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SPECTRAL AND ASYMPTOTIC PROPERTIES OF RESOLVENT-DOMINATED OPERATORS

FRANK RÄBIGER and MANFRED P. H. WOLFF

Dedicated to Professor H. H. Schaefer on the occasion of his 75th birthday

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Abstract

Let A and B be (not necessarily bounded) linear operators on a Banach lattice E such that $|(s-B)^{-1}x| \leq (s-A)^{-1}|x|$ for all x in E and sufficiently large $s \in \mathbb{R}$. The main purpose of this paper is to investigate the relation between the spectra $\sigma(B)$ and $\sigma(A)$ of B and A, respectively. We apply our results to study asymptotic properties of dominated C_0-semigroups.

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1. Introduction

A linear operator A with domain $\mathscr{D}(A)$ on a Banach lattice E is called *resolvent-positive* if the resolvent $R(s, A) := (s-A)^{-1}$ of A at s is positive for sufficiently large $s \in \mathbb{R}$. Resolvent-positive operators were studied in detail by Arendt [4]. In particular, he showed that positivity of the resolvent has a strong influence on the existence and uniqueness of solutions of the associated Cauchy problem

$$(CP)_A \qquad \begin{aligned} \dot{u}(t) &= Au(t), \quad t \geq 0, \\ u(0) &= x. \end{aligned}$$

On the other hand, it is well-known that there is a close connection between properties of the spectrum $\sigma(A)$ of an operator A and the asymptotic behaviour of solutions of $(CP)_A$. In [12, 13] (see also [19]) Greiner showed that the spectrum of most resolvent-positive operators exhibits a particular symmetry. Especially for well-posed Cauchy problems, that is, if A generates a C_0-semigroup $(T(t))_{t\geq 0}$ of operators on E, this has far-reaching consequences concerning the asymptotic behaviour of the semigroup $(T(t))_{t\geq 0}$ (see [12, 13, 19]).

In applications as well as for theoretical reasons it is often important to replace $(CP)_A$ by a perturbed Cauchy problem:

$$(CP)_B \qquad \begin{aligned} \dot{u}(t) &= Bu(t), \quad t \geq 0, \\ u(0) &= x. \end{aligned}$$

In many such situations it happens that the resolvents of A and B are comparable for the order induced by the Banach lattice E (see for example [4, 6, 7, 12, 13, 19, 26]).

The present paper is the continuation of our investigations in [22]. We consider operators A and B on a Banach lattice E such that the resolvent of B is dominated by the resolvent of A, that is,

$$|R(s, B)x| \leq R(s, A)|x|$$

for $x \in E$ and sufficiently large $s \in \mathbb{R}$. Our aim is to show that in such a situation certain spectral properties of A are inherited by B. This allows to deduce asymptotic properties of the solutions of $(CP)_B$ from asymptotic properties of the solutions of $(CP)_A$. Our approach is very general and based on pseudo-resolvents. In Section 2 we first recall some basic facts on pseudo-resolvents and discuss special properties of positive and dominated pseudo-resolvents. Section 3, Section 4 and Section 5 are devoted to the inheritance of spectral properties of dominated pseudo-resolvents. The special case of dominated resolvents and dominated C_0-semigroups is discussed in Section 6. Applications to the asymptotic behaviour of dominated semigroups are given in Section 7.

We point out that often resolvent-positivity and domination between resolvents can be verified without any knowledge of the resolvents themselves. For instance, resolvent-positivity of a densely defined operator $(A, \mathscr{D}(A))$ on a Banach lattice E is closely connected with the *Kato inequality*

$$(K) \qquad \operatorname{Re}\langle sg(x)Ax, \varphi\rangle \leq \langle |x|, A'\varphi\rangle, \quad x \in \mathscr{D}(A),\ 0 \leq \varphi \in \mathscr{D}(A'),$$

(see [3, 4, 9, 19, 24] and the references therein). If $(A, \mathscr{D}(A))$ and $(B, \mathscr{D}(B))$ are densely defined operators on a Banach lattice E and A is resolvent-positive, then the resolvent of B is dominated by the resolvent of A if the *generalized Kato inequality*

$$(GK) \qquad \operatorname{Re}\langle sg(x)Bx, \varphi\rangle \leq \langle |x|, A'\varphi\rangle, \quad x \in \mathscr{D}(B),\ 0 \leq \varphi \in \mathscr{D}(A')$$

holds. If A and B are generators of strongly continuous semigroups this has been shown by Arendt and Schep (see [4, 19, 24]), and an easy modification of the proof in [19, C-II.4.2] yields the general case.

Our notation is standard and follows mainly the books [23] and [19]. Unexplained notions can be found there. Throughout the whole paper we consider spaces over the complex field $\mathbb{C}$. If $r \in \mathbb{R}$ we set $\mathbb{C}_r := \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > r\}$. For a given Banach space E we denote by $\mathscr{L}(E)$ the space of bounded linear operators on E and by E' the (*topological*) *dual* of E. If A is a linear operator on E with domain $\mathscr{D}(A)$, then $\sigma(A)$ denotes the *spectrum*, $\sigma_p(A)$ the *point spectrum*, $r(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}$ the *spectral radius*, $s(A) := \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\}$ the *spectral bound*, $\rho(A) := \mathbb{C} \setminus \sigma(A)$ the *resolvent set* and $R(\cdot, A) = (\cdot - A)^{-1} : \rho(A) \to \mathscr{L}(E)$ the *resolvent* of A. We call $\sigma_\pi(A) := \sigma(A) \cap (s(A) + i\mathbb{R})$ the *peripheral spectrum* and $\sigma_u(A) := \sigma(A) \cap i\mathbb{R}$ the *unitary spectrum* of A. Analogously, the *peripheral point spectrum* $\sigma_{p,\pi}(A)$ and the *unitary point spectrum* $\sigma_{p,u}(A)$ is defined.

If E is a complex Banach lattice with modulus $|\cdot|$, then $E_+ := \{x \in E : x = |x|\}$ is the set of *positive* elements in E. The dual E' is again a Banach lattice and $x' \in E'$ is positive if and only if $\langle x', x\rangle \geq 0$ for all $x \in E_+$. For operators $S, T \in \mathscr{L}(E)$ we write $S \leq T$ if $(T - S)E_+ \subseteq E_+$ and T is called *positive* if $0 \leq T$. We say that S is *dominated* by T if $|Sx| \leq T|x|$ for $x \in E$.

2. Pseudo-resolvents

2.1. Elementary results on pseudo-resolvents In this section we introduce pseudo-resolvents on Banach spaces and collect their most important properties. In the following E always denotes a Banach space.

DEFINITION 2.1. Let $\emptyset \neq D \subseteq \mathbb{C}$. A mapping $\mathscr{R} : D \to \mathscr{L}(E)$ is called a *pseudo-resolvent* on E if $\mathscr{R}$ satisfies the resolvent equation

$$(1) \qquad \mathscr{R}(\lambda) - \mathscr{R}(\mu) = -(\lambda - \mu)\mathscr{R}(\lambda)\mathscr{R}(\mu) \quad \text{for} \quad \lambda, \mu \in D.$$

We give some examples of pseudo-resolvents.

EXAMPLE 2.2. (a) Let $(A, \mathscr{D}(A))$ be an operator on E with non-empty resolvent set $\rho(A)$. Then the resolvent $\mathscr{R}_A = R(\cdot, A) : \rho(A) \to \mathscr{L}(E)$ is a pseudo-resolvent. Note that not every pseudo-resolvent is the restriction of the resolvent of an operator.
(b) Let $(T(t))_{t>0}$ be a locally integrable semigroup in $\mathscr{L}(E)$, that is, $T(\cdot)x$ is integrable for all $x \in E$ on every finite subinterval of $(0, \infty)$. In this case $\mathscr{T} = (T(t))_{t>0}$ is strongly continuous and the growth bound $\omega(\mathscr{T}) = \lim_{t\to\infty} 1/t \log \|T(t)\|$ is finite (see [14, Theorem 10.2.3 and page 306]). For $\operatorname{Re} \lambda > \omega(\mathscr{T})$ and $x \in E$

the Bochner integral $\mathscr{R}(\lambda)x := \int_0^\infty T(t)x\,dt$ exists and $\mathscr{R}(\lambda) \in \mathscr{L}(E)$. Then the mapping $\mathscr{R} : \mathbb{C}_{\omega(\mathscr{T})} \to \mathscr{L}(E)$ is a pseudo-resolvent (see [14, Theorem 18.4.1]).

The following extension property of pseudo-resolvents is well-known (see [14, Theorem 5.8.6]).

PROPOSITION 2.3. *Let* $\mathscr{R} : D \to \mathscr{L}(E)$ *be a pseudo-resolvent. Then* $\mathscr{R}$ *has a unique maximal extension* $\mathscr{R}_{\max} : D_{\max} \to \mathscr{L}(E)$ *to a pseudo-resolvent. Moreover, the following assertions hold*:

(a) $D_{\max} \subseteq \mathbb{C}$ *is open.*

(b) *For fixed* $\lambda_0 \in D$ *we have* $\lambda \in D_{\max} \setminus \{\lambda_0\}$ *if and only if* $(\lambda_0 - \lambda)^{-1} \in \rho(\mathscr{R}(\lambda_0))$, *and* $\mathscr{R}_{\max}$ *is given by*

$$\begin{aligned} \mathscr{R}_{\max}(\lambda) &= \mathscr{R}(\lambda_0)(I - (\lambda_0 - \lambda)\mathscr{R}(\lambda_0))^{-1} \\ &= \frac{1}{\lambda_0 - \lambda}\mathscr{R}(\lambda_0)\left(\frac{1}{\lambda_0 - \lambda} - \mathscr{R}(\lambda_0)\right)^{-1} \quad \text{for } \lambda \in D_{\max}. \end{aligned} \tag{2}$$

Note that (2) implies that $\mathscr{R}_{\max} : D_{\max} \to \mathscr{L}(E)$ is analytic, and hence every pseudo-resolvent is the restriction of an analytic $\mathscr{L}(E)$-valued function. The previous proposition has the following immediate consequence.

COROLLARY 2.4. *Let* $\mathscr{R} : D \to \mathscr{L}(E)$ *be a pseudo-resolvent with maximal extension* $\mathscr{R}_{\max} : D_{\max} \to \mathscr{L}(E)$ *and let* $\lambda_0 \in D$. *Then the following assertions hold*:

(a) $\rho(\mathscr{R}(\lambda_0)) = \{(\lambda_0 - \lambda)^{-1} : \lambda \in D_{\max} \setminus \{\lambda_0\}\}$ *and* $D_{\max} = \{\lambda_0 - 1/\mu : \mu \in \rho(R(\lambda_0))\} \cup \{\lambda_0\}$.

(b) $\mathscr{R}_{\max}(\lambda) = \sum_{n\geq 0}(\lambda_0 - \lambda)^n \mathscr{R}(\lambda_0)^{n+1}$ *for* $|\lambda - \lambda_0| < r(\mathscr{R}(\lambda_0))^{-1}$.

(c) *If* $\mathscr{R}(\lambda_0) = R(\lambda_0, A)$ *for some operator* $(A, D(A))$ *on* E, *then* $D_{\max} = \rho(A)$ *and* $\mathscr{R}_{\max}(\lambda) = R(\lambda, A)$ *for* $\lambda \in D_{\max}$.

We now define the singular set of a pseudo-resolvent.

DEFINITION 2.5. Let $\mathscr{R} : D \to \mathscr{L}(E)$ be a pseudo-resolvent on the Banach space E with maximal extension $\mathscr{R}_{\max} : D_{\max} \to \mathscr{L}(E)$.

(a) The set $\operatorname{sing}(\mathscr{R}) := \mathbb{C} \setminus D_{\max}$ is called the *singular set* or *set of singular values* of $\mathscr{R}$.

(b) By $s(\mathscr{R}) := \inf\{r \in \mathbb{R} : \mathbb{C}_r \subseteq D_{\max}\}$ we denote the *singular bound* of $\mathscr{R}$.

(c) We call $\operatorname{sing}_\pi(\mathscr{R}) := \operatorname{sing}(\mathscr{R}) \cap (s(\mathscr{R}) + i\mathbb{R})$ the *peripheral singular set* and $\operatorname{sing}_u(\mathscr{R}) := \operatorname{sing}(\mathscr{R}) \cap i\mathbb{R}$ the *unitary singular set* of $\mathscr{R}$.

(d) A complex number λ is said to be a *pole* of $\mathscr{R}$ if $\lambda \in D_{\max}$ and λ is a pole of $\mathscr{R}_{\max}$. If the associated residuum is of finite rank r, then λ is called a *Riesz point of order* r.

If $\mathscr{R}$ is (the restriction of) the resolvent of an operator A on E, then $\operatorname{sing}(\mathscr{R})$ coincides with $\sigma(A)$ and the singular bound $s(\mathscr{R})$ is exactly the spectral bound $s(A)$ of A.

Note that for a pseudo-resolvent $\mathscr{R} : D \to \mathscr{L}(E)$ and $\lambda \in D$ Corollary 2.4 yields

$$\sigma(\mathscr{R}(\lambda)) \setminus \{0\} = \left\{ \frac{1}{\lambda - \mu} : \mu \in \operatorname{sing}(\mathscr{R}) \right\} \quad \text{and} \tag{3}$$

$$\operatorname{sing}(\mathscr{R}) = \left\{ \lambda - \frac{1}{\mu} : \mu \in \sigma(\mathscr{R}(\lambda)) \setminus \{0\} \right\}. \tag{4}$$

Next we define eigenvalues and eigenvectors of a pseudo-resolvent $\mathscr{R} : D \to \mathscr{L}(E)$.

DEFINITION 2.6. Let $\alpha \in \mathbb{C}$ and $0 \neq z \in E$. Then α is called an *eigenvalue* of $\mathscr{R}$ with corresponding *eigenvector* z if

$$(\lambda - \alpha)\mathscr{R}(\lambda)z = z \tag{5}$$

for all $\lambda \in D$. We denote by $\operatorname{sing}_p(\mathscr{R})$ the set of eigenvalues of $\mathscr{R}$, by $\operatorname{sing}_{p,\pi}(\mathscr{R}) := \operatorname{sing}_p(\mathscr{R}) \cap (s(\mathscr{R}) + i\mathbb{R})$ the set of peripheral eigenvalues, and by $\operatorname{sing}_{p,u}(\mathscr{R}) := \operatorname{sing}_p(\mathscr{R}) \cap i\mathbb{R}$ the set of unitary eigenvalues.

Note that (3) implies $\operatorname{sing}_p(\mathscr{R}) \subseteq \operatorname{sing}(\mathscr{R})$. If $\mathscr{R} = \mathscr{R}_A$ is the resolvent of an operator A, then $\operatorname{sing}_p(\mathscr{R}_A)$ is exactly the point spectrum $\sigma_p(A)$ of A. Equation (1) leads to the following observation (see [19, C-III.2.6]).

LEMMA 2.7. *Let $\mathscr{R} : D \to \mathscr{L}(E)$ be a pseudo-resolvent on the Banach space E and let $z \in E$, $\lambda_0 \in D$ and $\alpha \in \mathbb{C}$ such that $(\lambda_0 - \alpha)\mathscr{R}(\lambda_0)z = z$. Then $(\lambda - \alpha)\mathscr{R}(\lambda)z = z$ for all $\lambda \in D$.*

Equation (2) is an identity between holomorphic functions. Thus we obtain the following proposition (see [19, A-III.2.5]).

PROPOSITION 2.8. *Let $\mathscr{R} : D \to \mathscr{L}(E)$ be a pseudo-resolvent on the Banach space X, $\lambda_0 \in D$ and $\mu_0 \in \mathbb{C} \setminus D$. Then μ_0 is a pole of $\mathscr{R}$ if and only if $(\lambda_0 - \mu_0)^{-1}$ is a pole of the resolvent of $\mathscr{R}(\lambda_0)$. Moreover, the pole orders and the corresponding residues at μ_0 and $(\lambda_0 - \mu_0)^{-1}$, respectively, coincide. In particular, every pole of $\mathscr{R}$ is an eigenvalue of $\mathscr{R}$.*

2.2. Pseudo-resolvents on subspaces and quotients Let $\mathscr{R} : D \to \mathscr{L}(E)$ be a pseudo-resolvent on the Banach space E and let F be a closed *$\mathscr{R}$-invariant* subspace of E, that is, $\mathscr{R}(\lambda)F \subseteq F$ for all $\lambda \in D$. Denote by $\mathscr{R}_|(\lambda) \in \mathscr{L}(F)$ the restriction of $\mathscr{R}(\lambda)$ to F and by $\mathscr{R}_/(\lambda) \in \mathscr{L}(E/F)$ the operator on E/F induced by $\mathscr{R}(\lambda)$. The following result is shown in [9, Proposition A.3.10].

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PROPOSITION 2.9. *Under the above assumptions the following holds*:

(a) $\mathscr{R}_| : D \to \mathscr{L}(F)$ *and* $\mathscr{R}_/ : D \to \mathscr{L}(E/F)$ *are pseudo-resolvents.*

(b) *For* $\lambda_0 \in \overline{D} \setminus D$ *the following assertions are equivalent*:

(i) $\lambda_0 \notin \operatorname{sing}(\mathscr{R})$;

(ii) $\lambda_0 \notin \operatorname{sing}(\mathscr{R}_|) \cup \operatorname{sing}(\mathscr{R}_/)$.

(c) $\mathscr{R}$ *has a pole at* $\lambda_0 \in \overline{D} \setminus D$ *if and only if both* $\mathscr{R}_|$ *and* $\mathscr{R}_/$ *have a pole at* λ_0. *If* p, $p_|$ *and* $p_/$ *are the respective orders, then* $\max\{p_|, p_/\} \le p \le p_| + p_/$.

2.3. Pseudo-resolvents on Banach lattices In the following we are mainly interested in pseudo-resolvents on Banach lattices. First we introduce the notion of a positive and a dominated pseudo-resolvent, respectively.

DEFINITION 2.10. Let $\mathscr{R} : D(\mathscr{R}) \to \mathscr{L}(E)$ and $\mathscr{Q} : D(\mathscr{Q}) \to \mathscr{L}(E)$ be pseudo-resolvents on the Banach lattice E. Then $\mathscr{Q}$ is *dominated by* $\mathscr{R}$ if there exists $r \in \mathbb{R}$ such that $(r, \infty) \subseteq D(\mathscr{R}) \cap D(\mathscr{Q})$ and $|\mathscr{Q}(s)x| \le \mathscr{R}(s)|x|$ for $s \in (r, \infty)$ and $x \in E$. The pseudo-resolvent $\mathscr{R}$ is called *positive* if $\mathscr{R}$ dominates the pseudo-resolvent identically zero.

In the next proposition we collect some particular properties of dominated and positive pseudo-resolvents. A similar result has been shown for the resolvent of the generator of a positive C_0-semigroup (see [19, C-III.1.1, C-III.1.3]).

PROPOSITION 2.11. *Let* $\mathscr{R} : D(\mathscr{R}) \to \mathscr{L}(E)$ *and* $\mathscr{Q} : D(\mathscr{Q}) \to \mathscr{L}(E)$ *be pseudo-resolvents on the Banach lattice E such that* $\mathscr{Q}$ *is dominated by* $\mathscr{R}$ *and let* $r \in \mathbb{R}$ *be such that* $(r, \infty) \subseteq D(\mathscr{R}) \cap D(\mathscr{Q})$ *and* $|\mathscr{Q}(s)x| \le \mathscr{R}(s)|x|$ *for* $s \in (r, \infty)$ *and* $x \in E$. *Denote by* $\mathscr{R}_{\max} : D(\mathscr{R}_{\max}) \to \mathscr{L}(E)$ *and* $\mathscr{Q}_{\max} : D(\mathscr{Q}_{\max}) \to \mathscr{L}(E)$ *the maximal extensions of* $\mathscr{R}$ *and* $\mathscr{Q}$, *respectively. Then the following holds*:

(a) $\mathbb{C}_r \subseteq D(\mathscr{R}_{\max}) \cap D(\mathscr{Q}_{\max})$ *and* $s(\mathscr{Q}) \le s(\mathscr{R}) \le r < \infty$.

(b) *Either* $s(\mathscr{R}) = -\infty$ *or* $s(\mathscr{R}) \in \operatorname{sing}(\mathscr{R})$.

(c) $|\mathscr{Q}_{\max}(\lambda)x| \le \mathscr{R}_{\max}(\operatorname{Re}\lambda)|x|$ *and* $|\mathscr{R}_{\max}(\lambda)x| \le \mathscr{R}_{\max}(\operatorname{Re}\lambda)|x|$ *for* $\lambda \in \mathbb{C}_{s(\mathscr{R})}$ *and* $x \in E$.

(d) $r(\mathscr{R}_{\max}(s)) = (s - s(\mathscr{R}))^{-1}$ *for* $s > s(\mathscr{R})$.

PROOF. (I) $s(\mathscr{R}) < \infty$.
Let $s \in (r, \infty)$. Then $\mathscr{R}(s) \ge 0$, and hence $r(\mathscr{R}(s)) \in \sigma(\mathscr{R}(s))$ (see [23, V.4.1]). On the other hand by Corollary 2.4 (a) we have $((s-r)^{-1}, \infty) \subseteq \rho(\mathscr{R}(s))$. Thus $r(\mathscr{R}(s)) \le (s-r)^{-1}$. Another application of Corollary 2.4 (a) yields $B_{s-r}(s) := \{\lambda \in \mathbb{C} : |\lambda - s| < s - r\} \subseteq D(\mathscr{R}_{\max})$. Since this is true for every $s \in (r, \infty)$ we obtain $\mathbb{C}_r \subseteq D(\mathscr{R}_{\max})$, and hence $s(\mathscr{R}) \le r < \infty$.

(II) Either $s(\mathscr{R}) = -\infty$ or $s(\mathscr{R}) \in \operatorname{sing}(\mathscr{R})$.
Suppose $s(\mathscr{R}) > -\infty$ and $s(\mathscr{R}) \notin \operatorname{sing}(\mathscr{R})$. Then there exist $\varepsilon > 0$ and $\lambda_0 \in \operatorname{sing}(\mathscr{R})$

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such that $s(\mathscr{R})-\varepsilon < \operatorname{Re}\lambda_0$ and $[s(\mathscr{R})-\varepsilon,\infty)\cap\operatorname{sing}(\mathscr{R}) = \emptyset$. Choose $s > r$ such that $\lambda_0 \in B_{s-(s(\mathscr{R})-\varepsilon)}(s)$. Then $\mathscr{R}(s) \geq 0$, and hence $r(\mathscr{R}(s)) \in \sigma(\mathscr{R}(s))$. By (4) we have $s - r(\mathscr{R}(s))^{-1} \in \operatorname{sing}(\mathscr{R})$ and (3) implies $r(\mathscr{R}(s)) \geq |s-\lambda_0|^{-1} \geq (s-(s(\mathscr{R})-\varepsilon))^{-1}$. Thus $s - r(\mathscr{R}(s))^{-1} \in [s(\mathscr{R})-\varepsilon,\infty)\cap\operatorname{sing}(\mathscr{R})$ which is a contradiction.

(III) $\mathscr{R}_{\max}(s) \geq 0$ and $r(\mathscr{R}_{\max}(s)) = (s - s(\mathscr{R}))^{-1}$ for $s \in (s(\mathscr{R}),\infty)$.
Fix $s_0 \in (r,\infty)$. As in (I) we obtain $r(\mathscr{R}(s_0)) \leq (s_0 - s(\mathscr{R}))^{-1}$. (II) and equation (3) imply $(s_0 - s(\mathscr{R}))^{-1} \in \sigma(\mathscr{R}(s_0))$, and hence $r(\mathscr{R}(s_0)) = (s_0 - s(\mathscr{R}))^{-1}$. By Corollary 2.4 (b) we have $\mathscr{R}_{\max}(s) = \sum_{n\geq 0}(s_0 - s)^n\mathscr{R}_{\max}(s_0)^{n+1} \geq 0$ for $s \in (s(\mathscr{R}), s_0]$. Since $\mathscr{R}(s) \geq 0$ for every $s \in (r,\infty)$ we obtain $\mathscr{R}_{\max}(s) \geq 0$ for $s \in (s(\mathscr{R}),\infty)$.

(IV) $s(\mathscr{Q}) \leq s(\mathscr{R})$.
Fix $s \in (r,\infty)$. From $|\mathscr{Q}(s)x| \leq \mathscr{R}(s)|x|$, $x \in E$, we obtain $r(\mathscr{Q}(s)) \leq r(\mathscr{R}(s)) = (s - s(\mathscr{R}))^{-1}$. Corollary 2.4 (a) then implies $B_{s-s(\mathscr{R})}(s) \subseteq D(\mathscr{Q}_{\max})$. This holds for every $s \in (r,\infty)$, and hence $\mathbb{C}_{s(\mathscr{R})} \subseteq D(\mathscr{Q}_{\max})$. In particular, $s(\mathscr{Q}) \leq s(\mathscr{R})$.

(V) $|\mathscr{Q}_{\max}(\lambda)x| \leq \mathscr{R}_{\max}(\operatorname{Re}\lambda)|x|$ and $|\mathscr{R}_{\max}(\lambda)x| \leq \mathscr{R}_{\max}(\operatorname{Re}\lambda)|x|$ for $\lambda \in \mathbb{C}_{s(\mathscr{R})}$ and $x \in E$ (see also [19, proof of C-III.2.7]).
Fix $\lambda \in \mathbb{C}_{s(\mathscr{R})}$ and $x \in E$. Choose $t \geq r$ such that $\lambda \in B_{s-s(\mathscr{R})}(s)$ for all $s \in [t,\infty)$. Since $r(\mathscr{Q}(s)) \leq r(\mathscr{R}(s)) = (s - s(\mathscr{R}))^{-1}$ Corollary 2.4 (b) implies

$$\begin{aligned}|\mathscr{Q}_{\max}(\lambda)x| &\leq \sum_{n\geq 0}|s-\lambda|^n|\mathscr{Q}(s)^{n+1}x| \leq \sum_{n\geq 0}|s-\lambda|^n\mathscr{R}(s)^{n+1}|x|\\ &= \sum_{n\geq 0}(s-(s-|s-\lambda|))^n\mathscr{R}(s)^{n+1}|x| = \mathscr{R}(s-|s-\lambda|)|x|\end{aligned}$$

for $s \in [t,\infty)$. Notice that $\lim_{s\to\infty}(s - |s-\lambda|) = \operatorname{Re}\lambda$. Thus $|\mathscr{Q}_{\max}(\lambda)x| \leq \mathscr{R}_{\max}(\operatorname{Re}\lambda)|x|$. If we replace $\mathscr{Q}$ by $\mathscr{R}$ we obtain $|\mathscr{R}_{\max}(\lambda)x| \leq \mathscr{R}_{\max}(\operatorname{Re}\lambda)|x|$. □

In our later results we frequently impose the following growth conditions on a pseudo-resolvent (see [19, C-III.2.8]).

DEFINITION 2.12. Let $\mathscr{R} : D \to \mathscr{L}(E)$ be a positive pseudo-resolvent on the Banach lattice E such that $s(\mathscr{R}) > -\infty$ and let $\mathscr{R}_{\max} : D_{\max} \to \mathscr{L}(E)$ be the maximal extension of $\mathscr{R}$.

(a) $\mathscr{R}$ satisfies *the growth condition* (G) if

$$\limsup_{r\downarrow s(\mathscr{R})}\|(r - s(\mathscr{R}))\mathscr{R}_{\max}(r)\| < \infty. \tag{6}$$

(b) We say that $\mathscr{R}$ is (G)*-solvable* if there are closed ideals $\{0\} = I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = E$ such that

(i) $\mathscr{R}_{\max}(\lambda)I_k \subseteq I_k$ for $\lambda \in D_{\max}$ and $1 \leq k \leq n$, and

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(ii) the pseudo-resolvents $\mathscr{R}_k : D_{\max} \to \mathscr{L}(I_k/I_{k-1})$, $1 < k \le n$, induced by $\mathscr{R}_{\max}$ satisfy

$$\limsup_{r \downarrow s(\mathscr{R})} \|(r - s(\mathscr{R}))\mathscr{R}_k(r)\| < \infty.$$

Note that a positive pseudo-resolvent $\mathscr{R} : D \to \mathscr{L}(E)$ is (G)-solvable provided that $s(\mathscr{R}) > -\infty$ is a pole of $\mathscr{R}$. This is an immediate consequence of Proposition 2.8 and [23, V.4, Example 4], applied to $\mathscr{R}(s)$ for some $s > s(\mathscr{R})$.

The above growth conditions have strong influence on the structure of the singular set of a pseudo-resolvent. The following result is due to Greiner (see [19, C-III.2.10, C-III.2.12, C-III.2.15 (a)]).

PROPOSITION 2.13. *Let $\mathscr{R} : D \to \mathscr{L}(E)$ be a positive pseudo-resolvent on the Banach lattice E such that $\mathscr{R}$ is (G)-solvable. Then the peripheral singular set* $\mathrm{sing}_\pi(\mathscr{R})$ *is* imaginary additively cyclic, *that is, if $s(\mathscr{R}) + i\alpha \in \mathrm{sing}(\mathscr{R})$, $\alpha \in \mathbb{R}$, then $s(\mathscr{R}) + ik\alpha \in \mathrm{sing}(\mathscr{R})$ for all $k \in \mathbb{Z}$. In particular, this holds if $s(\mathscr{R}) > -\infty$ is a pole of $\mathscr{R}$.*

2.4. Pseudo-resolvents on ultrapowers We need the following construction described in [23, V.1], in detail. For a Banach space E denote by $l^\infty(E)$ the space of bounded E-valued sequences endowed with the sup-norm. Let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$ and consider the closed linear subspace $c_{\mathscr{U}}(E) := \{(x_n) \in l^\infty(E) : \lim_{\mathscr{U}} \|x_n\| = 0\}$. The quotient space $E_{\mathscr{U}} := l^\infty(E)/c_{\mathscr{U}}(E)$ is called *ultrapower* or *$\mathscr{U}$-power* of E. Instead of $(x_n) + c_{\mathscr{U}}(E) \in E_{\mathscr{U}}$ we also write $\widehat{(x_n)}$. The space E is isometrically embedded into $E_{\mathscr{U}}$ by means of $x \mapsto (x, x, \dots)\hat{}$. Every operator $T \in \mathscr{L}(E)$ has a canonical extension $T_{\mathscr{U}} \in \mathscr{L}(E_{\mathscr{U}})$ given by $T_{\mathscr{U}}\widehat{(x_n)} := \widehat{(Tx_n)}$. The mapping $T \mapsto T_{\mathscr{U}}$ from $\mathscr{L}(E)$ into $\mathscr{L}(E_{\mathscr{U}})$ is an isometric Banach algebra homomorphism and

$$\sigma(T_{\mathscr{U}}) = \sigma(T) \quad \text{for } T \in \mathscr{L}(E). \tag{7}$$

If E is a Banach lattice, then $E_{\mathscr{U}}$ is also a Banach lattice and $|\widehat{(x_n)}| = \widehat{(|x_n|)}$. Moreover, if $T \in \mathscr{L}(E)$ is positive, then $T_{\mathscr{U}}$ is positive as well.

The ultrapower extension of a pseudo-resolvent has the following properties.

PROPOSITION 2.14. *Let $\mathscr{R} : D \to \mathscr{L}(E)$ be a pseudo-resolvent on the Banach space E and set $\mathscr{R}_{\mathscr{U}}(\lambda) := \mathscr{R}(\lambda)_{\mathscr{U}} \in \mathscr{L}(E_{\mathscr{U}})$, $\lambda \in D$. Then the following holds*:

(a) *$\mathscr{R}_{\mathscr{U}} : D \to \mathscr{L}(E_{\mathscr{U}})$ is a pseudo-resolvent and $\|\mathscr{R}_{\mathscr{U}}(\lambda)\| = \|\mathscr{R}(\lambda)\|$ for $\lambda \in D$;*
(b) *$D_{\max}(\mathscr{R}) = D_{\max}(\mathscr{R}_{\mathscr{U}})$, and $\mathrm{sing}(\mathscr{R}_{\mathscr{U}}) = \mathrm{sing}(\mathscr{R})$;*
(c) *$\mathrm{sing}(\mathscr{R}) \cap \partial D \subseteq \mathrm{sing}_p(\mathscr{R}_{\mathscr{U}})$;*

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(d) $\lambda_0 \in \mathbb{C}$ *is a pole of* $\mathscr{R}$ *if and only if it is a pole of* $\mathscr{R}_{\mathscr{U}}$, *and then the orders of the poles are equal*;

(e) *If* E *is a Banach lattice and* $\mathscr{R}$ *is a positive pseudo-resolvent, then* $\mathscr{R}_{\mathscr{U}}$ *is positive.*

PROOF. (a) and (d) follow immediately from the fact that $T \mapsto T_{\mathscr{U}}$ is an isometric algebra homomorphism from $\mathscr{L}(E)$ into $\mathscr{L}(E_{\mathscr{U}})$.

(b) follows from (7) and (3) applied to $\mathscr{R}(\lambda)$ for fixed $\lambda \in D$.

In order to prove (c) fix $\mu \in \operatorname{sing}(\mathscr{R}) \cap \partial D$ and $\lambda_0 \in D$. From (3) we obtain that $(\lambda_0 - \mu)^{-1}$ is in the boundary of $\sigma(\mathscr{R}(\lambda_0))$, hence it is an approximate eigenvalue of $\mathscr{R}(\lambda_0)$. An application of [23, V.1.4] shows that $(\lambda_0 - \mu)^{-1}$ is an eigenvalue of $\mathscr{R}_{\mathscr{U}}(\lambda_0)$, that is, $\mu \in \operatorname{sing}_p(\mathscr{R}_{\mathscr{U}})$ by Lemma 2.7.

Finally, (e) follows from the fact that for positive $T \in \mathscr{L}(E)$ also $T_{\mathscr{U}} \in \mathscr{L}(E_{\mathscr{U}})$ is positive. □

3. The peripheral singular set of a positive dominated pseudo-resolvent

In this section we show that for positive pseudo-resolvents $\mathscr{Q}$ and $\mathscr{R}$ on a Banach lattice E such that $\mathscr{Q}$ is dominated by $\mathscr{R}$ we always have

$$\operatorname{sing}(\mathscr{Q}) \cap (s(\mathscr{R}) + i\mathbb{R}) \subseteq \operatorname{sing}_\pi(\mathscr{R}) \tag{8}$$

provided that $\mathscr{R}$ satisfies the growth condition (G) or, more general, is (G)-solvable. In view of Proposition 2.11 it suffices to consider pseudo-resolvents $\mathscr{R}, \mathscr{Q} : \mathbb{C}_0 \to \mathscr{L}(E)$ and to show

$$\operatorname{sing}_u(\mathscr{Q}) \subseteq \operatorname{sing}_u(\mathscr{R}). \tag{9}$$

At first we present a condition under which a unitary eigenvalue of $\mathscr{Q}$ is also an eigenvalue of $\mathscr{R}$.

LEMMA 3.1. *Let* E *be a Banach lattice and let* $\mathscr{Q}, \mathscr{R} : \mathbb{C}_0 \to \mathscr{L}(E)$ *be positive pseudo-resolvents such that* $\mathscr{Q}$ *is dominated by* $\mathscr{R}$. *Suppose that there exist* $z \in E$, $r_0 > 0$ *and* $\beta \in \mathbb{R}$ *such that* $(r_0 - i\beta)\mathscr{Q}(r_0)z = z$ *and* $r_0\mathscr{R}(r_0)|z| = |z|$. *Then* $(r_0 - i\beta)\mathscr{R}(r_0)z = z$.

PROOF. By Lemma 2.7 we have $r_0\mathscr{Q}(r_0 + i\beta)z = z$ and from Proposition 2.11 (c) we obtain $|r_0\mathscr{Q}(r_0 + i\beta)z| \le r_0\mathscr{Q}(r_0)|z|$. Then $|z| = |r_0\mathscr{Q}(r_0 + i\beta)z| \le r_0\mathscr{Q}(r_0)|z| \le r_0\mathscr{R}(r_0)|z| = |z|$, and hence $|z| = r_0\mathscr{Q}(r_0)|z| = r_0\mathscr{R}(r_0)|z| = |z|$. Thus

$$0 \le |(r_0 - i\beta)(R(r_0)z - Q(r_0)z)| \le |r_0 - i\beta| r_0^{-1}(r_0\mathscr{R}(r_0) - r_0\mathscr{Q}(r_0))|z| = 0.$$

This implies $(r_0 - i\beta)\mathscr{R}(r_0)z = (r_0 - i\beta)\mathscr{Q}(r_0)z = z$. □

We now come to the main result of this section. Note that a pseudo-resolvent $\mathscr{R}: \mathbb{C}_0 \to \mathscr{L}(E)$ satisfies the growth condition (G) if and only if on every ultrapower $E_{\mathscr{U}}$ the induced pseudo-resolvent $\mathscr{R}_{\mathscr{U}}: \mathbb{C}_0 \to \mathscr{L}(E_{\mathscr{U}})$ satisfies (G) (see Proposition 2.14 (a)).

THEOREM 3.2. *Let E be a Banach lattice and let $\mathscr{Q}, \mathscr{R}: \mathbb{C}_0 \to \mathscr{L}(E)$ be positive pseudo-resolvents such that $\mathscr{Q}$ is dominated by $\mathscr{R}$ and $\mathscr{R}$ satisfies (G). Then* $\operatorname{sing}_u(\mathscr{Q}) \subseteq \operatorname{sing}_u(\mathscr{R})$.

PROOF. Let $i\beta \in \operatorname{sing}_u(\mathscr{Q})$. By passing to an ultrapower we may assume $i\beta \in \operatorname{sing}_p(\mathscr{Q})$ (see Proposition 2.14). Thus, by Lemma 2.7 there exists $0 \neq z \in E$ such that $\lambda\mathscr{Q}(\lambda + i\beta)z = z$ for all $\lambda \in \mathbb{C}_0$. Proposition 2.11 yields

$$|z| = |\lambda\mathscr{Q}(\lambda + i\beta)z| \leq |\lambda|\mathscr{R}(\operatorname{Re}\lambda)|z| \quad \text{for} \quad \lambda \in \mathbb{C}_0.$$

Since $\mathscr{R}$ satisfies (G) the function $p(x) := \limsup_{r\downarrow 0} \|r\mathscr{R}(r)|x|\|$, $x \in E$, is a continuous lattice seminorm on E. In particular, $J := \ker p$ is a closed ideal in E. For $\lambda \in \mathbb{C}_0$ and $x \in E$ we have $\mathscr{R}(r)|\mathscr{R}(\lambda)x| \leq \mathscr{R}(r)\mathscr{R}(\operatorname{Re}\lambda)|x|$, and hence $p(\mathscr{R}(\lambda)x) \leq \|\mathscr{R}(\operatorname{Re}\lambda)\|p(x)$. Thus $\mathscr{R}(\lambda)J \subseteq J$ for $\lambda \in \mathbb{C}_0$. Moreover, $\mathscr{Q}(\lambda)J \subseteq J$, $\lambda \in \mathbb{C}_0$, since $\mathscr{Q}$ is dominated by $\mathscr{R}$. Consider now the positive pseudo-resolvents $\mathscr{Q}_/$, $\mathscr{R}_/: \mathbb{C}_0 \to \mathscr{L}(E/J)$ induced by $\mathscr{Q}$ and $\mathscr{R}$, respectively. Clearly, $\mathscr{Q}_/$ is dominated by $\mathscr{R}_/$. From $r\mathscr{R}(r)|z| \geq |z|$ for $r > 0$ we obtain $p(z) \geq \|z\| > 0$, and hence $\tilde{z} := z + J \in E/J$ is non-zero. Moreover, $\lambda\mathscr{Q}_/(\lambda + i\beta)\tilde{z} = \tilde{z}$ for $\lambda \in \mathbb{C}_0$. Since $\mathscr{R}$ satisfies (G) we have

$$\begin{aligned} p(s\mathscr{R}(s)|z| - |z|) &= \limsup_{r\downarrow 0} \|rs\mathscr{R}(r)\mathscr{R}(s)|z| - r\mathscr{R}(r)|z|\| \\ &= \limsup_{r\downarrow 0} \left\| \frac{s}{s-r} r\mathscr{R}(r)|z| - \frac{r}{s-r} s\mathscr{R}(s)|z| - r\mathscr{R}(r)|z| \right\| \\ &= \limsup_{r\downarrow 0} \left\| \frac{r}{s-r} (r\mathscr{R}(r)|z| - s\mathscr{R}(s)|z|) \right\| \\ &= 0 \end{aligned}$$

for $s \in (0, \infty)$. Hence $s\mathscr{R}_/(s)|\tilde{z}| = |\tilde{z}|$ for $s \in (0, \infty)$ and Lemma 3.1 yields $s\mathscr{R}_/(s + i\beta)\tilde{z} = \tilde{z}$. Thus $\|\mathscr{R}(s + i\beta)\| \geq \|\mathscr{R}_/(s + i\beta)\| \geq s^{-1} \to \infty$ as $s \to 0$, and we obtain $i\beta \in \operatorname{sing}(\mathscr{R})$. □

We can extend Theorem 3.2 to pseudo-resolvents $\mathscr{R}$ which are (G)-solvable. In fact, Proposition 2.9 permits to reduce this more general situation to pseudo-resolvents satisfying (G).

COROLLARY 3.3. *Let E be a Banach lattice and let $\mathcal{Q}, \mathcal{R} : \mathbb{C}_0 \to \mathcal{L}(E)$ be positive pseudo-resolvents such that $\mathcal{Q}$ is dominated by $\mathcal{R}$ and $\mathcal{R}$ is (G)-solvable. Then* $\operatorname{sing}_u(\mathcal{Q}) \subseteq \operatorname{sing}_u(\mathcal{R})$. *In particular, this holds if* 0 *is a pole of* $\mathcal{R}$.

4. Peripheral eigenvalues of dominated positive pseudo-resolvents

In this section we investigate under which conditions the inclusion

$$\operatorname{sing}_p(\mathcal{Q}) \cap (s(\mathcal{R}) + i\mathbb{R}) \subseteq \operatorname{sing}_{p,\pi}(\mathcal{R}) \tag{10}$$

holds, where $\mathcal{Q}$ and $\mathcal{R}$ are positive pseudo-resolvents on a Banach lattice E such that $\mathcal{Q}$ is dominated by $\mathcal{R}$. As in the previous section it suffices to consider pseudo-resolvents $\mathcal{R}, \mathcal{Q} : \mathbb{C}_0 \to \mathcal{L}(E)$ and to ask if

$$\operatorname{sing}_{p,u}(\mathcal{Q}) \subseteq \operatorname{sing}_{p,u}(\mathcal{R}) \tag{11}$$

holds. It turns out that ergodicity properties of the dominating pseudo-resolvent play a central role. We recall the following result of Yosida ([29, VIII.4, Theorem 2]).

PROPOSITION 4.1. *Let $\mathcal{R} : \mathbb{C}_0 \to \mathcal{L}(E)$ be a pseudo-resolvent on the Banach space E and assume that $\mathcal{R}$ satisfies the growth condition (G). Then for $x \in E$ the following assertions are equivalent*:

(a) $\lim_{s\downarrow 0} s\mathcal{R}(s)x$ *exists in* E;
(b) $(s\mathcal{R}(s)x)_{s>0}$ *has a weak cluster point as* $s \to 0$.

In this case $y := \lim_{s\downarrow 0} s\mathcal{R}(s)x$ *satisfies* $\lambda\mathcal{R}(\lambda)y = y$ *for all* $\lambda \in \mathbb{C}_0$.

A pseudo-resolvent $\mathcal{R} : \mathbb{C}_0 \to \mathcal{L}(E)$ is called *Abel-ergodic* if $P_{\mathcal{R}}x := \lim_{s\downarrow 0} s\mathcal{R}(s)x$ exists for all $x \in E$. Then $P_{\mathcal{R}} \in \mathcal{L}(E)$ is a projection, $P_{\mathcal{R}}E = \operatorname{Fix}(\lambda\mathcal{R}(\lambda))$ and $\ker P_{\mathcal{R}} = \overline{(I - \lambda\mathcal{R}(\lambda))E}$ for $\lambda \in \mathbb{C}_0$ (see [29, VIII.4]). Note that by the principle of uniform boundedness an Abel-ergodic pseudo-resolvent with $s(\mathcal{R}) = 0$ always satisfies (G).

In the following we use the following construction (see [23, II.8, Example 1]). Let E be a Banach lattice and $y' \in E'_+$. Then $p : E \to \mathbb{R}_+ : x \mapsto \langle y', |x|\rangle$ is a lattice seminorm with kernel $\ker p = N(y') := \{x \in E : \langle y', |x|\rangle = 0\}$. The induced norm on $E/\ker p$ is a lattice norm and the completion (E, y') of $E/\ker p$ is a Banach lattice. Moreover, (E, y') is an AL-space, that is, the norm is additive on $(E, y')_+$, and the mapping $j_{y'} : E \to (E, y')$ induced by the quotient map $q : E \to E/\ker p$ is a lattice homomorphism. If $\mathcal{R} : D \to \mathcal{L}(E)$ is a positive pseudo-resolvent such that $s\mathcal{R}(s)'y' \leq y'$ for $s > 0$, then $\lambda\mathcal{R}(\lambda)N(y') \subseteq N(y')$ for $\lambda \in \mathbb{C}_0$. Hence $\mathcal{R}(\lambda)$ induces an operator $\mathcal{R}(\lambda)_/$ on $E/\ker p$ which is a positive contraction. Thus $\mathcal{R}(\lambda)_/$ has a

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unique contractive positive extension $\tilde{\mathcal{R}}(\lambda) \in \mathcal{L}((E, y'))$ and $\tilde{\mathcal{R}} : \mathbb{C}_0 \to \mathcal{L}((E, y'))$ is a pseudo-resolvent.

Now we can state the following inheritance result on unitary eigenvalues.

THEOREM 4.2. *Let E be a Banach lattice and let $\mathcal{Q}, \mathcal{R} : \mathbb{C}_0 \to \mathcal{L}(E)$ be positive pseudo-resolvents such that $\mathcal{Q}$ is dominated by $\mathcal{R}$ and $\mathcal{R}_\alpha := \mathcal{R}(\cdot + i\alpha)$ is Abel-ergodic for all $\alpha \in \mathbb{R}$. Then* $\operatorname{sing}_{p,u}(\mathcal{Q}) \subseteq \operatorname{sing}_{p,u}(\mathcal{R})$.

PROOF. Let $i\beta \in \operatorname{sing}_{p,u}(\mathcal{Q})$ and $0 \neq x \in E$ such that $(\lambda - i\beta)\mathcal{Q}(\lambda)x = x$ for $\lambda \in \mathbb{C}_0$. Then $|x| \leq |s\mathcal{Q}(s + i\beta)x| \leq s\mathcal{R}(s)|x|$ for $s > 0$. Since $\mathcal{R}$ is Abel-ergodic $y := \lim_{s\downarrow 0} s\mathcal{R}(s)|x|$ exists and $0 \leq |x| \leq y = s\mathcal{R}(s)y$, $s > 0$. Choose $x' \in E'_+$ such that $\langle x', |x|\rangle > 0$. Another application of the Abel-ergodicity of $\mathcal{R}$ implies that $y' := \sigma(E', E) - \lim_{s\downarrow 0} s\mathcal{R}(s)'x'$ exists and $0 \leq s\mathcal{Q}(s)'y' \leq s\mathcal{R}(s)'y' = y'$, $s > 0$. Moreover, $\langle y', |x|\rangle = \lim_{s\downarrow 0}\langle s\mathcal{R}(s)'x', |x|\rangle = \langle x', y\rangle \geq \langle x', |x|\rangle > 0$. In particular, $\tilde{x} := j_{y'}x \in (E, y') \setminus \{0\}$.

Now let $\tilde{\mathcal{R}}, \tilde{\mathcal{Q}} : \mathbb{C}_0 \to \mathcal{L}((E, y'))$ be the positive pseudo-resolvents on (E, y') induced by $\mathcal{R}$ and $\mathcal{Q}$, respectively. Then $\tilde{\mathcal{Q}}$ is dominated by $\tilde{\mathcal{R}}$ and $(\lambda - i\beta)\tilde{\mathcal{Q}}(\lambda)\tilde{x} = \tilde{x}$ for $\lambda \in \mathbb{C}_0$, that is, $i\beta \in \operatorname{sing}_{p,u}(\tilde{\mathcal{Q}})$. Moreover, $|\tilde{x}| \leq s\tilde{\mathcal{Q}}(s)|\tilde{x}| \leq s\tilde{\mathcal{R}}(s)|\tilde{x}|$, $s > 0$. From $s\mathcal{R}(s)'y' = y'$ it follows that $s\tilde{\mathcal{R}}(s)$ is a contraction on (E, y'). Hence the strict monotonicity of the norm on (E, y') yields $s\tilde{\mathcal{R}}(s)|\tilde{x}| = |\tilde{x}|$, $s > 0$. Lemma 3.1 implies $(s - i\beta)\tilde{\mathcal{R}}\tilde{x} = \tilde{x}$, $s > 0$. Since $\mathcal{R}_\beta$ is Abel-ergodic $z := \lim_{s\downarrow 0} s\mathcal{R}(s + i\beta)x$ exists in E and $(s - i\beta)\mathcal{R}(s)z = z$, $s > 0$. Moreover,

$$j_{y'}z = \lim_{s\downarrow 0} s\mathcal{R}(s + i\beta)x = \lim_{s\downarrow 0} s\tilde{\mathcal{R}}(s + i\beta)j_{y'}x = \lim_{s\downarrow 0} s\tilde{\mathcal{R}}(s + i\beta)\tilde{x} = \tilde{x} \neq 0.$$

Thus $z \neq 0$ and this shows $i\beta \in \operatorname{sing}_{p,u}(\mathcal{R})$. □

If the Banach lattice E has order continuous norm we can relax the conditions on the pseudo-resolvent $\mathcal{R}$. Note that order continuity of the norm of E is equivalent to the fact that for every relatively weakly compact set $C \subseteq E_+$ the *solid hull* so $C := \{y \in E : |y| \leq x \text{ for some } x \in C\}$ is relatively weakly compact (see [1, 13.8]). Examples of such spaces are c_0, L^p, $1 \leq p < \infty$, and all reflexive Banach lattices.

COROLLARY 4.3. *Let E be a Banach lattice with order continuous norm and let $\mathcal{Q}, \mathcal{R} : \mathbb{C}_0 \to \mathcal{L}(E)$ be positive pseudo-resolvents such that $\mathcal{Q}$ is dominated by $\mathcal{R}$ and $\mathcal{R}$ is Abel-ergodic. Then* $\operatorname{sing}_{p,u}(\mathcal{Q}) \subseteq \operatorname{sing}_{p,u}(\mathcal{R})$.

PROOF. Let $\alpha \in \mathbb{R}$. For $\lambda \in \mathbb{C}_0$ and $x \in E$ we have $|\lambda\mathcal{R}(\lambda + i\alpha)x| \leq |\lambda|\mathcal{R}(\operatorname{Re}\lambda)|x|$. Since $\mathcal{R}$ is Abel-ergodic, $\mathcal{R}$ and hence $\mathcal{R}_\alpha = \mathcal{R}(\cdot + i\alpha)$ satisfies the growth condition (G). Moreover, $\{s\mathcal{R}(s + i\alpha)x : 0 < s \leq 1\}$ is contained in the solid hull of $\{s\mathcal{R}(s)|x| : 0 < s \leq 1\}$. Thus $\{s\mathcal{R}(s + i\alpha)x : 0 < s \leq 1\}$ is relatively weakly compact and Proposition 4.1 implies that $\mathscr{R}_\alpha$ is Abel-ergodic. The assertion now follows from Theorem 4.2. □

If E is a *KB-space*, that is, E is a (projection) band in its bidual, we can even skip the ergodicity condition on $\mathscr{R}$. Note that in a KB-space every norm bounded increasing sequence in E_+ converges in norm (see [23, II.5.15]) and every KB-space has order continuous norm (see [23, II.5, Example 7]). Examples of KB-spaces are L^p, $1 \le p < \infty$, and all reflexive Banach lattices.

THEOREM 4.4. *Let E be a KB-space and let $\mathscr{Q}$, $\mathscr{R} : \mathbb{C}_0 \to \mathscr{L}(E)$ be positive pseudo-resolvents such that $\mathscr{Q}$ is dominated by $\mathscr{R}$ and $\mathscr{R}$ satisfies* (G). *Then* $\operatorname{sing}_{p,u}(\mathscr{Q}) \subseteq \operatorname{sing}_{p,u}(\mathscr{R})$.

PROOF. Let $i\beta \in \operatorname{sing}_{p,u}(\mathscr{Q})$ and choose $0 \neq x \in E$ such that $(\lambda - i\beta)\mathscr{Q}(\lambda)x = x$, $\lambda \in \mathbb{C}_0$. Lemma 2.7 and Proposition 2.11 yield $|x| = |\lambda||\mathscr{Q}(\lambda + i\beta)x| \le |\lambda|\mathscr{R}(\operatorname{Re}\lambda)|x|$ for $\lambda \in \mathbb{C}_0$. In particular, $\mathscr{R}(1)|x| \ge |x|$, and hence $(\mathscr{R}(1)^n|x|)$ is an increasing sequence in E. On the other hand, the power series expansion of $\mathscr{R}(\cdot)|x|$ at 1 yields (see Corollary 2.4)

$$\begin{aligned}\mathscr{R}(s)|x| &= \sum_{n\ge 0}(1-s)^n\mathscr{R}(1)^{n+1}|x| \\ &\ge \sum_{n\ge m}(1-s)^n\mathscr{R}(1)^{m+1}|x| = (1-s)^m s^{-1}\mathscr{R}(1)^{m+1}|x|\end{aligned}$$

for $0 < s \le 1$ and $m \in \mathbb{N}$. Thus $s\mathscr{R}(s)|x| \ge (1-s)^m\mathscr{R}(1)^{m+1}|x| \ge 0$. Letting $s \downarrow 0$ and using the fact that $\mathscr{R}$ satisfies (G) we obtain that the sequence $(\mathscr{R}(1)^n|x|)$ is bounded. Since E is a KB-space $y := \lim_n \mathscr{R}(1)^n|x|$ exists in E and $y \ge |x|$. Clearly, $\mathscr{R}(1)y = y$ and by Lemma 2.7

$$\lambda\mathscr{R}(\lambda)y = y, \quad \lambda \in \mathbb{C}_0. \tag{12}$$

Let F be the closed ideal in E generated by y. From (12) and Proposition 2.11 we obtain $\mathscr{R}(\lambda)F \subseteq F$ and $\mathscr{Q}(\lambda)F \subseteq F$. Let $\mathscr{R}_|$, $\mathscr{Q}_| : \mathbb{C}_0 \to \mathscr{L}(F)$ be the pseudo-resolvents defined by restricting $\mathscr{R}(\lambda)$ and $\mathscr{Q}(\lambda)$ to F. Since $x \in F$ we have $i\beta \in \operatorname{sing}_{p,u}(\mathscr{Q}_|)$. On the other hand F as a closed ideal of E is a KB-space. In particular, F has order continuous norm. Clearly, $\{s\mathscr{R}_|(s)y : 0 < s \le 1\}$, and hence $\{s\mathscr{R}_|(s)z : 0 < s \le 1\}$ is relatively weakly compact for all $z \in F$ (note that $\mathscr{R}$ and hence $\mathscr{R}_|$ satisfies the growth condition). Proposition 4.1 implies that $\mathscr{R}_|$ is Abel-ergodic. Now an application of Corollary 4.3 yields $i\beta \in \operatorname{sing}_{p,u}(\mathscr{R}_|) \subseteq \operatorname{sing}_{p,u}(\mathscr{R})$. □

The following example shows that in Corollary 4.3 the condition on $\mathscr{R}$ (Abel-ergodicity) and in Theorem 4.4 the condition on E (KB-space) cannot be omitted.

EXAMPLE 4.5. In [22, Example 2.7] we constructed contractions $0 \le S \le T$ on $E = c_0$(= space of sequences converging to 0) such that $1 \in \sigma_p(S)$ and $1 \notin \sigma_p(T)$. For $\lambda \in \mathbb{C}_0$ set $\mathscr{R}(\lambda) := R(1+\lambda, T)$ and $\mathscr{Q}(\lambda) := R(1+\lambda, S)$. Then $\mathscr{Q}, \mathscr{R} : \mathbb{C}_0 \to \mathscr{L}(E)$ are pseudo-resolvents such that $0 \le \mathscr{Q} \le \mathscr{R}$, $0 \in \operatorname{sing}_p(\mathscr{Q})$ and $0 \notin \operatorname{sing}_p(\mathscr{R})$.

5. The essential singular set of a dominated pseudo-resolvent

In analogy with the definition of the essential spectrum of an operator we introduce the following notion.

DEFINITION 5.1. Let $\mathscr{R}$ be a pseudo-resolvent on the Banach space E. Then

$$\operatorname{sing}_{\mathrm{ess}}(\mathscr{R}) := \{\lambda \in \operatorname{sing}(\mathscr{R}) : \lambda \text{ is not a Riesz point of } \mathscr{R}\}$$

is called the *essential singular set* of $\mathscr{R}$ and $\operatorname{sing}_{\mathrm{ess},u}(\mathscr{R}) := \operatorname{sing}_{\mathrm{ess}}(\mathscr{R}) \cap i\mathbb{R}$ is the *unitary part* of the essential singular set. The pseudo-resolvent is said to be *quasi-compact* if the *essential singular bound*

$$s_{\mathrm{ess}}(\mathscr{R}) := \sup\{\operatorname{Re} \lambda : \lambda \in \operatorname{sing}_{\mathrm{ess}}(\mathscr{R})\}$$

is negative and $\operatorname{sing}(\mathscr{R}) \cap \mathbb{C}_r$ is finite for some $s_{\mathrm{ess}}(\mathscr{R}) < r < 0$.

We have the following result on the essential singular set of a dominated pseudo-resolvent.

THEOREM 5.2. *Let E be a Banach lattice and let $\mathscr{Q}, \mathscr{R} : \mathbb{C}_0 \to \mathscr{L}(E)$ be pseudo-resolvents such that $\mathscr{Q}$ is dominated by $\mathscr{R}$. Then the following holds*:

(a) *$s_{\mathrm{ess}}(\mathscr{R}) < 0$ if and only if $s(\mathscr{R}) < 0$ or 0 is a Riesz point of $\mathscr{R}$.*

(b) *If 0 is a Riesz point of $\mathscr{R}$, then there exists $\delta > 0$ (only dependent on $\mathscr{R}$) such that* $\operatorname{sing}(\mathscr{Q}) \cap \mathbb{C}_{-\delta}$ *contains only Riesz points, that is, $s_{\mathrm{ess}}(\mathscr{Q}) \le -\delta$. In particular, $s_{\mathrm{ess}}(\mathscr{R}) \le -\delta$.*

PROOF. Let 0 be a Riesz point of $\mathscr{R}$. Proposition 2.8 and Proposition 2.11 imply that 1 is a Riesz point of $T := \mathscr{R}(1) \ge 0$ and that $r(T) = 1$. By [21, Corollary 1.6] there exists $0 \le c < 1$ such that every operator $S \in \mathscr{L}(E)$ dominated by T satisfies

$$r_{\mathrm{ess}}(S) \le c. \tag{13}$$

Now fix $\delta > 0$ such that $(1+\delta)^{-1} > c$ and let $\alpha + i\beta \in \operatorname{sing}(\mathscr{Q}) \cap \mathbb{C}_{-\delta}$, $\alpha, \beta \in \mathbb{R}$. From (3) we obtain $(1-\alpha)^{-1} \in \sigma(\mathscr{Q}(1+i\beta))$ and $|(1-\alpha)^{-1}| \ge (1+\delta)^{-1} > c$. Proposition 2.11 implies that $\mathscr{Q}(1+i\beta)$ is dominated by $\mathscr{R}(1)$, and hence $r_{\mathrm{ess}}(\mathscr{Q}(1+i\beta)) \le c$ by (13). Thus $(1-\alpha)^{-1}$ is a Riesz point of $\mathscr{Q}(1+i\beta)$ and from Proposition 2.8 it follows that $\alpha + i\beta$ is a Riesz point of $\mathscr{Q}$. This proves $s_{\mathrm{ess}}(\mathscr{Q}) \le -\delta$. Now the remaining assertions are obvious. □

If the dominating pseudo-resolvent is quasi-compact we obtain the following result.

PROPOSITION 5.3. *Let* $\mathscr{Q}, \mathscr{R} : \mathbb{C}_0 \to \mathscr{L}(E)$ *be positive pseudo-resolvents on the Banach lattice* E *such that* $\mathscr{Q}$ *is dominated by* $\mathscr{R}$ *and* $\mathscr{R}$ *is quasi-compact. Then there exists* $\delta > 0$ *(only dependent on* $\mathscr{R}$*) such that* $s_{\mathrm{ess}}(\mathscr{Q}) \le -\delta$ *and* $\mathrm{sing}_u(\mathscr{Q}) \subseteq \mathrm{sing}_u(\mathscr{R}) \subseteq \{0\}$.

PROOF. We only have to prove the second assertion. If $\mathrm{sing}_u(\mathscr{R}) = \emptyset$, then Proposition 2.11 yields $s(\mathscr{Q}) \le s(\mathscr{R}) < 0$ and the assertion follows. Otherwise 0 is a Riesz point of $\mathscr{R}$. Proposition 2.13 implies that $\mathrm{sing}_u(\mathscr{R})$ is imaginary additively cyclic. Since $\mathscr{R}$ is quasi-compact, $\mathrm{sing}_u(\mathscr{R}) = \{0\}$. Then by Corollary 3.3 we have $\mathrm{sing}_u(\mathscr{Q}) \subseteq \mathrm{sing}_u(\mathscr{R}) = \{0\}$. □

We do not know if in Proposition 5.3 the pseudo-resolvent $\mathscr{Q}$ is even quasi-compact.

6. The spectrum of resolvent-dominated operators and dominated semigroups

In this section we apply the results of Section 3, Section 4 and Section 5 to operators A and B on a Banach lattice E such that the resolvent of B is dominated by the resolvent of A, that is,

$$|R(s, B)x| \le R(s, A)|x| \tag{14}$$

for $x \in E$ and $s \in (s_0, \infty)$ for some $s_0 \in \mathbb{R}$. In this case we shortly say that B is *resolvent-dominated* or *r-dominated* by A. Recall that A is *resolvent-positive*, or *r-positive* for short, if $(s_0, \infty) \subseteq \rho(A)$ for some $s_0 \in \mathbb{R}$ and $R(s, A) \ge 0$ for $s \in (s_0, \infty)$. From Section 2 we know that the singular set of the resolvent $\mathscr{R}_A = R(\cdot, A)$ coincides with the spectrum $\sigma(A)$ of A, and the singular bound $s(\mathscr{R}_A)$ coincides with the spectral bound $s(A)$. An r-positive operator A is called *(G)-solvable* if its resolvent is (G)-solvable (see Definition 2.12).

Now Theorem 3.2 leads at once to the following result.

THEOREM 6.1. *Let* E *be a Banach lattice and let* A *and* B *be r-positive operators on* E *such that* B *is r-dominated by* A *and* A *is* (G)*-solvable. Then* $\sigma(B) \cap (s(A) + i\mathbb{R}) \subseteq \sigma_\pi(A)$.

If A is the generator of a positive C_0-semigroup $\mathscr{T} = (T(t))_{t \ge 0}$ on E, then A is r-positive (see [19, C-III.1.1]). Moreover, if B is the generator of a C_0-semigroup $\mathscr{S} = (S(t))_{t \ge 0}$ such that $\mathscr{S}$ is *dominated* by $\mathscr{T}$, that is, $|S(t)x| \le T(t)|x|$ for $t \ge 0$ and $x \in E$, then B is r-dominated by A (see [19, C-II.4.1]). The semigroup $\mathscr{T}$ is said to be *(G)-solvable* if A is (G)-solvable. With these notions Corollary 3.3 yields the following generalization of [2, Theorem 2.2].

COROLLARY 6.2. *Let E be a Banach lattice and let $\mathscr{T} = (T(t))_{t\geq 0}$ and $\mathscr{S} = (S(t))_{t\geq 0}$ be positive C_0-semigroups on E with generator A and B, respectively, such that $\mathscr{S}$ is dominated by $\mathscr{T}$. If $\mathscr{T}$ is (G)-solvable, then $\sigma(B) \cap (s(A) + i\mathbb{R}) \subseteq \sigma_\pi(A)$. In particular, this holds if $s(A)$ is a pole of the resolvent $R(\cdot, A)$.*

The results of Section 4 lead to the following assertion on the point spectra.

THEOREM 6.3. *Let A and B be r-positive operators on the Banach lattice E such that B is r-dominated by A. Suppose, in addition, that one of following conditions is satisfied:*

(a) *$R(\cdot + i\alpha, A)$ is Abel-ergodic for all $\alpha \in \mathbb{R}$.*
(b) *E has order continuous norm and $R(\cdot, A)$ is Abel-ergodic.*
(c) *E is a KB-space and A satisfies (G).*

Then $\sigma_p(B) \cap (s(A) + i\mathbb{R}) \subseteq \sigma_{p,\pi}(A)$. In particular, this holds if A and B are the generators of positive C_0-semigroups $\mathscr{T} = (T(t))_{t\geq 0}$ and $\mathscr{S} = (S(t))_{t\geq 0}$, respectively, such that $\mathscr{S}$ is dominated by $\mathscr{T}$.

Our next result is a consequence of Theorem 5.2. Note that for an operator A the essential singular set of the resolvent $\mathscr{R}_A = R(\cdot, A)$ coincides with the *essential spectrum* $\sigma_{\mathrm{ess}}(A)$, and hence $s_{\mathrm{ess}}(\mathscr{R}_A)$ and the *essential spectral bound* $s_{\mathrm{ess}}(A) := \sup\{\mathrm{Re}\ \lambda : \lambda \in \sigma_{\mathrm{ess}}(A)\}$ are equal. In contrast to the previous results only A has to be r-positive.

THEOREM 6.4. *Let A and B be operators on the Banach lattice E. Suppose that B is r-dominated by A. Then the following holds:*

(a) *$s_{\mathrm{ess}}(A) < s(A)$ if and only if $s(A)$ is finite and a Riesz point of A.*
(b) *If $s(A)$ is a Riesz point of A, then there exists $\delta > 0$ (only dependent on A) such that $s_{\mathrm{ess}}(B) \leq s(A) - \delta$. In particular, $s_{\mathrm{ess}}(A) \leq -\delta$.*

As in the previous cases there is an obvious reformulation of Proposition 5.3 for r-dominated operators. For dominated semigroups we obtain a slightly different result. Recall that a C_0-semigroup $\mathscr{T} = (T(t))_{t\geq 0}$ is *quasi-compact* if there is $t_0 > 0$ such that $r_{\mathrm{ess}}(T(t_0)) < 1$, where $r_{\mathrm{ess}}(T(t_0)) := \sup\{|\lambda| : \lambda \in \sigma_{\mathrm{ess}}(T(t_0))\}$ is the *essential spectral radius* of $T(t_0)$. Note that for a quasi-compact C_0-semigroup $\mathscr{T}$ the resolvent of its generator A is quasi-compact in the sense of Definition 5.1 (see [19, B-IV.2.10]). The converse is not true in general. The following result generalizes [17, Proposition 3.3], where the semigroups were assumed to be positive.

THEOREM 6.5. *Let E be a Banach lattice and let $\mathscr{T} = (T(t))_{t\geq 0}$ and $\mathscr{S} = (S(t))_{t\geq 0}$ be C_0-semigroups on E with generator A and B, respectively, such that $s(A) \leq 0$ and $\mathscr{S}$ is dominated by $\mathscr{T}$. If $\mathscr{T}$ is quasi-compact, then $\mathscr{S}$ is quasi-compact.*

PROOF. By the quasi-compactness of $\mathcal{T}$ there exists $t_0 > 0$ such that $r_{\text{ess}}(T(t_0)) < 1$. Thus $M := \sigma(T(t_0)) \cap \{\lambda \in \mathbb{C} : |\lambda| > r_{\text{ess}}(T(t_0))\}$ contains only Riesz points, and hence consists of eigenvalues of $T(t_0)$. By the spectral mapping theorem for the point spectrum (see [19, A-III.6.3]) for each $\lambda \in M$ there is an eigenvalue μ of A such that $\lambda = e^{t_0\mu}$. Since $s(A) \leq 0$ we have $|\lambda| \leq 1$ for each $\lambda \in M$. Thus $r(T(t_0)) \leq 1$. By our assumption $S(t_0)$ is dominated by $T(t_0)$ and then by [22, Theorem 3.1] we have $r_{\text{ess}}(S(t_0)) < 1$, that is, $\mathcal{S}$ is quasi-compact. □

7. Asymptotic properties of dominated semigroups

We now use the results of the previous sections to investigate asymptotic properties for dominated C_0-semigroups.

Our first result is a Katznelson-Tzafriri type theorem for dominated semigroups. Recall that $f \in L^1(\mathbb{R})$ is of *spectral synthesis* with respect to a closed set $F \subseteq \mathbb{R}$ if f is the limit of a sequence (f_n) in $L^1(\mathbb{R})$ such that for each $n \in \mathbb{N}$ the Fourier transform $\hat{f}_n$ vanishes in a neighbourhood of F. In the following $L^1(\mathbb{R}_+)$ is always considered as a subspace of $L^1(\mathbb{R})$ (by setting $f \in L^1(\mathbb{R}_+)$ identically zero on $\mathbb{R}_-$). For a bounded C_0-semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on a Banach space E and $f \in L^1(\mathbb{R}_+)$ we define $\hat{f}(\mathcal{T}) \in \mathcal{L}(E)$ by

$$\hat{f}(\mathcal{T})x := \int_0^\infty f(s)T(s)x\,ds, \quad x \in E.$$

THEOREM 7.1. *Let E be a Banach lattice and let $\mathcal{T} = (T(t))_{t\geq 0}$ and $\mathcal{S} = (S(t))_{t\geq 0}$ be positive bounded C_0-semigroups on E with generator A and B, respectively, such that $\mathcal{S}$ is dominated by $\mathcal{T}$. If $f \in L^1(\mathbb{R}_+)$ is of spectral synthesis with respect to $i\sigma_u(A)$, then* $\lim_{t\to\infty} \|S(t)\hat{f}(\mathcal{S})\| = 0$.

PROOF. From Corollary 6.2 we obtain $\sigma_u(B) \subseteq \sigma_u(A)$. Thus f is also of spectral synthesis with respect to $i\sigma_u(B)$. An application of the Katznelson-Tzafriri theorem for C_0-semigroups (see [11, Théorème 3.4] and [27, Theorem 3.2]) yields $\lim_{t\to\infty} \|S(t)\hat{f}(\mathcal{S})\| = 0$. □

As a special case we obtain the following result.

COROLLARY 7.2. *Let E be a Banach lattice and let $\mathcal{T} = (T(t))_{t\geq 0}$ and $\mathcal{S} = (S(t))_{t\geq 0}$ be positive bounded C_0-semigroups on E with generator A and B, respectively, such that $\mathcal{S}$ is dominated by $\mathcal{T}$.*

(a) *If $\sigma_u(A) \subseteq i\omega\mathbb{Z}$ for some $\omega > 0$, then* $\lim_{t\to\infty} \|(S(t+2\pi/\omega) - S(t))\hat{f}(\mathcal{S})\| = 0$ *for all $f \in L^1(\mathbb{R}_+)$.*

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(b) *If* $\sigma_u(A) \subseteq \{0\}$, *then* $\lim_{t\to\infty} \|(S(t+s) - S(t))\hat{f}(\mathscr{S})\| = 0$ *for all* $s > 0$ *and* $f \in L^1(\mathbb{R}_+)$.

PROOF. Let $f \in L^1(\mathbb{R}_+)$ and set $g := f_s - f$ where $f_s := f(\cdot - s)$ for $s > 0$. Then $\hat{g}(t) = (e^{its} - 1)\hat{f}(t)$, $t \in \mathbb{R}$. Thus $\hat{g}(0) = 0$ for every $s > 0$ and $\hat{g}|_{\omega\mathbb{Z}} = 0$ for $s = \frac{2\pi}{\omega}$. Since every countable closed set $F \subseteq \mathbb{R}$ is a set of spectral synthesis, that is, every function $h \in L^1(\mathbb{R})$ such that $\hat{h}|_F = 0$ is of spectral synthesis with respect F (see [16, 37C]), the assertion follows from Theorem 7.1. □

Next we discuss almost periodicity of dominated C_0-semigroups. Recall that a C_0-semigroup $\mathscr{T} = (T(t))_{t\geq 0}$ on a Banach space E is *almost periodic* if for each $x \in E$ the orbit $\{T(t)x : t \geq 0\}$ is relatively compact in E. In this case the Jacobs-Glicksberg-deLeeuw theorem (see [15, 2.4.4, 2.4.5]) yields a decomposition $E = E_0 \oplus E_r$ with $\mathscr{T}$-invariant spaces $E_0 = \{x \in E : \lim_{t\to\infty} \|T(t)x\| = 0\}$ and $E_r = \overline{\text{lin}}\{x \in E :$ there exists $\lambda \in i\mathbb{R}$ such that $Ax = \lambda x\}$, where A is the generator of $\mathscr{T}$. The semigroup $\mathscr{T}$ is called *stable* if $\lim_{t\to\infty} T(t)x$ exists for all $x \in E$. In this case $E_r = \ker A$. Finally, we say that $\mathscr{T}$ is *Abel-ergodic* if the resolvent $R(\cdot, A)$ is Abel-ergodic. By a theorem of Ljubich and Vũ [28, Theorem 2] (see also [8, Theorem 8]), a bounded C_0-semigroup with generator A is almost periodic if $\sigma_u(A)$ is countable and $\mathscr{T}_\alpha = (e^{i\alpha t}T(t))_{t\geq 0}$ is Abel-ergodic for all $i\alpha \in \sigma_u(A)$. Together with Corollary 6.2 this immediately leads to the following result.

THEOREM 7.3. *Let E be a Banach lattice and let* $\mathscr{T} = (T(t))_{t\geq 0}$ *and* $\mathscr{S} = (S(t))_{t\geq 0}$ *be positive bounded* C_0*-semigroups on E with generator A and B, respectively, such that* $\mathscr{S}$ *is dominated by* $\mathscr{T}$. *If* $\sigma_u(A)$ *is countable and* $\mathscr{S}_\alpha = (e^{i\alpha t}S(t))_{t\geq 0}$ *is Abel-ergodic for all* $i\alpha \in \sigma_u(A)$, *then* $\mathscr{S}$ *is almost periodic. If, in addition,* $\sigma_u(A) \subseteq \{0\}$ *or* $\mathscr{T}$ *is stable, then* $\mathscr{S}$ *is stable.*

Only recently, on Banach lattices with order continuous norm the following inheritance result on almost periodicity and stability of dominated semigroups has been shown (see [10]).

THEOREM 7.4. *Let E be a Banach lattice with order continuous norm and let* $\mathscr{T} = (T(t))_{t\geq 0}$ *and* $\mathscr{S} = (S(t))_{t\geq 0}$ *be positive* C_0*-semigroups on E with generator A and B, respectively, such that* $\mathscr{S}$ *is dominated by* $\mathscr{T}$. *If* $\mathscr{T}$ *is almost periodic, then* $\mathscr{S}$ *is almost periodic, and if* $\mathscr{T}$ *is stable, then* $\mathscr{S}$ *is stable.*

We now investigate uniform ergodicity of dominated semigroups. The following spectral characterization of uniformly Abel-ergodic pseudo-resolvents is an immediate consequence of [14, Theorem 18.8.1].

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LEMMA 7.5. *Let* $\mathcal{R} : \mathbb{C}_0 \to \mathcal{L}(E)$ *be a pseudo-resolvent on the Banach space* E. *Then the following assertions are equivalent*:

(a) $P_{\mathcal{R}} := \lim_{s\downarrow 0} s\mathcal{R}(s)$ *exists in* $\mathcal{L}(E)$, *that is,* $\mathcal{R}$ *is* uniformly Abel-ergodic.
(b) 0 *is a pole of* $\mathcal{R}$ *of order at most* 1.

A C_0-semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on a Banach space E with generator A is called *uniformly Abel-ergodic* if $s(A) \leq 0$ and $P_{\mathcal{T}} := \lim_{s\downarrow 0} s\mathcal{R}(s, A)$ exists in $\mathcal{L}(E)$. The operator $P_{\mathcal{T}}$ is called the *ergodic projection* corresponding to $\mathcal{T}$. We obtain the following inheritance result on uniform Abel-ergodicity.

THEOREM 7.6. *Let* E *be a Banach lattice and let* $\mathcal{T} = (T(t))_{t\geq 0}$ *and* $\mathcal{S} = (S(t))_{t\geq 0}$ *be* C_0*-semigroups on* E *with generator* A *and* B, *respectively, such that* $\mathcal{S}$ *is dominated by* $\mathcal{T}$. *If* $\mathcal{T}$ *is uniformly Abel-ergodic with ergodic projection* $P_{\mathcal{T}}$ *of finite rank, then* $\mathcal{S}$ *is uniformly Abel-ergodic with ergodic projection* $P_{\mathcal{S}}$ *of finite rank.*

PROOF. Lemma 7.5 implies that 0 is a Riesz point of A. From Theorem 6.4 we know that 0 is a Riesz point of B. In particular, 0 is a pole of the resolvent $R(\cdot, B)$. Since $R(\cdot, B)$ is dominated by $R(\cdot, A)$ we have $\limsup_{s\downarrow 0} \|sR(s, B)\| \leq \limsup_{s\downarrow 0} \|sR(s, A)\| < \infty$. Thus 0 is a pole of order at most one of $R(\cdot, B)$, and the assertion follows from Lemma 7.5. □

We point out that the same result can be shown for dominated pseudo-resolvents (instead of Theorem 6.4 one has to use Theorem 5.2). An example of Arendt and Batty (see [5, Example 3.1]) shows that in Theorem 7.6 the rank condition on $P_{\mathcal{T}}$ cannot be omitted.

For a C_0-semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on a Banach space E the Cesàro means $C(t) \in \mathcal{L}(E)$, $t > 0$, are defined by $C(t)x := (1/t)\int_0^t T(s)x\,ds$. The semigroup $\mathcal{T}$ is called *uniformly ergodic* if $P_{\mathcal{T}} := \lim_{t\to\infty} C(t)$ exists in $\mathcal{L}(E)$. As above we call $P_{\mathcal{T}}$ the *ergodic projection* corresponding to $\mathcal{T}$. The following result due to Shaw clarifies the connection between uniform ergodicity and uniform Abel-ergodicity (see [25, Theorem 4 and Proposition 7]). We set

$$\omega_1(\mathcal{T}) := \inf\left\{ r \in \mathbb{R} : \lim_{t\to\infty} \int_0^t e^{-\lambda s} T(s)x\,ds \text{ exists for all } \operatorname{Re}\lambda > r \text{ and all } x \in E \right\}.$$

PROPOSITION 7.7. *Let* $\mathcal{T} = (T(t))_{t\geq 0}$ *be a* C_0*-semigroup with generator* A *on the Banach space* E *such that* $\omega_1(\mathcal{T}) \leq 0$. *Then the following assertions are equivalent*:

(a) $\mathcal{T}$ *is uniformly ergodic.*
(b) $\lim_{t\to\infty} \|T(t)R(1, A)\| = 0$ *and* $\mathcal{T}$ *is uniformly Abel-ergodic.*

Moreover, the corresponding ergodic projections coincide.

Together with Theorem 7.6 this yields the following inheritance result on uniform ergodicity which generalizes [20, Theorem 3.4].

THEOREM 7.8. *Let E be a Banach lattice and let $\mathcal{T} = (T(t))_{t\geq 0}$ and $\mathcal{S} = (S(t))_{t\geq 0}$ be C_0-semigroups on E with generator A and B, respectively, such that $\mathcal{S}$ is dominated by $\mathcal{T}$. If $\mathcal{T}$ is uniformly ergodic with ergodic projection $P_{\mathcal{T}}$ of finite rank, then $\mathcal{S}$ is uniformly ergodic with ergodic projection $P_{\mathcal{S}}$ of finite rank.*

PROOF. The uniform ergodicity of $\mathcal{T}$ implies $\omega_1(\mathcal{T}) \leq 0$ (see [25, Proposition 8]). Since $\mathcal{S}$ is dominated by $\mathcal{T}$ we have $\omega_1(\mathcal{S}) \leq \omega_1(\mathcal{T}) \leq 0$. Moreover, $R(\cdot, B)$ is dominated by $R(\cdot, A)$. Thus $\|S(t)R(1, B)\| \leq \|T(t)R(1, A)\|$, and hence $\lim_{t\to\infty} \|S(t)R(1, B)\| = 0$. Theorem 7.6 implies that $\mathcal{S}$ is uniformly Abel-ergodic with ergodic projection $P_{\mathcal{S}}$ of finite rank. Now the assertion follows from Proposition 7.7. □

We point out that a corresponding result on the inheritance of uniform stability for dominated positive semigroups has been shown in [20, Theorem 3.6].

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Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
D-72076 Tübingen
Germany
e-mail: frra@michelangelo.mathematik.uni-tuebingen.de
e-mail: manfred.wolff@uni-tuebingen.de