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A CONVOLUTION-INDUCED TOPOLOGY ON THE ORLICZ SPACE OF A LOCALLY COMPACT GROUP

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Abstract

Let G be a locally compact group with a fixed left Haar measure. In this paper, given a strictly positive Young function Φ, we consider $L^{\Phi}(G)$ as a Banach left $L^1(G)$-module. Then we equip $L^{\Phi}(G)$ with the strict topology induced by $L^1(G)$ in the sense of Sentilles and Taylor. Some properties of this locally convex topology and a comparison with weak*, bounded weak* and norm topologies are presented.

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1. Introduction

Throughout let G be a locally compact group with a fixed left Haar measure. In [5], Crombez and Govaerts considered $L^{\infty}(G)$ as a Banach left $L^1(G)$-module with the convolution as module operation, and then introduced and studied a locally convex topology, denoted τ_c, on the Banach space $L^{\infty}(G)$ induced by $L^1(G)$. Among other things, they showed that τ_c is complete on norm-bounded subsets in $L^{\infty}(G)$; that is, $(L^{\infty}(G), \tau_c)$ is quasi-complete. Motivated by their work, we consider Orlicz spaces on a locally compact group G associated to a strictly positive Young function Φ as a Banach left $L^1(G)$-module with the convolution as module operation. Then we investigate some interesting properties of the induced topology. For our study we use the notion of the strict topology in the sense of Sentilles and Taylor [24]. This approach places our study in a correct frame, and enables us to generalize and extend some nice results in [5]. For instance, we show that $L^{\Phi}(G)$ with the induced topology is complete. We also examine when this topology coincides with norm, weak*, or bounded weak* topologies, and we also give a necessary and sufficient condition for its metrizability.

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Let us remark that Orlicz spaces are a genuine generalization of classical Lebesgue spaces. We would also like to mention [7, 19–21] in which certain linear topologies on Orlicz spaces and more general function spaces has been considered as well.

In the next section, we give necessary definitions and notations concerning Orlicz spaces and strict topology, and in Section 3 we present our results.

2. Preliminaries

Throughout this work, let G be a locally compact group with a fixed left Haar measure λ. By $\int_G f(x)\,dx$ we denote the integration of a function f defined on G with respect to λ. Also, let $L^0(G)$ denote the set of all equivalence classes of λ-measurable complex-valued functions on G. For measurable functions f and g on G, the *convolution* product

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)\,dy$$

is defined at each point $x \in G$ for which this makes sense. We denote by f^* the function defined by $f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$ for all $x \in G$, where Δ denotes the modular function on G. Also recall that the right and left translations of f by $t \in G$ defined as $R_t f(x) = \Delta(t^{-1})f(xt^{-1})$ and $L_t f(x) = f(t^{-1}x)$ for all $x \in G$, respectively. The characteristic function of a subset $A \subseteq G$ is denoted by χ_A. For background on analysis on locally compact groups, we refer to [9].

We refer to two excellent books [16, 22] for more details concerning Orlicz spaces. A function $\Phi : \mathbb{R} \to [0, \infty]$ is called a *Young function* if Φ is a convex, even, and left continuous function with $\Phi(0) = 0$ which is neither identically zero nor identically infinite.

For any Young function Φ let

$$\Psi(x) = \sup\{xy - \Phi(y) : y \in \mathbb{R}\} \quad (x \in \mathbb{R}).$$

It is easily verified that Ψ is a Young function called the *complementary Young function to* Φ. It should be remarked that Φ is also the complementary Young function to Ψ. Then (Φ, Ψ) is called a *complementary pair* of Young functions.

Let Φ be a Young function. For $f \in L^0(G)$ define

$$\rho_\Phi(f) = \int_G \Phi(|f(x)|)\,dx.$$

Then the *Orlicz space* $L^\Phi(G)$ is defined by

$$L^\Phi(G) = \{f \in L^0(G) : \rho_\Phi(af) < \infty, \text{ for some } a > 0\}.$$

We also set

$$M^\Phi(G) = \{f \in L^0(G) : \rho_\Phi(af) < \infty, \text{ for all } a > 0\}.$$

Then $L^\Phi(G)$ is a Banach space under the norm $N_\Phi(\cdot)$, called the *Luxemburg–Nakano norm*, defined for $f \in L^\Phi(G)$ by

$$N_\Phi(f) = \inf\{k > 0 : \rho_\Phi(f/k) \le 1\}.$$

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It is well known that $N_\Phi(f) \le 1$ if and only if $\rho_\Phi(f) \le 1$. Furthermore, if the Young function Φ is strictly increasing and continuous, then using the complementary Young function Ψ, another norm $\|\cdot\|_\Phi$, called the *Orlicz norm*, is defined on $L^\Phi(G)$ in the following way:

$$\|f\|_\Phi = \sup\Big\{\int_G |fg|\,d\lambda : \rho_\Psi(g) \le 1\Big\}.$$

Let us remark that $\|\cdot\|_\Phi$ is equivalent to $N_\Phi(\cdot)$; in fact, $N_\Phi(f) \le \|f\|_\Phi \le 2N_\Phi(f)$, for every $f \in L^\Phi(G)$.

For $1 \le p \le \infty$, classical Lebesgue spaces on G with respect to the Haar measure λ will be denoted by $L^p(G)$ with the norm $\|\cdot\|_p$ as defined in [9]. It is clear that $L^p(G)$ is an elementary example of the Orlicz space $L^\Phi(G)$.

We now give some definitions and facts about general strict topology. Strict topology has been studied extensively by many authors; see, for example, [8, 13–15, 24–26]. To introduce this topology, we need to recall and fix some definitions and notation concerning Banach modules. A bounded net (e_ι) in the Banach algebra A is called a *bounded approximate identity* if $\|e_\iota a - a\| \to 0$ and $\|ae_\iota - a\| \to 0$ for all $a \in A$. Let A be a Banach algebra, and let V be a Banach space. Then V together with the continuous bilinear map $(a, v) \mapsto a \cdot v$, $A \times V \to V$ is called a Banach left A-module if $a \cdot (b \cdot v) = ab \cdot v$ for all $a, b \in A$ and $v \in V$. A Banach right A-module is defined similarly.

If V is a Banach left A-module, then V is called *faithful* if, for each $v \in V \setminus \{0\}$, there exists $a \in A$ with $a \cdot v \ne 0$. If V is a Banach left A-module, then V^*, the topological dual of V, is a Banach right A-module with the dual module operation $(v^*, a) \mapsto v^* \diamond a$, $V^* \times A \to V^*$, specified by the formula $\langle v^* \diamond a, v\rangle = \langle v^*, a \cdot v\rangle$ for all $a \in A$, $v \in V$, and $v^* \in V^*$.

We are now prepared to introduce the strict topology for Banach modules. Let $(V, \|\cdot\|)$ be a faithful Banach left A-module, where A is a Banach algebra with a bounded approximate identity. Then the *strict topology* β on V induced by A is defined as the locally convex topology on V generated by the family of seminorms $\mathcal{P}_a(v) = \|a \cdot v\|$ for all $a \in A$, $v \in V$. In particular, the sets $U_a = \{v \in V : \|a \cdot v\| \le 1\}$, for $a \in A$, form a neighborhood base at zero for the topology β.

3. Results

Throughout this work let Φ be a strictly positive Young function. It is known from [4, Theorem 2.5] that $L^\Phi(G)$ can be considered as a Banach left $L^1(G)$-module under the convolution as module operation. Recall that the Banach algebra $L^1(G)$ contains a bounded approximate identity (e_ι) where $e_\iota = \chi_{U_\iota}/\lambda(U_\iota)$ and U_ι runs through the directed set of all compact symmetric neighborhoods of G, and $L^\Phi(G)$ is a faithful $L^1(G)$-module. Thus we can equip $L^\Phi(G)$ with the strict topology induced by $L^1(G)$, denoting it by β_c in the sequel.

We start with the following result on completeness of β_c.

PROPOSITION 3.1. *Suppose that the complementary function to Φ is finite-valued. Then the space $(L^\Phi(G), \beta_c)$ is a complete locally convex space.*

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Proof. By [24, Theorem 3.5], it is enough to prove that the mapping $f \mapsto T_f$, $T_f(g) = g * f$, from $L^\Phi(G)$ into $\mathrm{Hom}_{L^1(G)}(L^1(G), L^\Phi(G))$ is onto, where $\mathrm{Hom}_{L^1(G)}(L^1(G), L^\Phi(G))$ denotes the Banach space of all bounded module homomorphisms from $L^1(G)$ into $L^\Phi(G)$. To see this, let $T \in \mathrm{Hom}_{L^1(G)}(L^1(G), L^\Phi(G))$ be arbitrary. Let $(e_\iota)_\iota$ be an approximate identity for $L^1(G)$ with $\|e_\iota\|_1 = 1$ for all ι. Now from the duality $L^\Phi(G) = (M^\Psi(G), N_\Psi(\cdot))^*$ and the fact that Ψ is complementary to Φ together with the Banach–Alaoglu theorem, we can assume that the net $(T(e_\iota))_\iota$ converges in the weak* topology $\sigma(L^\Phi(G), M^\Psi(G))$ to some $h \in L^\Phi(G)$ with $N_\Phi(h) \leq \|T\|$. For all $f \in L^1(G)$ and $g \in M^\Psi(G)$, we have

$$\langle g, T(f * e_\iota)\rangle = \langle g, f * T(e_\iota)\rangle = \langle f^* * g, T(e_\iota)\rangle,$$

which converges to $\langle f^* * g, h\rangle = \langle g, f * h\rangle$. On the other hand,

$$|\langle g, T(f * e_\iota)\rangle - \langle g, T(f)\rangle| \leq N_\Psi(g)\|T\| \cdot \|f * e_\iota - f\|_1,$$

which tends to zero because (e_ι) is a bounded approximate identity for $L^1(G)$. It follows that, for all $f \in L^1(G)$ and $g \in M^\Psi(G)$, we have $\langle g, T(f)\rangle = \langle g, f * h\rangle$. This implies that $T(f) = f * h$, for every $f \in L^1(G)$, and hence $T = T_h$. □

As an immediate consequence of [24, Theorem 4.1 and Corollary 4.7] together with Proposition 3.1, we have the following corollary.

Corollary 3.2. *The dual of $(L^\Phi(G), \beta_c)$ can be identified with $L^\Phi(G)^* \diamond L^1(G)$. In particular, if $L^\Phi(G) = M^\Phi(G)$ and Ψ vanishes only at zero, then $(M^\Phi(G), \beta_c)^* = L^\Psi(G) \diamond L^1(G)$.*

Example. Let $\Phi(x) = |x|^p/p$, for $p \geq 1$. Then, by Corollary 3.2, we have $(L^p(G), \beta_c)^* = L^q(G)$ for $p > 1$, where $1/p + 1/q = 1$. Note that, by [24], we have $(L^1(G), \beta_c)^* = LUC(G)$, the Banach space of all bounded left uniformly continuous functions on G.

In the next result we show that β_c coincides with the norm topology only when the group is discrete. For the proof we need an interesting result due to Yap [29]. We state it for the convenience of the reader.

Theorem 3.3. *Let A be a Banach algebra, and V a Banach left A-module. Let Y be a closed subspace of V such that $a \cdot v \in Y$ for all $a \in A, v \in V$. Then:*

(i) *$\{v \in V : A \cdot v = Y\}$ is an open subset of V;*
(ii) *$\{a \in A : a \cdot V = Y\}$ is an open subset of A.*

Now we are ready to prove the next result.

Proposition 3.4. *The following assertions hold.*

(i) *The strict topology β_c coincides with the norm topology generated by $N_\Phi(\cdot)$ on $L^\Phi(G)$ if and only if G is discrete.*
(ii) *$(L^\Phi(G), \beta_c)$ is a metrizable space if and only if G is discrete.*

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Proof. (i) Let G be discrete. If e is the identity of G, then χ_e is an identity element for the module operation on $L^\Phi(G)$, and hence the map $f \mapsto \chi_e * f$ is the identity map on $L^\Phi(G)$. Now the result follows directly from [24, Theorems 2.4 and 3.2].

Conversely, suppose that the topology β_c coincides with the norm topology and G is not discrete. Then, again by [24, Theorems 2.4 and 3.2], there exists an element f in $L^1(G)$ such that $f * L^\Phi(G) = L^\Phi(G)$. According to Theorem 3.3 and the fact that $C_c(G)$, the space of continuous functions with compact support, is dense in $L^1(G)$, there is a function $h \in C_c(G)$ such that $h * L^\Phi(G) = L^\Phi(G)$. Also, for all $g \in L^\Phi(G)$ and $x \in G$,

$$\begin{aligned}|h * g(x)| &= \left|\int_G h(y)g(y^{-1}x)\,dy\right| \\ &= \left|\int_G h(y^{-1})g(yx)\Delta(y^{-1})\,dy\right| \\ &\leq 2(1+K)^2 \max\{1+\Delta(x^{-1})\}N_\Psi(h)N_\Phi(g)\end{aligned}$$

where $K = \sup\{\Delta(x^{-1}) : x \in \text{supp}(h)\}$. Since the modular function Δ is continuous, $h * g$ is bounded on every compact subset of G. Now let V be a compact neighborhood of the identity element of G. Since G is not discrete, there exists a sequence $(K_n)_{n=1}^\infty$ of pairwise disjoint measurable subsets of V with $\lambda(K_n) = 2^{-n}$. Define $g = \sum_{n=1}^\infty \Phi^{-1}(2^{n/2})\chi_{K_n}$. Then $g \in L^\Phi(G)$, but g is clearly not bounded λ-almost everywhere on V. Therefore $h * L^\Phi(G) \neq L^\Phi(G)$. Thus we have arrived at a contradiction.

(ii) This is immediate from part (i) and [24, Theorem 3.2]. □

In the next result we compare the topology β_c with the weak* topology on $L^\Phi(G)$.

Theorem 3.5. *The weak* topology* $\sigma(L^\Phi(G), M^\Psi(G))$ *coincides with the strict topology* β_c *if and only if G is finite.*

Proof. Let G be finite. Then $L^\Phi(G)$ is of finite dimension, and hence all locally convex topologies on $L^\Phi(G)$ are equal.

For the converse, first suppose that G is discrete. Then, by Theorem 3.4, the strict topology and the norm topology on $L^\Phi(G)$ are equal, and hence the weak* topology $\sigma(L^\Phi(G), M^\Psi(G))$ coincides with the norm topology generated by $N_\Phi(\cdot)$. This implies that $L^\Phi(G)$ is of finite dimension. Now, since G is discrete, it follows that G must be finite.

Second, we suppose that G is not discrete. The proof will be complete if we construct a β_c-neighborhood at zero such that it is not a weak* neighborhood of zero. To do this, we borrow the method used in part (b) of [28, Theorem 5]. Fix an $x \in G$ that is not the identity element of G. Then there is a compact symmetric neighborhood U of the identity element of G such that $\lambda(U) < \frac{1}{2}$ and $U \cap xU = \varnothing$. Let $F = U \cup \{x\}$. Let $\mathcal{A}$ be the collection of all open symmetric neighborhoods V of the identity with $V \subset U$. For $V \in \mathcal{A}$ let $F_V = F\overline{V}$ and $E_V = xV \cap (G \backslash F)$. Since G is not discrete, x is a boundary point of F. It follows that E_V is a nonempty open subset of G and thus $\lambda(E_V) > 0$. Hence, for all $V \in \mathcal{A}$, we have $\lambda(F) < \lambda(F) + \lambda(E_V) \leq \lambda(F_V) \leq \lambda(F^2)$. Also, $\{F_V : V \in \mathcal{A}\}$ forms a neighborhood base for F. Thus, by the regularity of Haar measure λ, $\lambda(F_V) \to \lambda(F)$, and so there is a decreasing sequence (V_n) in $\mathcal{A}$ with $\lambda(F_{V_n})$ all distinct and $\lambda(F_{V_n}) \to \lambda(F)$. In particular, $\lambda(F_{V_n} \setminus F_{V_{n+1}}) > 0$ for all $n \geq 1$.

Let $f_0 = \chi_F$ and $g_n = \chi_{\overline{V}_n}/N_\Phi(\chi_{\overline{V}_n})$. Since, for each $s \in G$, we have $f_0 * g_n(s) = \lambda(F \cap s\overline{V}_n)/N_\Phi(\chi_{\overline{V}_n})$, $\operatorname{supp}(f_0 * g_n) = F_{V_n}$, we infer that the sequence $\{f_0 * g_n : n \geq 1\}$ is linearly independent and contained in the strict neighborhood U_{f_0}. Let A be the subspace of $L^\Phi(G)$ consisting of all $g \in L^\Phi(G)$ with $f_0 * g = 0$, for all $n \geq 1$. Note that $g_n \notin A$ for all $n \geq 1$. Thus A has infinite codimension. This implies that any subspace B of $L^\Phi(G)$ contained in U_{f_0} has infinite codimension. This is because for any scalar c, $cg \in B$ whenever $g \in B$, and we obtain $|c|\,N_\Phi(f_0 * g) \leq 1$, for $g \in B$ and any scalar c. Therefore $f_0 * g = 0$ for $g \in B$, and hence $B \subseteq A$. However, any weak* neighborhood of zero contains a subspace of $L^\Phi(G)$ with finite codimension. It follows that U_{f_0} is not a weak* neighborhood of zero, whereas it is a strict neighborhood of zero. This contradiction implies that G is discrete. Finally, together with the first case, this implies that G is finite. □

If X is a Banach space, then the *bounded weak* topology* $b\sigma(X^*, X)$ on its dual X^* is the strongest topology which coincides with $\sigma(X^*, X)$ on bounded subsets; see [6, Section V.5]. For a study of this topology in the setting of analytic functions; see [23].

The following theorem generalizes and extends [27, Theorem 4.3].

Theorem 3.6. *Let G be a locally compact group. Then G is compact if and only if the bounded weak* topology $b\sigma(L^\Phi(G), M^\Psi(G))$ on $L^\Phi(G)$ coincides with the topology β_c.*

Proof. Let G be a compact group with the normalized Haar measure; that is, $\lambda(G) = 1$, and $B = \{h \in L^\Phi(G) : N_\Phi(h) \leq C\}$ for any $C > 0$. It suffices to establish that for any β_c-neighborhood V of zero, there exists a weak* neighborhood E of zero such that $E \cap B \subset V$. But, by virtue of [24, Theorem 3.3], the β_c topology coincides with the κ-topology on B. Here κ is the locally convex topology on $L^\Phi(G)$ generated by the seminorms $f \mapsto \|e_\iota * f\|_\Phi$ in which (e_ι) is a bounded approximate identity of $L^1(G)$. Hence we can work only with the κ-neighborhoods of zero. For any ι, let $V_\iota = \{g \in L^\Phi(G) : N_\Phi(e_\iota * g) \leq 1\}$. By the compactness of G and the continuity of the right translation mapping $x \mapsto R_x g$ from G into $(M^\Psi(G), N_\Psi(\cdot))$, [2, Lemma 4.1], it follows that each e_ι is an almost periodic function on $(M^\Psi(G), N_\Psi(\cdot))$. This means that, given ϵ with $\Phi(\epsilon) \leq 1$, there are elements $x_1, x_2, \ldots, x_n$ in G such that, for any $x \in G$, we can choose an element x_i $(1 \leq i \leq n)$ such that

$$N_\Psi(R_x e_\iota - R_{x_i} e_\iota) < \frac{\epsilon}{4C}.$$

This implies that for every $g \in B$ and the corresponding x_i to x,

$$\begin{aligned} |e_\iota * g(x) - e_\iota * g(x_i)| &= |\langle R_x e_\iota, g\rangle - \langle R_{x_i} e_\iota, g\rangle| \\ &\leq 2N_\Psi(R_x e_\iota - R_{x_i} e_\iota) N_\Phi(g) < \frac{\epsilon}{2}, \end{aligned}$$

whence we obtain $|e_\iota * g(x)| < |e_\iota * g(x_i)| + \epsilon/2$ for all ι. Put

$$E = \Big\{h \in L^\Phi(G) : |\langle R_{x_i} e_\iota, h\rangle| < \frac{\epsilon}{2}\Big\}.$$

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Then E is a weak* neighborhood of zero. Also, for each $h \in E \cap B$, we get

$$\int_G \Phi(|e_\iota * h(x)|)\,dx = \int_G \Phi(|\langle R_x e_\iota, h\rangle|)\,dx \leq \int_G \Phi(\epsilon/2 + |\langle R_{x_i} e_\iota, g\rangle|)\,dx \leq \int_G \Phi(\epsilon)\,dx \leq 1.$$

Therefore $E \cap B \subset V_\iota$, for all ι, as required.

To prove the converse, we assume, by way of contradiction, that G is not compact. Then there are an infinite sequence (x_n) in G and symmetric compact neighborhoods U and V of the identity element of G such that

$$\Phi(\lambda(U))\lambda(U) > 1, \quad U^2 \subset V, \quad x_n V \cap x_m V = \varnothing \quad (n \neq m).$$

For every $n \geq 1$, set $f_n = \chi_{x_n V}$. Then the sequence (f_n) is a norm-bounded sequence in $L^\Phi(G)$. By invoking Hölder's inequality, for any $g \in M^\Psi(G)$,

$$\left|\int_G f_n(x)g(x)\,dx\right| \leq 2\left[\Phi^{-1}\left(\frac{1}{\lambda(V)}\right)\right]^{-1} N_\psi(g\chi_{x_n V}) \quad (n \geq 1).$$

However, the sequence $(N_\psi(g\chi_{V x_n}))$ tends to zero as $n \to \infty$. To show this, let $a > 0$ be arbitrarily chosen. Then

$$\sum_{n=1}^{\infty} \int_{x_n V} \Psi(a|g(x)|)\,dx \leq \int_G \Psi(a|g(x)|)\,dx < \infty.$$

Thus $f_n \to 0$ in the weak* topology. But, for $\chi_U \in L^1(G)$,

$$\begin{aligned}\int_G \Phi(\chi_U * f_n(x))\,dx &\geq \int_U \Phi\left(\int_U \chi_U(y)\chi_{x_n V}(y^{-1}x)\,dy\right)dx \\ &= \int_U \Phi\left(\int_U \chi_U(yx_n^{-1})\chi_V(y^{-1}x)\Delta(x_n^{-1})\,dy\right)dx \\ &= \int_U \Phi\left(\int_U \chi_U(yx_n^{-1})\Delta(x_n^{-1})\,dy\right)dx \\ &\geq \Phi(\lambda(U))\lambda(U) > 1.\end{aligned}$$

It follows that $N_\Phi(\chi_U * f_n) \geq 1$ for all $n \geq 1$. Hence (f_n) does not converge to zero in the topology κ and hence in the β_c topology. □

As a direct consequence of the preceding theorem we show that the weak* and strict topologies are sequentially equivalent if and only if G is compact.

Corollary 3.7. *Any weak* convergent sequence is β_c convergent if and only if G is compact.*

Proof. Let G be compact and let $(g_n)_n$ be a sequence in $L^\Phi(G)$ that converges in the weak* topology to some $g \in L^\Phi(G)$. Then the set $A = \{g_n : n \geq 1\} \cup \{g\}$ is weak*

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compact in $L^\Phi(G)$, and hence norm-bounded. Now, by Theorem 3.6, we get that $g_n \to g$ in the β_c topology.

For the converse, as in Theorem 3.6 we can construct a sequence $(g_n)_n$ in $L^\Phi(G)$ such that for every $n \in \mathbb{N}$, $N_\Phi(g_n) \geq 1$ and $g_n \to 0$ in the weak* topology but (g_n) does not converge to zero in the β_c topology. □

Remark 3.8. Corollary 3.7 does not hold if the norm topology is replaced by the strict topology. Indeed, by [11] or [18], weak* convergence is equivalent to norm convergence for sequences in $L^\Phi(G)$ if and only if G is finite.

Theorem 3.9. *Suppose that the complementary function to Φ is finite-valued. Then a weak* closed linear subspace of $L^\Phi(G)$ is an $L^1(G)$-submodule if and only if it is left translation invariant.*

Proof. Suppose that I is a left translation invariant subspace. We have to show that $f * g \in I$ for every $f \in L^1(G)$ and $g \in I$. Let $h \in M^\Psi(G)$ be such that $\int_G h(x)g(x)\,dx = 0$ for all $f \in I$. Then, for $g \in I$ and any $f \in L^1(G)$,

$$\begin{aligned}\int_G (f * g)(x)h(x)\,dx &= \int_G h(x)\Big(\int_G f(y)g(y^{-1}x)\,dy\Big)\,dx\\ &= \int_G f(y)\Big(\int_G L_y g(x)h(x)\,dx\Big)\,dy = 0.\end{aligned}$$

Since $(M^\Psi(G), N_\Psi(\cdot))^* = L^\Phi(G)$, the Banach–Alaoglu theorem implies that $f * g \in I$ for all $f \in L^1(G)$ and $g \in I$. Hence I is an $L^1(G)$-submodule.

Conversely, let I be an $L^1(G)$-submodule of $L^\Phi(G)$ and $s \in G$ be arbitrary. Let $g \in I$ and let V be a symmetric compact neighborhood of the identity in G. Then for any $h \in M^\Psi(G)$ we have

$$\begin{aligned}\Big\langle h, L_x g - \frac{\chi_{xV} * g}{\lambda(V)}\Big\rangle &= \int_G h(x)\Big(\int_G \frac{g(s^{-1}x)\chi_V(s^{-1}y)}{\lambda(V)}\,dy - \int_G \frac{\chi_V(s^{-1}y)g(y^{-1}x)}{\lambda(V)}\,dy\Big)\,dx\\ &= \int_V \frac{\chi_V(y)}{\lambda(V)} \int_G h(x)(g(s^{-1}x) - g(y^{-1}s^{-1}x))\,dx\,dy\\ &= \int_V \frac{\chi_V(y)}{\lambda(V)} \int_G (h(sx) - h(syx))g(x)\,dx\,dy\\ &\leq 2 \sup_{y\in V} N_\Phi(g)N_\Psi(L_{s^{-1}}h - L_{y^{-1}}(L_{s^{-1}}h)).\end{aligned}$$

Since, for any $f \in M^\Psi(G)$, the map $y \mapsto L_{y^{-1}}f$ from G into $M^\Psi(G)$ is continuous [2], for every $\epsilon > 0$ there exists V such that

$$\sup_{y\in V} N_\Psi(L_{s^{-1}}h - L_{y^{-1}}(L_{s^{-1}}h)) < \epsilon.$$

This implies that each $g \in I$ belongs to the weak* closed subset $L^1(G) * g$. As I is a weak* closed $L^1(G)$-submodule, it follows that $L_s g \in I$. This completes the proof. □

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We conclude this work with the following observation. It is interesting to compare this result with [28, Theorem 5.d] in which it was proved that, for a unimodular locally compact group G, $L^1(G)$ with the locally convex topology β^1, introduced there, is a topological algebra if and only if G is compact. The second named author has shown in [17] that the topology β^1 is, in fact, a strict topology in the sense of Sentilles and Taylor. We refer the interested reader to [3] for a more recent study of a generalization of the topology β^1 for Orlicz spaces. Here, our results show that β_c is different from β^1.

Let us remark that $L^\Phi(G)$ is not, in general, closed under the convolution product; for more information see [1, 2, 12]. If G is locally compact abelian group, it was proved in [10] that $L^\Phi(G)$ is closed under the convolution product if and only if $\lim_{x\to 0} \Phi(x)/x > 0$ or G is compact. With this in mind, we present our final result.

Proposition 3.10. *Let G be a locally compact abelian group and let $L^\Phi(G) * L^\Phi(G) \subseteq L^\Phi(G)$. Then $(L^\Phi(G), \beta_c)$ is a topological algebra with the convolution as multiplication.*

Proof. It is well known, and easily follows from the Cohen–Hewitt factorization theorem, that $L^1(G) * L^1(G) = L^1(G)$; see [9, Corollary 32.30]. Thus each $\varphi \in L^1(G)$ can be written as $\varphi = \varphi_1 * \varphi_2$ for some $\varphi_1, \varphi_2 \in L^1(G)$. Now if $f, g \in L^\Phi(G)$ then

$$N_\Phi(\varphi * (f * g)) = N_\Phi((f * \varphi_1) * (g * \varphi_2)) \leq N_\Phi(f * \varphi_1) N_\Phi(g * \varphi_2),$$

from which the result follows. □

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