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ON SOME EFFECTIVE RESULTS INVOLVING ZEROS OF THE RIEMANN ZETA FUNCTION

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In this thesis, we present a collection of explicit results relating to the size of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 1$$

(where $s = \sigma + it$), a zero-free region, upper bounds on the associated functions $\zeta'(s)/\zeta(s)$ and $1/\zeta(s)$, and by extension, Mertens function

$$M(x) = \sum_{n \le x} \mu(n),$$

where μ denotes the Möbius function. In doing so, we showcase how all these results are deeply linked together. Amongst other things, information regarding the above-mentioned items is of central importance in the study of primes.

First, we provide a list of effective upper bounds on $\zeta(s)$ for some $\sigma \ge 1/2$. Both point-wise and bounds uniform in σ are presented. We give bounds of various orders, for example,

$$\log t, \quad \frac{\log t}{\log\log t}, \quad (\log t)^{2/3},$$

and so on. Results given in this chapter are taken from [1, 3, 5, 6].

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Next, current results in the literature concerning the location of the zeros are discussed, in the form of zero-free regions. We define the three commonly seen types of zero-free regions: for $t \geq 3$, we have $\zeta(s) \neq 0$ in the regions

$$\sigma \geq 1 - \frac{1}{Z_1 \log t}, \quad \sigma \geq 1 - \frac{\log\log t}{Z_2 \log t} \quad \text{and} \quad \sigma \geq 1 - \frac{1}{Z_3 (\log t)^{2/3} (\log\log t)^{1/3}}$$

for some positive constants Z_1, Z_2, Z_3. These regions are known as the *classical zero-free region*, the *Littlewood zero-free region* and the *Korobov–Vinogradov zero-free region*, respectively. We also discuss a method of Heath-Brown [2] using a function theoretic result related to a theorem of Jensen, which gives us a zero-free region better than what one can expect with a commonly used device of Stechkin [8].

Using a zero-free region together with our results on upper bounds for $\zeta(s)$, we prove explicit upper bounds for $\zeta'(s)/\zeta(s)$ and $1/\zeta(s)$ by the classical method, which uses complex function theoretic results. This was first made explicit by Trudgian [9] and normally yields bounds of order $\log t$. Working in the Littlewood zero-free region, we extend the method to produce effective versions of

$$\frac{\zeta'(s)}{\zeta(s)} \ll \frac{\log t}{\log\log t} \quad \text{and} \quad \frac{1}{\zeta(s)} \ll \frac{\log t}{\log\log t}.$$

We note that zero-free regions for $\zeta(s)$ are essential here, since such bounds are only valid wherever $\zeta(s)$ does not vanish. This illustrates the importance of effective results, since any constants involved tend to rapidly increase as σ approaches the edge of the zero-free region, that is, a possible pole. The results here are from [1, 3].

We then improve on the classical method just mentioned, obtaining better leading constants for effective bounds of the form

$$\frac{\zeta'(s)}{\zeta(s)} \ll \log t \quad \text{and} \quad \frac{1}{\zeta(s)} \ll \log t$$

in the classical zero-free region. In addition, we use an idea involving nonnegative trigonometric polynomials to get a power saving in the case of upper bounds for $1/\zeta(s)$. In particular, we provide explicit bounds of the type

$$\frac{1}{\zeta(s)} \ll (\log t)^{11/12} \quad \text{and} \quad \frac{1}{\zeta(s)} \ll (\log t)^{2/3} (\log\log t)^{1/4},$$

which hold in the classical and Korobov–Vinogradov zero-free region, respectively. The results here come from [5, 6].

A brief discussion on trigonometric polynomials is then conducted, since they make recurring appearances throughout the thesis. They are useful in widening zero-free regions, as well as improving upper bounds on $1/\zeta(s)$. We prove that for the purposes of improving explicit constants for bounds on $1/\zeta(s)$, the classical polynomial is optimal, which is surprising. The material here is based on [7].

Finally, we turn to the summatory Möbius function $M(x)$. The oscillatory nature of μ makes estimating $M(x)$ tricky and, to date, its true order is still unknown. This again highlights the importance of effective estimates, which we provide here. The main technique used is Perron's formula applied to $M(x)$, for which we require estimates for $1/\zeta(s)$ and, by extension, a zero-free region. We mention a number of different types of bounds for $M(x)$ here, one example being

$$|M(x)| \leq \frac{2.9189x}{(\log x)^2} \quad \text{for } x > 1$$

(see [4]), and another being an explicit bound of the type

$$M(x) \ll x \exp(-c_1\sqrt{\log x})$$

for some constant $c_1 > 0$. More significantly, we prove an effective version of

$$M(x) \ll x \exp(-c_2(\log x)^{3/5}(\log \log x)^{-1/5})$$

for some constant $c_2 > 0$, by using the Korobov–Vinogradov zero-free region. This is asymptotically the strongest unconditional estimate known. The result is based on [5].

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