathrm{DivVal}(X,p)$ of divisorial valuations centered at p is homogeneous with respect to the scaling of $\mathbb{Q}_{>0}$, and this yields a topological vector space isomorphism between $\mathrm{Div}_{\mathrm{b}}(X,p)$ and the space of homogeneous functions on $\mathrm{DivVal}(X,p)$, endowed with the topology of pointwise convergence.

Pick a b-divisor B over p. If B is Cartier, we say that B is *(relatively) nef* if B_π is π-nef for some (hence, any) determination π. In the general case, we say that B is nef if it can be written as a limit of nef Cartier b-divisors. By the negativity lemma, any nef b-divisor $B \in \mathrm{Div}_{\mathrm{b}}(X,p)$ is automatically antieffective, i.e. $v(B) \leq 0$ for all $v \in \mathrm{DivVal}(X,p)$. By [BdFF12, Lemma 2.10], we further have the following result.

Lemma A.1. *A b-divisor B over p is nef if and only if, for each birational model π, the numerical class of B_π in $\mathrm{N}^1(X_\pi/X)$ is* nef in codimension 1 *(also known as* movable*).*

Example A.2. Consider an ideal $\mathfrak{a} \subset \mathcal{O}_{X,p}$, and assume that $\mathfrak{a}$ is *primary*, i.e. containing some power of the maximal ideal. Then $\mathfrak{a}$ determines a nef Cartier b-divisor $Z(\mathfrak{a})$, defined by $v(Z(\mathfrak{a})) = -v(\mathfrak{a})$ for each $v \in \mathrm{DivVal}(X,p)$, and determined on the normalized blow-up of $\mathfrak{a}$. For any tuple of primary ideals $\mathfrak{a}_1,\dots,\mathfrak{a}_n$,

$$e(\mathfrak{a}_1,\dots,\mathfrak{a}_n) = -(Z(\mathfrak{a}_1) \cdot \dots \cdot Z(\mathfrak{a}_n))$$

further coincides with the mixed multiplicity of the $\mathfrak{a}_i$.

Example A.3. For any valuation v centered at p, the valuation ideals

$$\mathfrak{a}_m(v) := \{f \in \mathcal{O}_{X,p} \mid v(f) \geq m\}$$

define a graded sequence of primary ideals $\mathfrak{a}_\bullet(v)$, and hence a nef b-divisor over p

$$Z(v) := Z(\mathfrak{a}_\bullet(v)) = \lim_m m^{-1} Z(\mathfrak{a}_m(v)),$$

(see [BdFF12, Lemma 2.11]), which is not Cartier in general.

Lemma A.4. *If $B \in \mathrm{Div}_{\mathrm{b}}(X,p)$ is nef, then $B \le -v(B)Z(v)$ for all $v \in \mathrm{DivVal}(X,p)$,*

Proof. Write $v = c\,\mathrm{ord}_E$ for a prime divisor E on X_π and $c \in \mathbb{Q}_{>0}$. Then $Z(v)$ coincides with $\mathrm{Env}_\pi(-c^{-1}E)$ (see [BdFF12, Definition 2.3]), and the result thus follows from [BdFF12, Proposition 2.12]. □

A.2 Normalized volume and *b*-divisors

From now on, we assume that the normal singularity $p \in X$ is further *isolated*.

By [BdFF12, Theorem 4.14], the intersection pairing (A.1) then extends to arbitrary tuples of nef b-divisors over p. This extended pairing takes values in $\mathbb{R} \cup \{-\infty\}$, and is symmetric, additive and non-decreasing in each variable, and continuous along decreasing nets.

Definition A.5. For any nef b-divisor B over p, we define the *Hilbert–Samuel multiplicity* of B as

$$e(B) := -B^n \in [0, +\infty].$$

When (X,p) is further klt, we define the *log canonical threshold* of B as

$$\mathrm{lct}(B) := \inf_{v \in \mathrm{DivVal}(X,p)} \frac{\mathrm{A}_X(v)}{-v(B)} \in [0, +\infty),$$

where $\mathrm{A}_X(v) \ge 0$ denotes the log discrepancy of v.

Example A.6. For any primary ideal $\mathfrak{a} \subset \mathcal{O}_{X,p}$, the associated nef Cartier b-divisor $B := Z(\mathfrak{a})$ (see Example A.2) satisfies $e(B) = e(\mathfrak{a})$, and $\mathrm{lct}(B) = \mathrm{lct}(\mathfrak{a})$ when (X,p) is klt.

Example A.7. Pick any valuation v centered at p, with associated nef b-divisor $Z(v)$ (see Example A.3). Then it follows from [BdFF12, Remark 4.17] that the volume $\mathrm{Vol}(v) := \lim_{m\to\infty}(n!/m^n)\dim \mathcal{O}_{X,p}/\mathfrak{a}_m(v)$ satisfies

$$\mathrm{Vol}(v) = e(Z(v)). \tag{A.2}$$

Lemma A.8. *For each nef b-divisor B over p, we have $e(B) = \sup_{C \ge B} e(C)$, where C ranges over all nef Cartier b-divisors of the form $C = m^{-1}Z(\mathfrak{a})$ for a primary ideal $\mathfrak{a} \subset \mathcal{O}_{X,p}$ and $m \in \mathbb{Z}_{>0}$, and such that $C \ge B$.*

Proof. Since B is the limit of the decreasing net $(\mathrm{Env}_\pi(B_\pi))$ (see [BdFF12, Remark 2.17]), it is enough to prove the result when $B = \mathrm{Env}_\pi(B_\pi)$, by continuity of the intersection pairing along decreasing nets. By [BdFF12, Theorem 4.11], we can then write B as the limit of a decreasing sequence (C_i) of nef Cartier b-divisors of the desired form, and we are done since $e(C_i) \to e(B)$. □

Consider now a psh function φ on X. The collection of its Lelong numbers on all birational models defines a homogeneous function $v \mapsto v(\varphi)$ on $\mathrm{DivVal}(X,p)$, and hence an antieffective b-divisor $Z(\varphi)$ over p, such that $v(Z(\varphi)) = -v(\varphi)$.

Proposition A.9. *The b-divisor $Z(\varphi)$ is nef. Furthermore:*

(i) *if φ is locally bounded outside p, then*

$$e(Z(\varphi)) \le e(\varphi) := (dd^c\varphi)^n(\{p\});$$

(ii) *if (X,p) is klt, then $\mathrm{lct}(Z(\varphi)) = \mathrm{lct}(\varphi)$.*

Proof. Consider the closed positive $(1,1)$-current $T := dd^c\varphi$, and pick a log resolution $\pi\colon X_\pi \to X$ of (X,p). The Siu decomposition of $\pi^\star T = dd^c\pi^\star\varphi$ shows that $\pi^\star T + [Z(\varphi)_\pi]$ is a positive current with zero generic Lelong numbers along each component of $\pi^{-1}(p)$. By Demailly regularization, it follows that the class of $Z(\varphi)_\pi$ in $\mathrm{N}^1(X_\pi/X)$ is nef in codimension 1, and hence that $Z(\varphi)$ is nef (see Lemma A.1).

Assume next that φ is locally bounded outside p, and pick a primary ideal $\mathfrak{a}\subset\mathcal{O}_{X,p}$ and $m\in\mathbb{Z}_{>0}$ such that $C:=m^{-1}Z(\mathfrak{a})\geq Z(\varphi)$. Choose a finite set of local generators (f_i) of $\mathfrak{a}$, and consider the psh function $\psi:=m^{-1}\log\sum_i|f_i|$. Then $Z(\varphi)\leq C=Z(\psi)$ and, hence, $\varphi\leq\psi+O(1)$ (to see this, pull back φ and ψ to a log resolution of $\mathfrak{a}$, and use the Siu decomposition). By Demailly's comparison theorem, it follows that $e(C)=e(\psi)\leq e(\varphi)$, and taking the supremum over C yields part (i), by Lemma A.8.

Finally, part (ii) is a rather simple consequence of [BBJ21, Theorem B.5] applied to the pullback of φ to a log resolution of (X,p). ☐

We can now state the following variant of Proposition 5.8.

Theorem A.10. *Let (X,p) be an isolated klt singularity. Then*

$$\widehat{\mathrm{vol}}(X,p)=\inf_B e(B)\,\mathrm{lct}(B)^n=\inf_\varphi e(\varphi)\,\mathrm{lct}(\varphi)^n,$$

where B runs over all nef b-divisors over p, and φ runs over all psh functions on X that are locally bounded outside p.

Proof. By Theorem 2.16 we have $\widehat{\mathrm{vol}}(X,p)=\inf_{\mathfrak{a}} e(\mathfrak{a})\,\mathrm{lct}(\mathfrak{a})^n$, where $\mathfrak{a}\subset\mathcal{O}_{X,p}$ runs over all primary divisors, and hence $\widehat{\mathrm{vol}}(X,p)\geq\inf_B e(B)\,\mathrm{lct}(B)^n$, by Example A.6. Conversely, pick a nef b-divisor B over p. For any $v\in\mathrm{DivVal}(X,p)$, Lemma A.4 yields $B\leq -v(B)Z(v)$. By monotonicity and homogeneity of the intersection pairing, this yields $B^n\leq(-v(B))^nZ(v)^n$, i.e. $e(B)\geq(-v(B))^n\mathrm{Vol}(v)$, by (A.2). Thus,

$$e(B)\left(\frac{\mathrm{A}_X(v)}{-v(B)}\right)^n\geq \mathrm{A}_X(v)^n\mathrm{Vol}(v)\geq\widehat{\mathrm{vol}}(X,p).$$

Taking the infimum over v yields $e(B)\,\mathrm{lct}(B)^n\geq\widehat{\mathrm{vol}}(X,p)$ for any nef b-divisor B over p, and hence also $e(\varphi)\,\mathrm{lct}(\varphi)^n\geq\widehat{\mathrm{vol}}(X,p)$ for any psh function φ locally bounded outside p, by Proposition A.9. ☐

Acknowledgements

We thank Sébastien Boucksom for writing Appendix A, which enhances Proposition 5.8. We are grateful to the referee for suggesting many improvements.

Conflicts of interest

None.

Financial support

The authors are supported by the project Hermetic (ANR-11-LABX-0040) and the ANR project Paraplui. The second author is supported by a grant from the Knut and Alice Wallenberg foundation.

Journal information

Compositio Mathematica is owned by the Foundation Compositio Mathematica and published by the London Mathematical Society in partnership with Cambridge University Press. All surplus income from the publication of *Compositio Mathematica* is returned to mathematics and higher education through the charitable activities of the Foundation, the London Mathematical Society and Cambridge University Press.

References

ACC12 P. Åhag, U. Cegrell and R. Czyż, *On Dirichlet's principle and problem*, Math. Scand. **110** (2012), 235–250.

ACKPZ09 P. Åhag, U. Cegrell, S. Kołodziej, H. H. Pham and A. Zeriahi, *Partial pluricomplex energy and integrability exponents of PSH functions*, Adv. Math. **222** (2009), 2036–2058.

BGL22 B. Bakker, H. Guenancia and C. Lehn, *Algebraic approximation and decomposition theorem for Kähler Calabi-Yau varieties*, Invent. Math. **228** (2022), 1255–1308.

BT82 E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. **149** (1982), 1–40.

BB17 R. Berman and B. Berndtsson, *The volume of Kähler–Einstein Fano varieties and convex bodies*, J. Reine Angew. Math. **723** (2017), 127–152.

BB22 R. Berman and B. Berndtsson, *Moser–Trudinger type inequalities for complex Monge–Ampère operators and Aubin's "hypothèse fondamentale"*, Ann. Fac. Sci. Toulouse Math. (6) **31** (2022), 595–645.

BB10 R. J. Berman and S. Boucksom, *Growth of balls of holomorphic sections and energy at equilibrium*, Invent. Math. **181** (2010), 337–394.

BBEGZ19 R. J. Berman, S. Boucksom, P. Eyssidieux, V. Guedj and A. Zeriahi, *Kähler–Einstein metrics and Kähler–Ricci flow on log Fano varieties*, J. Reine Angew. Math. **751** (2019), 27–89.

BBGZ13 R. J. Berman, S. Boucksom, V. Guedj and A. Zeriahi, *A variational approach to complex Monge–Ampère equations*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 179–245.

BBJ21 R. J. Berman, S. Boucksom and M. Jonsson, *A variational approach to the Yau–Tian–Donaldson conjecture*, J. Amer. Math. Soc. **34** (2021), 605–652.

Blo06 Z. Błocki, *The domain of definition of the complex Monge–Ampère operator*, Amer. J. Math. **128** (2006), 519–530.

Blu18 H. Blum, *Existence of valuations with smallest normalized volume*, Compositio Math. **154** (2018), 820–849.

BdFF12 S. Boucksom, T. de Fernex and C. Favre, *The volume of an isolated singularity*, Duke Math. J. **161** (2012), 1455–1520.

BFJ08 S. Boucksom, C. Favre and M. Jonsson, *Valuations and plurisubharmonic singularities*, Publ. Res. Inst. Math. Sci. **44** (2008), 449–494.

CKNS85 L. Caffarelli, J. J. Kohn, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge–Ampère, and uniformly elliptic, equations*, Comm. Pure Appl. Math. **38** (1985), 209–252.

Ceg98 U. Cegrell, *Pluricomplex energy*, Acta Math. **180** (1998), 187–217.

Ceg04 U. Cegrell, *The general definition of the complex Monge–Ampère operator*, Ann. Inst. Fourier (Grenoble) **54** (2004), 159–179.

CDS15 X. X. Chen, S. Donaldson and S. Sun, *Kähler–Einstein metrics on Fano manifolds, I, II & III*, J. Amer. Math. Soc. **28** (2015), 183–197, 199–234 & 235–278.

DFS23 V. Datar, X. Fu and J. Song, *Kähler–Einstein metric near an isolated log canonical singularity*, J. Reine Angew. Math. **797** (2023), 79–116.

Dem85 J.-P. Demailly, *Mesures de Monge–Ampère et caractérisation géométrique des variétés algébriques affines*, Mém. Soc. Math. Fr. (N.S.) **19** (1985).

Dem09 J.-P. Demailly, Estimates on Monge–Ampère operators derived from a local algebra inequality, in *Complex analysis and digital geometry* (Uppsala Universitet, Uppsala, 2009), 131–143.

Dem12 J.-P. Demailly, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, vol. 1 (International Press–Higher Education Press, Beijing, 2012).

DK01 J.-P. Demailly and J. Kollàr, *Semi-continuity of complex singularity exponents and Kähler–Einstein metrics on Fano orbifolds*, Ann. Sci. Éc. Norm. Supéer. (4) **34** (2001), 525–556.

DP14 J.-P. Demailly and H. H. Pham, *A sharp lower bound for the log canonical threshold*, Acta Math. **212** (2014), 1–9.

Dru18 S. Druel, *A decomposition theorem for singular spaces with trivial canonical class of dimension at most five*, Invent. Math. **211** (2018), 245–296.

ELS03 L. Ein, R. Lazarsfeld and K. E. Smith, *Uniform approximation of Abhyankar valuation ideals in smooth function fields*, Amer. J. Math. **125** (2003), 409–440.

EGZ09 P. Eyssidieux, V. Guedj and A. Zeriahi, *Singular Kähler–Einstein metrics*, J. Amer. Math. Soc. **22** (2009), 607–639.

FN80 J.-E. Fornæss and R. Narasimhan, *The Levi problem on complex spaces with singularities*, Math. Ann. **248** (1980), 47–72.

Fu23 X. Fu, *Dirichlet problem of complex Monge–Ampère equation near an isolated KLT singularity*, Commun. Contemp. Math. **25** (2023), 2250019.

Fuj19 K. Fujita, *Uniform K-stability and plt blowups of log Fano pairs*, Kyoto J. Math. **59** (2019), 399–418.

GL10 B. Guan and Q. Li, *Complex Monge–Ampère equations and totally real submanifolds*, Adv. Math. **225** (2010), 1185–1223.

GZ15 Q. Guan and X. Zhou, *A proof of Demailly's strong openness conjecture*, Ann. of Math. (2) **182** (2015), 605–616.

GGZ23 V. Guedj, H. Guenancia and A. Zeriahi, *Continuity of singular Kähler–Einstein potentials*, Int. Math. Res. Not. IMRN **2** (2023), 1355–1377.

GKY13 V. Guedj, B. Kolev and N. Yeganefar, *Kähler–Einstein fillings*, J. Lond. Math. Soc. (2) **88** (2013), 737–760.

GZ17 V. Guedj and A. Zeriahi, *Degenerate complex Monge–Ampère equations*, EMS Tracts in Mathematics, vol. 26 (European Mathematical Society (EMS), Zürich, 2017).

Har77 R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52 (Springer, New York–Heidelberg, 1977).

HS17 H.-J. Hein and S. Sun, *Calabi-Yau manifolds with isolated conical singularities*, Publ. Math. Inst. Hautes Études Sci. **126** (2017), 73–130.

HP19 A. Höring and T. Peternell, *Algebraic integrability of foliations with numerically trivial canonical bundle*, Invent. Math. **216** (2019), 395–419.

Kol98 S. Kołodziej, *The complex Monge–Ampère equation*, Acta Math. **180** (1998), 69–117.

Li18 C. Li, *Minimizing normalized volumes of valuations*, Math. Z. **289** (2018), 491–513.

Li22 C. Li, *G-uniform stability and Kähler–Einstein metrics on Fano varieties*, Invent. Math. **227** (2022), 661–744.

LTW21 C. Li, G. Tian and F. Wang, *On Yau–Tian–Donaldson conjecture for singular Fano varieties*, Comm. Pure Appl. Math. **74** (2021), 1748–1800.

LWX21 C. Li, X. Wang and C. Xu, *Algebraicity of the metric tangent cones and equivariant K-stability*, J. Amer. Math. Soc. **34** (2021), 1175–1214.

Liu18 Y. Liu, *The volume of singular Kähler–Einstein Fano varieties*, Compositio Math. **154** (2018), 1131–1158.

LXZ22 Y. Liu, C. Xu and Z. Zhang, *Finite generation for valuations computing stability thresholds and applications to K-stability*, Ann. of Math. (2) **196** (2022), 507–566.

Pha14 H. H. Pham, *The weighted log canonical threshold*, C. R. Math. Acad. Sci. Paris **352** (2014), 283–288.

Pha18 H. H. Pham, *Log canonical thresholds and Monge–Ampère masses*, Math. Ann. **370** (2018), 555–566.

PSS12 D. H. Phong, J. Song and K. Sturm, *Complex Monge–Ampère equations*, Surv. Differ. Geom. **17** (2012), 327–410.

Siu87 Y.-T. Siu, *Lectures on Hermitian–Einstein metrics for stable bundles and Kähler–Einstein metrics*, DMV Seminar, vol. 8 (Birkhäuser, Basel, 1987).

Tia87 G. Tian, *On Kähler–Einstein metrics on certain Kähler manifolds with* $C_1(M) > 0$, Invent. Math. **89** (1987), 225–246.

Yau78 S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. **31** (1978), 339–411.

Zer09 A. Zeriahi, *A stronger version of Demailly's estimate on Monge–Ampère operators*, in *Complex analysis and digital geometry* (Uppsala Universitet, Uppsala, 2009), 144–146.

Vincent Guedj vincent.guedj@math.univ-toulouse.fr
Institut de Mathématiques de Toulouse et Institut Universitaire de France, Université de Toulouse, 118 route de Narbonne, 31400 Toulouse, France

Antonio Trusiani trusiani@chalmers.se
Department of Mathematical Sciences, Chalmers University of Technology, Chalmers tvärgata 3, 41258 Göteborg, Sweden