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NON-DEGENERATE REAL HYPERSURFACES IN COMPLEX MANIFOLDS ADMITTING LARGE GROUPS OF PSEUDO-CONFORMAL TRANSFORMATIONS. I

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Introduction

Let S (resp. S') be a (real) hypersurface (i.e. a real analytic submanifold of codimension 1) of an n-dimensional complex manifold M (resp. M'). A homeomorphism f of S onto S' is called a pseudo-conformal homeomorphism if it can be extended to a holomorphic homeomorphism of a neighborhood of S in M onto a neighborhood of S' in M. In case such an f exists, we say that S and S' are pseudo-conformally equivalent. A hypersurface S is called non-degenerate (index r) if its Levi-form is non-degenerate (and its index is equal to r) at each point of S.

In his paper [6], N. Tanaka has shown that if a hypersurface S is connected and non-degenerate at a point, then the group $A(S)$ of all pseudo-conformal transformations of S becomes a Lie transformation group of S with dim. $A(S) \leqq n^2 + 2n$.

The purpose of this paper is to determine, under pseudo-conformal equivalence, non-degenerate hypersurfaces S for which the groups $A(S)$ have either the largest dimension $n^2 + 2n$ or the second largest dimension.

Our main results are stated as follows;

THEOREM 7.2. *Let M be a complex manifold of dimension n. Let S be a connected non-degenerate* (*index r*) *homogeneous hypersurface* $\left(0 \leqq r \leqq \left[\frac{n-1}{2}\right]\right)$. *Then we have the following classification table*:

$$Q_r = \left\{(z_0, \cdots, z_n) \in P^n(C) \mid -\sqrt{-1}z_0\bar{z}_n - \sum_{i=1}^{r} z_i\bar{z}_i + \sum_{i=r+1}^{n-1} z_i\bar{z}_i + \sqrt{-1}z_n\bar{z}_0 = 0\right\},$$

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(n,r)	the case of the largest dimension		the case of the second largest dimension	
	dim. $A(S)$	S	dim. $A(S)$	S
$n=3$ & $r=1$	$15(=n^2+2n)$	Q_1	$11(=n^2+2)$	$Q_1{}^*(1)$
$n=5$ & $r=2$	$35(=n^2+2n)$	Q_2	$26(=n^2+1)$	$Q_2{}^*(2)$ or $Q_2{}^*$
otherwise	n^2+2n	Q_r	n^2+1	$Q_r{}^*$

$$Q_r^* = \{(z_0, \cdots, z_n) \in Q_r \mid z_0 \neq 0\} ,$$
$$Q_1^*(1) = \{(z_0, \cdots, z_3) \in Q_1 \mid |z_0| + |z_1 - z_2| \neq 0\} ,$$
$$Q_2^*(2) = \{(z_0, \cdots, z_5) \in Q_2 \mid |z_0| + |z_1 - z_4| + |z_2 - z_3| \neq 0\} ,$$

where $P^n(C)$ is the complex projective space of dimension n with its homogeneous coordinate $(z_0, \cdots, z_n)$.

This is a partial generalization of the results of E. Cartan [2] in the case $n = 2$.

THEOREM 7.4. *Let M be a complex manifold of dimension n. Let S be a connected hypersurface of M which is non-degenerate of index r at a point of S. If* dim. $A(S) = n^2 + 2n$, *then S is pseudo-conformally equivalent to Q_r.*

Now we will describe the method of proving our theorems. Let S be a non-degenerate (index r) hypersurface of a complex manifold, and let $A(S)$ be the group of all pseudo-conformal transformations of S and $\mathfrak{a}(S)$ be its Lie algebra. Then according to N. Tanaka [6], [7] we can associate with S a principal fibre bundle $P(S, G'(r))$ together with an infinitesimal structure ω on it, which is a Cartan connection of type $(G(r), G'(r))$, the so-called normal pseudo-conformal connection. Here $G(r)$ is the group of all projective transformations leaving Q_r invariant and $G'(r)$ is the isotropy subgroup of it at a point o of Q_r (cf. I). Let $\mathfrak{g}(r)$ be the Lie algebra of $G(r)$. If we fix a point p_0 of S, then the connection form ω induces an injective linear map of $\mathfrak{a}(S)$ (identified with the Lie algebra of right invariant vector fields of P leaving the Cartan connection invariant) into the graded Lie algebra $\mathfrak{g}(r) = \sum_{k=-2}^{2} \mathfrak{g}_k(r)$. So we can induce a filtration of $\mathfrak{a}(S)$ at p_0 via the map ω. With respect to this filtration $\mathfrak{a}(S) = \mathfrak{h}$ becomes a filtered Lie algebra. Moreover it is seen that the associated graded Lie algebra $\bar{\mathfrak{h}}$ of $\mathfrak{h}$ becomes a graded subalgebra of $\mathfrak{g}(r)$ (cf. II). So under the dimension hypothesis of $A(S)$

and the homogeneity assumption, we can determine explicitly the possibilities of $\tilde{\mathfrak{h}}$. In fact we determine the graded subalgebras of $\mathfrak{g}(r)$ of the minimum codimension satisfying a certain (homogeneity) condition (cf. IV). Moreover under the dimension hypothesis of $A(S)$ (more precisely if $\tilde{\mathfrak{h}}$ coincides with one of the graded subalgebras of $\mathfrak{g}(r)$ obtained in IV) we will see that S is flat, that is, the curvature form of the connection vanishes identically and that $\mathfrak{a}(S)$ is isomorphic with $\tilde{\mathfrak{h}}$ (cf V). Conversely let $\mathfrak{g}$ be one of the graded subalgebras of $\mathfrak{g}(r)$ obtained in IV. Then we can construct a model space Q corresponding to $\mathfrak{g}$ as follows; let G be the analytic subgroup of $G(r)$ corresponding to $\mathfrak{g}$. Q is defined as the orbit of G passing through $o \in Q_r$. Then Q is a connected non-degenerate (index r) homogeneous flat hypersurface of $P^n(C)$ for which G is the identity component of $A(Q)$ (cf. VI). On the other hand, the bundle $A(S)(S, A_{p_0}(S))$ can be regarded as a subbundle of $P(S, G'(r))$, if we assume that S is homogeneous. Moreover the structure equation of the connection determines the Maurer-Cartan equation of $A(S)$. From these facts we see that, in order to find a pseudo-conformal homeomorphism between two homogeneous hypersurfaces S and S', we have only to find a group isomorphism between $A(S)$ and $A(S')$ which satisfies certain additional conditions (cf. III). So under the dimension hypothesis we compare $A^0(S)$ with the corresponding G satisfying $\mathfrak{g} \cong \mathfrak{a}(S)$. In this way we see that S is pseudo-conformally equivalent to the corresponding Q (cf. VII).

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Preliminary remarks.

Throughout this paper we always assume the differentiability of class C^∞. We use the notations and terminology in S. Kobayashi-K. Nomizu [5] without special references (e.g. the differential of a mapping, fundamental vector fields, homomorphisms of fibre bundles).

Let I be a hermitian matrix of degree n. We denote by $U(I)$ the unitary group defined by I; $U(I) = \{\sigma \in GL(n, C) \mid {}^t\bar{\sigma} I \sigma = I\}$, where ${}^t\sigma$ is the transposed matrix of σ and $\bar{\sigma}$ is the complex conjugate matrix of σ. We denote by $\mathfrak{u}(I)$ the Lie algebra of $U(I)$. Moreover we denote by

$SU(I)$ the special unitary group defined by I; $SU(I) = \{\sigma \in U(I) \mid \det \sigma = 1\}$. We denote by $\mathfrak{su}(I)$ the Lie algebra of $SU(I)$.

I. Pseudo-conformal geometry.

In this section we will review the fundamental concepts of the pseudo-conformal geometry and state the results of Tanaka, following N. Tanaka [6], [7], which are necessary for later considerations.

1. The *H*-structure. Let M and M' be complex manifolds of dimension n ($n \geqq 2$). Let S (resp. S') be a (real) hypersurface, that is a $(2n-1)$-dimensional real analytic regular submanifold, of M (resp. M').

DEFINITION 1.1. A homeomorphism f of S onto S' is called a pseudo-conformal homeomorphism if it can be extended to a holomorphic homeomorphism of a neighborhood of S in M onto a neighborhood of S' in M'.

Let p be an arbitrary point of S. We denote by $T_p(S)$ the tangent space to S at p and by J the complex structure of M. We set

$$D_p = T_p(S) \cap J(T_p(S)) .$$

Then D_p is a maximal complex vector subspace of $T_p(M)$ contained in $T_p(S)$ and $\dim_{.C} D_p = n - 1$.

Take the natural base $\{e_i\}_{1 \leqq i \leqq n}$ of the n-dimensional complex number space C^n. We denote by $\mathfrak{m}$ the $(2n-1)$-dimensional real vector subspace of C^n spanned by the $2n-1$ vectors $e_1, \cdots, e_n, \sqrt{-1}e_1, \cdots, \sqrt{-1}e_{n-1}$ and by $\mathfrak{m}_*$ the $(n-1)$-dimensional complex vector subspace of C^n spanned by the $n-1$ vectors $e_1, \cdots, e_{n-1}$. We define a closed subgroup H of the general linear group $GL(n, C)$ by setting

$$H = \{\sigma \in GL(n, C) \mid \sigma(\mathfrak{m}) = \mathfrak{m}\} .$$

Each element of H is represented as a matrix of the following form

$$\begin{pmatrix} B & C \\ 0 & a \end{pmatrix}$$

where $a \in R\backslash\{0\}$, $B \in GL(n-1, C)$ and $C \in C^{n-1}$. Hence we get

$$H = \{\sigma \in GL(\mathfrak{m}) \mid \sigma(\mathfrak{m}_*) = \mathfrak{m}_* \text{ and } \sigma \mid \mathfrak{m}_* \text{ is complex linear}\}$$

We denote by $L(S)$ the bundle of linear frames of S. A linear frame

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x at a point p of S is a linear isomorphism of $\mathfrak{m}$ onto $T_p(S)$, where we identify $\mathfrak{m}$ with $\boldsymbol{R}^{2n-1}$ through the natural isomorphism. We define a subbundle F of $L(S)$ by

$$F = \{x \in L(S) \,|\, x(\mathfrak{m}_*) = D_{\varpi(x)} \text{ and } x|\mathfrak{m}_* \text{ is complex linear}\} ,$$

where ϖ is the bundle projection of $L(S)$ onto S. Then F becomes a principal fibre bundle over S with the structure group H. $F(S, H)$ is called the pseudo-conformal H-bundle associated with the hypersurface S (cf. [6]).

Remark 1.2. The "Fundamental theorem" (i.e. Theorem 1 [6]) says that a C^ω-homeomorphism f of a hypersurface S onto another hypersurface S' is a pseudo-conformal homeomorphism if and only if f induces an isomorphism between the corresponding pseudo-conformal H-bundles, preserving the canonical 1-forms.

2. The Levi-form. Let θ^* be the canonical 1-form on F (cf. [5]), that is,

$$\theta^*_x(X) = x^{-1}(\varpi_*(X)) = \begin{pmatrix} \theta^*_1(X) \\ \vdots \\ \theta^*_n(X) \end{pmatrix} \in \mathfrak{m} \subset \boldsymbol{C}^n \qquad \text{for } x \in F, X \in T_x(F) ,$$

where θ^*_i $(i = 1, 2, \cdots, n)$ is the i-th component of θ^*. Note that θ^*_i $(i = 1, \cdots, n-1)$ is a $\boldsymbol{C}$-valued 1-form on F and θ^*_n is a $\boldsymbol{R}$-valued 1-form on F. We pay attention to θ^*_n, which characterizes the maximal complex tangent space D_p of $T_p(S)$. First we notice

LEMMA 1.3. *Let x be an arbitrary point of F, and let X and Y be tangent vectors at x. Then we have*

(i) $\theta^*_n(X) = 0$ *if and only if* $\varpi_*(X) \in D_{\varpi(x)}$

(ii) $d\theta^*_n(X, Y) = 0$ *if* $\varpi_*(X) \in D_{\varpi(x)}$ *and* $\varpi_*(Y) = 0$.

Lemma 1.3 is easily proved from the definition of F and the following

$$\begin{cases} R^*_\sigma \theta^*_n = a^{-1}\theta^*_n & \text{for } \sigma = \begin{pmatrix} B & C \\ 0 & a \end{pmatrix} \in H \\ \theta^*_n(A^*) = 0 & \text{for } A \in \text{the Lie algebra of } H \end{cases}$$

where R_σ is a right action on F induced by $\sigma \in H$ and A^* is the fundamental vector field corresponding to A (cf. [5]).

From Lemma 1.3 we can define a skew-symmetric bilinear mapping K_x of $D_p \times D_p$ into $\boldsymbol{R}$ by

$$K_x(X, Y) = -2\, d\theta^*_{nx}(X^*, Y^*) \qquad p = \varpi(x),\ X, Y \in D_p\ ,$$

where X^* (resp. Y^*) is any vector at x such that $\varpi_*(X^*) = X$ (resp. $\varpi_*(Y^*) = Y$). One should note that we can also write

$$K_x(X, Y) = \theta^*_{nx}([X^*, Y^*])\ ,$$

where X^* (resp. Y^*) is any vector field around x such that $\theta^*_n(X^*) = 0$ (resp. $\theta^*_n(Y^*) = 0$) and $\varpi_*(X^*_x) = X$ (resp. $\varpi_*(Y^*_x) = Y$). Hence from the integrability condition of the complex structure of the ambient space M we have

LEMMA 1.4. *Let x be an arbitrary point of F. Then*

$$K_x(X, Y) = K_x(JX, JY) \qquad \text{for } X, Y \in D_{\varpi(x)}\ ,$$

where J is the complex structure of M.

Now Lemma 1.3 and Lemma 1.4 imply

LEMMA 1.5 ([6]). *There exist a 1-form β and unique $\boldsymbol{C}$-valued functions L_{ij} $(i, j = 1, 2, \cdots, n-1)$ on F such that*

$$d\theta^*_n + \sum_{i,j=1}^{n-1} L_{ij}\theta^*_i \wedge \bar\theta^*_j + \beta \wedge \theta^*_n = 0 \ \ (L_{ij} + \bar L_{ji} = 0)\ ,$$

where $\bar\theta^*_j$ is the complex conjugate 1-form of θ^*_j.

For $x \in F$, we set $L(x) = (L_{ij}(x))$. Then $\sqrt{-1}L(x)$ is a hermitian matrix of degree $n-1$. We call $\sqrt{-1}L(x)$ the Levi-form at $x \in F$. The Levi-form at x defines a hermitian inner product of $D_{\varpi(x)}$. In fact if we set;

$$L_x(X, Y) = K_x(JX, Y) + \sqrt{-1}K_x(X, Y) \qquad \text{for } X, Y \in D_{\varpi(x)}\ ,$$

then we have easily

$$L_x(X, Y) = 2\sum_{i,j=1}^{n-1} \sqrt{-1}L_{ij}(x)\xi_i\bar\eta_j\ ,$$

where

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$$x^{-1}(X) = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_{n-1} \\ 0 \end{bmatrix}, \qquad x^{-1}(Y) = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_{n-1} \\ 0 \end{bmatrix} \in \mathfrak{m}_* .$$

Now we will define the notion of a non-degenerate hypersurface and its index. Let p be a point of S. For $x \in \pi^{-1}(p)$, L_x is a hermitian inner product of D_p. Let $k(x)$ (resp. $l(x)$) be the dimension of a maximal subspace on which L_x is positive definite (resp. negative definite). We define an integer valued function $\lambda(p)$ on S by $\lambda(p) =$ minimum of $k(x)$ and $l(x)$. The integer $\lambda(p)$ is well-defined, that is, $\lambda(p)$ is independent of the choise of $x \in \pi^{-1}(p)$ ([6]), and satisfies $0 \leqq \lambda(p) \leqq \left[\frac{n-1}{2}\right]$.

DEFINITION 1.6. Let p be a point of S.

(1) S is called non-degenerate at p if the Levi-form is non-degenerate at p.

(2) S is called of index r at p if $\lambda(p) = r$.

S is called a non-degenerate hypersurface if its Levi-form is non-degenerate at each point of S. Obviously the index of a non-degenerate hypersurface S is constant on each connected component of S.

3. Quadrics. Let us fix an integer r satisfying $0 \leqq r \leqq \left[\frac{n-1}{2}\right]$. We will give the model space of non-degenerate (index r) hypersurface ([6]).

Let $P^n(C)$ be the n-dimensional complex projective space, and let $z_0, z_1, \cdots, z_n$ be the system of its homogeneous coordinates. We define the hermitian matrices I_r and $\tilde{I}_r$ of degree $n-1$ and $n+1$ by

$$I_r = \begin{pmatrix} -E_r & 0 \\ 0 & E_{n-r-1} \end{pmatrix}, \qquad \tilde{I}_r = \begin{pmatrix} 0 & 0 & \sqrt{-1} \\ 0 & I_r & 0 \\ -\sqrt{-1} & 0 & 0 \end{pmatrix}$$

where E_s is the unit matrix of degree s.

Let Q_r be the quadric of $P^n(C)$ defined by $\tilde{I}_r$, that is,

$$Q_r = \Big\{(z_0, \cdots, z_n) \in P^n(C) \mid -\sqrt{-1} z_0 \bar{z}_n - \sum_{i=1}^{r} z_i \bar{z}_i + \sum_{i=r+1}^{n-1} z_i \bar{z}_i + \sqrt{-1} z_n \bar{z}_0 = 0\Big\} .$$

It is known [6] that Q_r is a connected non-degenerate hypersurface of $P^n(C)$ and its index is r.

Let $P(n, C)$ be the group of all projective transformations. We consider the subgroup $G(r)$ of $P(n, C)$ which consists of all projective transformations leaving Q_r invariant. $G(r)$ acts effectively and transitively on Q_r as a group of pseudo-conformal transformations. Moreover if we identify $P(n, C)$ with $GL(n+1, C)/GL(1, C)$, the identity component of $G(r)$ is $U(\tilde{I}_r)/U(1) = SU(\tilde{I}_r)/\mathfrak{n}$, where $U(1)$ (resp. $\mathfrak{n}$) is the center of $U(\tilde{I}_r)$ (resp. $SU(\tilde{I}_r)$). $G(r)$ is connected in case $r \neq \frac{n-1}{2}$ and it has two connected components in case $r = \frac{n-1}{2}$ (n: odd integer). We denote by $G'(r)$ the isotropy subgroup of $G(r)$ at $o = (1, 0, \cdots, 0) \in Q_r$.

Now we will explain the graded structure of the Lie algebra $\mathfrak{g}(r)$ of $G(r)$. Since the identity component of $G(r)$ is $SU(\tilde{I}_r)/\mathfrak{n}, \mathfrak{g}(r)$ can be identified with $\mathfrak{su}(\tilde{I}_r)$, that is,

$$\mathfrak{g}(r) = \{X \in \mathfrak{gl}(n+1, C) \mid {}^t\bar{X}\tilde{I}_r + \tilde{I}_r X = 0, \text{ trace } X = 0\} .$$

$\mathfrak{g}(r)$ is isomorphic with $\mathfrak{su}(r+1, n-r)$, and so it is simple. Each element X of $\mathfrak{g}(r)$ can be written explicitly as a matrix of the form

$$\begin{pmatrix} -\bar{u} & -\sqrt{-1}\,{}^t\bar{w}I_r & w_n \\ \xi & v & w \\ \xi_n & \sqrt{-1}\,{}^t\bar{\xi}I_r & u \end{pmatrix}$$

where $\xi_n, w_n \in \boldsymbol{R}$, $\xi, w \in \boldsymbol{C}^{n-1}$, $v \in \mathfrak{u}(I_r)$, and $u - \bar{u} + \text{trace } v = 0$. For an element $E_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ of $\mathfrak{g}(r)$, $\text{ad}(E_0)$ (i.e. $\text{ad}(E_0)(X) = [E_0, X]$) is a semi-simple endomorphism of $\mathfrak{g}(r)$. Its eigenvalues are $-2, -1, 0, 1$, and 2. We set $\mathfrak{g}_k(r) = \{X \in \mathfrak{g}(r) \mid \text{ad}(E_0)(X) = kX\}$. Then $\mathfrak{g}(r) = \sum_{k=-2}^{2} \mathfrak{g}_k(r)$, and $\mathfrak{g}(r)$ becomes a graded Lie algebra with respect to this decomposition. More precisely $\{\mathfrak{g}_k(r)\}_{k \in Z}$ satisfies

$$[\mathfrak{g}_k(r), \mathfrak{g}_l(r)] \subset \mathfrak{g}_{k+l}(r) ,$$

where we set $\mathfrak{g}_k(r) = \{0\}$ for $|k| \geqq 3$. Moreover if we set

$$\begin{cases} \mathfrak{m}(r) = \sum_{k=-2}^{-1} \mathfrak{g}_k(r) , \\ \mathfrak{g}'(r) = \sum_{k=0}^{2} \mathfrak{g}_k(r) , \end{cases}$$

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then we have $\mathfrak{g}(r) = \mathfrak{m}(r) \oplus \mathfrak{g}'(r)$. $\mathfrak{m}(r)$ and $\mathfrak{g}'(r)$ are subalgebras of $\mathfrak{g}(r)$. It is easily seen that $\mathfrak{g}'(r)$ coincides with the Lie algebra of $G'(r)$.

Remark 1.7. Let χ be the natural homomorphism of $GL(n+1, \boldsymbol{C})$ onto $P(n, \boldsymbol{C}) = GL(n+1, \boldsymbol{C})/GL(1, \boldsymbol{C})$. Setting $\hat{G}(r) = \chi^{-1}(G(r))$, we have

$$\hat{G}(r) = \{\sigma \in GL(n+1, \boldsymbol{C}) \,|\, {}^t\bar{\sigma}\tilde{I}_r\sigma = \pm \tilde{I}_r\} .$$

Hence we get

(1) if $r \neq \dfrac{n-1}{2}$ $\qquad \hat{G}(r) = U(\tilde{I}_r)$,

(2) if $r = \dfrac{n-1}{2}$ (n: odd integer) $\qquad \hat{G}(r) = U(\tilde{I}_r) \cup \sigma_0(U(\tilde{I}_r))$,

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_r^* & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad I_r^* = \begin{pmatrix} 0 & E_r \\ E_r & 0 \end{pmatrix}.$$

In particular the Lie algebra of $\hat{G}(r)$ is $\mathfrak{u}(\tilde{I}_r)$. Note that the kernel of χ_* coincides with the center $\mathfrak{u}(1)$ of $\mathfrak{u}(\tilde{I}_r)$ and $\mathfrak{u}(\tilde{I}_r) = \mathfrak{u}(1) \oplus \mathfrak{su}(\tilde{I}_r)$ (direct sum). Moreover we have $\chi_* \circ \mathrm{Ad}_{\hat{G}(r)}(\sigma) = \mathrm{Ad}_{G(r)}(\chi(\sigma)) \circ \chi_*$ from $\chi \circ I_\sigma = I_{\chi(\sigma)} \circ \chi$ (I_σ is the inner automorphism induced by σ). Since we are identifying $\mathfrak{g}(r)$ with $\mathfrak{su}(\tilde{I}_r)$, $\mathrm{Ad}_{G(r)}(\chi(\sigma))$ is identified with the restriction of $\mathrm{Ad}_{\hat{G}(r)}(\sigma)$ to $\mathfrak{su}(\tilde{I}_r)$.

4. Pseudo-conformal $G'(r)$-bundles. First we consider the linear isotropy group of $G'(r)$. We identify the tangent space at o to $Q_r = G(r)/G'(r)$ with $\mathfrak{m}(r)$ ($\cong \mathfrak{g}(r)/\mathfrak{g}'(r)$). Moreover we identify $\mathfrak{m}(r)$ with $\mathfrak{m}$ via

$$\mathfrak{m} \ni \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ \xi & 0 & 0 \\ \xi_n & \sqrt{-1}\,{}^t\bar{\xi}I_r & 0 \end{pmatrix} \in \mathfrak{m}(r) \qquad \xi_n \in \boldsymbol{R},\ \xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{n-1} \end{pmatrix} \in \boldsymbol{C}^{n-1} .$$

We consider the linear isotropy representation $l; G'(r) \to GL(\mathfrak{m})$. Let $\tilde{G}(r) = l(G'(r))$ be the linear isotropy group of $G'(r)$. Then $\tilde{G}(r)$ is a closed subgroup of H. In fact let $\tau = \chi(\sigma)$ be an element of $G'(r)$, where σ is given by

$$\sigma = \begin{pmatrix} \bar{a}^{-1} & -\varepsilon\sqrt{-1}\bar{a}^{-1}\,{}^t\bar{C}I_rB & d \\ 0 & B & C \\ 0 & 0 & \varepsilon a \end{pmatrix}$$

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$(\varepsilon = \pm 1, a, d \in \boldsymbol{C}\ , \quad C \in \boldsymbol{C}^{n-1}\ , \quad {}^t\bar{B}I_rB = \varepsilon I_r\ , \quad \sqrt{-1}(\bar{a}d - a\bar{d}) = {}^t\bar{C}I_rC)\ .$

Then we have

$$l(\tau) = \begin{pmatrix} \bar{a}B & \bar{a}C \\ 0 & \varepsilon|a|^2 \end{pmatrix},$$

which is easily seen from the following commutative diagram

$$\begin{array}{ccc} \mathfrak{g}(r) & \xrightarrow{\mathrm{Ad}\,(\tau)} & \mathfrak{g}(r) \\ p\downarrow & & \downarrow p \\ \mathfrak{m}(r) & \xrightarrow{l(\tau)} & \mathfrak{m}(r) \end{array} \qquad \tau \in G'(r)$$

(p is the projection of $\mathfrak{g}(r)$ onto $\mathfrak{m}(r)$ corresponding to $\mathfrak{g}(r) = \mathfrak{m}(r) \oplus \mathfrak{g}'(r)$). From this we get easily ([6])

$$\tilde{G}(r) = \left\{\sigma = \begin{pmatrix} B & C \\ 0 & a \end{pmatrix} \in H \,\middle|\, a^{-1}\,{}^t\bar{B}I_rB = I_r\right\}.$$

Let S be a hypersurface which is non-degenerate of index r at every point. Then at each point x of F the Levi-form $\sqrt{-1}L(x)$ is a hermitian matrix of signature $(r, n-r-1)$ or $(n-r-1, r)$, where we say that a hermitian matrix L is of signature (p, q) if L has p negative eigenvalues and q positive eigenvalues. We set

$$\tilde{F} = \{x \in F \mid \sqrt{-1}L(x) = I_r\}\ .$$

Then since $L(x\sigma) = a^{-1}\,{}^t\bar{B}L(x)B$ for $\sigma = \begin{pmatrix} B & C \\ 0 & a \end{pmatrix} \in H$ (cf. Lemma 4 [6]), $\tilde{F}$ becomes a principal fibre bundle over S with the structure group $\tilde{G}(r)$. Obviously $\tilde{F}(S, \tilde{G}(r))$ is a subbundle of $F(S, H)$ (therefore of $L(S)$). $\tilde{F}(S, \tilde{G}(r))$ is called the pseudo-conformal $\tilde{G}(r)$-bundle associated with S ([6], [7]).

Remark 1.8 (cf. [7]). Let $\tilde{\theta}_1, \cdots, \tilde{\theta}_n$ be the components of the canonical 1-form $\tilde{\theta}$ on $\tilde{F}$. Then from the definition of $\sqrt{-1}L(x)$ (cf. Lemma 1.5), we have

$$d\tilde{\theta}_n + \sqrt{-1}\sum_{i=1}^{n-1}\varepsilon_i\tilde{\theta}_i \wedge \bar{\tilde{\theta}}_i \equiv 0 \qquad \mathrm{mod}\ \tilde{\theta}_n\ ,$$

where

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$$\varepsilon_i = \begin{cases} -1 & 1 \leqq i \leqq r\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

Identifying $\mathfrak{m}$ with $\mathfrak{m}(r)$, we write the $\mathfrak{m}(r)$-valued 1-form $\tilde{\theta}$ in the form $\tilde{\theta} = \tilde{\theta}_{-2} + \tilde{\theta}_{-1}$, where $\tilde{\theta}_k$ is the $\mathfrak{g}_k(r)$-component of $\tilde{\theta}$ ($k = -2, -1$). Then we can write

$$d\tilde{\theta}_{-2} + \tfrac{1}{2}[\tilde{\theta}_{-1} \wedge \tilde{\theta}_{-1}] \equiv 0 \qquad \operatorname{mod} \tilde{\theta}_{-2}\,,$$

where [,] is the bracket operation of $\mathfrak{m}(r)$.

5. Tanaka's theorem. Digressing from hypersurfaces we will now mention about the Cartan connection and its curvature (cf. [4]).

Let M be a manifold of dimension n. Let G be a Lie group, and G' be a closed subgroup of G with $\dim. G/G' = n$. We denote by $\mathfrak{g}, \mathfrak{g}'$ the Lie algebras of G and G' respectively.

DEFINITION 1.9. Let M, G and G' be as above. (P, ω) is called a Cartan connection of type (G, G') over M if P and ω satisfy the following

(1) P is a principal fibre bundle over M with the structure group G'.

(2) ω is a $\mathfrak{g}$-valued 1-form on P satisfying the following conditions.

(a) $R_a^*\omega = \operatorname{Ad}(a^{-1})\omega \qquad$ for $a \in G'$,

(b) $\omega(A^*) = A \qquad$ for $A \in \mathfrak{g}'$,

where A^* is the fundamental vector field corresponding to A.

(c) $\omega(X) = 0$ implies $X = 0$.

From (c) ω defines an absolute parallelism on P. Hence for $U \in \mathfrak{g}$, we can define a vector field U^* on P by $U_z^* = \omega_z^{-1}(U)$, $z \in P$. For $A \in \mathfrak{g}'$ it is obvious from (b) that A^* above coincides with the fundamental vector field corresponding to A.

The curvature form Ω of a Cartan connection (P, ω) is defined by

$$\Omega = d\omega + \tfrac{1}{2}[\omega \wedge \omega]\,.$$

DEFINITION 1.10. Let S be a non-degenerate (index r) hypersurface, and let $\tilde{F}(S, \tilde{G}(r))$ be the corresponding $\tilde{G}(r)$-bundle over S. A triplet $(P, \omega, \bar{l})$ is called a pseudo-conformal connection over S if P, ω and $\bar{l}$ satisfy the following

(1) (P, ω) is a Cartan connection of type $(G(r), G'(r))$ over S.

(2) $\bar{l}$ is a bundle homomorphism of $P(S, G'(r))$ onto $\tilde{F}(S, \tilde{G}(r))$ corresponding to $l; G'(r) \to \tilde{G}(r)$, which preserves the base space and satisfies

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$\bar{l}^*\tilde{\theta} = \theta$, where $\tilde{\theta}$ is the canonical 1-form on $\tilde{F}$ and θ is the $\mathfrak{m}(r)$-component of ω.

Let Ω be the curvature form of a pseudo-conformal connection $(P, \omega, \bar{l})$. Let B be the Killing form of $\mathfrak{g}(r)$. We have $B(\mathfrak{g}_k(r), \mathfrak{g}_l(r)) = 0$ if $k + l \neq 0$. Moreover the bilinear mapping $\mathfrak{g}_k(r) \times \mathfrak{g}_{-k}(r) \ni (X, Y) \mapsto B(X, Y) \in \boldsymbol{R}$ gives a duality between $\mathfrak{g}_k(r)$ and $\mathfrak{g}_{-k}(r)$. Then the "Ricci" curvature Ω^*, which is a $\mathfrak{g}(r)$-valued 1-form on P, is defined by

$$\Omega^*_z(X) = \sum_{k=-2}^{-1} \sum_i [u_i^{-k}, \Omega_z((u_i^{-k})^*, X)] \qquad X \in T_z(P) ,$$

where $\{u_i^k\}_i$ is a base of $\mathfrak{g}_k(r)$ and $\{u_i^{-k}\}_i$ is the dual base of $\{u_i^k\}_i$.

Now we state the results of Tanaka.

THEOREM A [7]. *Let M and M' be complex manifolds of dimension n. Let S (resp. S') be a non-degenerate (index r) hypersurface of M (resp. M'). Then there exists a pseudo-conformal connection $(P, \omega, \bar{l})$ (resp. $(P', \omega', \bar{l}')$) over S (resp. S'), which satisfies*

$$\Omega_{-2} = \Omega_{-1} = \Omega^* = 0 \qquad (resp.\ \Omega'_{-2} = \Omega'_{-1} = \Omega'^* = 0) ,$$

where Ω_k (resp. Ω'_k) is the $\mathfrak{g}_k(r)$-component of Ω (resp. Ω').

And suppose that f is a pseudo-conformal homeomorphism of S onto S'. Then there corresponds a unique bundle isomorphism $\tilde{f}$ of $P(S, G'(r))$ onto $P'(S', G'(r))$ which induces the given f on S and satisfies $\tilde{f}^*\omega' = \omega$. Conversely every bundle isomorphism $\tilde{f}$ of $P(S, G'(r))$ onto $P'(S', G'(r))$ satisfying $\tilde{f}^*\omega' = \omega$ induces a pseudo-conformal homeomorphism of S onto S'.

The above $P(S, G'(r))$, whose existence and uniqueness (up to a isomorphism commuting with $\bar{l}$) are guaranteed in the theorem, is called the pseudo-conformal $G'(r)$-bundle associated with S and (P, ω) is called the normal pseudo-conformal connection.

Let S be a non-degenerate (index r) hypersurface, and let $P(S, G'(r))$ be the corresponding $G'(r)$-bundle over S. We now consider the Lie algebra $\tilde{\mathfrak{a}}(S)$ of all infinitesimal pseudo-conformal transformations of S. We set $\tilde{\mathfrak{a}}(P) = \{X \in \mathfrak{X}(P) \mid L_X\omega = 0, R_{a*}X = X \text{ for } a \in G'(r)\}$, where $\mathfrak{X}(P)$ is the Lie algebra of all vector fields on P and L_X is the Lie differentiation with respect to X. Then the infinitesimal version of Theorem A reads;

THEOREM A′. *Let S be a non-degenerate (index r) hypersurface, and*

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let $P(S, G'(r))$ be the corresponding $G'(r)$-bundle over S. Let π be the bundle projection of P onto S. Then π_* is a Lie algebra isomorphism of $\tilde{\mathfrak{a}}(P)$ onto $\tilde{\mathfrak{a}}(S)$.

II. Filtration of $\mathfrak{a}(S)$.

First we will examine the filtration of $\mathfrak{g}(r)$. For $\mathfrak{g}(r) = \sum_{k=-2}^{2} \mathfrak{g}_k(r)$, we set for each integer l

$$\begin{cases} \mathscr{L}_l(r) = \sum_{k=l}^{2} \mathfrak{g}_k(r) & (l = -2, -1, 0, 1, 2) , \\ \mathscr{L}_l(r) = \mathscr{L}_{-2}(r) & (l \leqq -3) , \qquad \mathscr{L}_l(r) = 0 \ \ (l \geqq 3) . \end{cases}$$

With respect to this filtration $\mathfrak{g}(r) = \mathscr{L}_{-2}(r)$ becomes a filtered Lie algebra, that is, $\{\mathscr{L}_k(r)\}_{k \in Z}$ satisfy $[\mathscr{L}_k(r), \mathscr{L}_l(r)] \subset \mathscr{L}_{k+l}(r)$.

LEMMA 2.1. *For $a \in G'(r)$, $\mathrm{Ad}\,(a)$ preserves this filtration.*

Proof. Recall that the Lie algebra of $G'(r)$ coincides with $\mathfrak{g}'(r) = \mathscr{L}_0(r)$.

(1) in case $G'(r)$ is connected $\left(\text{i.e. } r \neq \frac{n-1}{2}\right)$. For $X \in \mathfrak{g}'(r) = \mathscr{L}_0(r)$, $\mathrm{ad}\,(X)$ preserves the filtration. Hence $\mathrm{Ad}\,(\exp X) = \exp \mathrm{ad}\,(X)$ preserves the filtration.

(2) in case $G'(r)$ is not connected $\left(\text{i.e. } r = \frac{n-1}{2}\right)$. $G'(r)$ has two connected components. But in this case we can find an element $\tau_0 = \chi(\sigma_0)$ of $G'(r)$, which does not belong to the identity component, such that $\mathrm{Ad}\,(\tau_0)$ preserves the filtration, e.g.

$$\sigma_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_r^* & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad I_r^* = \begin{pmatrix} 0 & E_r \\ E_r & 0 \end{pmatrix} .$$

(In fact $\mathrm{Ad}\,(\tau_0)$ preserves also the grading of $\mathfrak{g}(r)$.) Q.E.D.

From now on in this section let S be a non-degenerate (index r) hypersurface. And let $(P, \omega, \bar{l})$ be the normal pseudo-conformal connection over S.

Let us fix an arbitrary point z of P. Since each element of $\tilde{\mathfrak{a}}(P)$ is an infinitesimal automorphism of the absolute parallelism defined by (P, ω), it is known (cf. [5; p232 Lemma]) that the linear map $\omega_z \colon \tilde{\mathfrak{a}}(P) \ni X \mapsto \omega_z(X_z) \in \mathfrak{g}(r)$, is injective.

LEMMA 2.2. *For $X, Y \in T_z(P)$, we have*

(1) $\omega_z(X) \in \mathscr{L}_{-1}(r)$ *if and only if* $\pi_*(X) \in D_{\pi(z)}$,

(2) $\omega_z(x) \in \mathscr{L}_0(r) = \mathfrak{g}'(r)$ *if and only if* $\pi_*(X) = 0$,

(3) $\Omega_z(X, Y) = 0$ *if* $\pi_*(X) = 0$ *or* $\pi_*(Y) = 0$,

where Ω is the curvature form of the connection.

Proof. (1) and (2) follow immediately from $\bar{l}(z)\,(\mathfrak{g}_{-1}(r)) = D_{\pi(z)}$ and the following commutative diagram which is a direct consequence of the equality $\bar{l}^*\tilde{\theta} = \theta$ $(= p\omega)$;

$$\begin{array}{ccc} T_z(P) & \xrightarrow{\omega_z} & \mathfrak{g}(r) \\ \pi_* \downarrow & & \downarrow p \\ T_{\pi(z)}(S) & \xleftarrow{l(z)} & \mathfrak{m}(r) \end{array} .$$

In fact for $X \in T_z(P)$ we have

$$p\omega_z(X) = \theta_z(X) = \tilde{\theta}_{\bar{l}(z)}(\bar{l}_*X) = (\bar{l}(z))^{-1}(\varpi_*\bar{l}_*X) = (\bar{l}(z))^{-1}(\pi_*X) .$$

In order to prove (3), we have only to show $\Omega(U^*, A^*) = 0$ for $U \in \mathfrak{g}(r)$ and $A \in \mathfrak{g}'(r)$. First we note that $[U^*, A^*] = [U, A]^*$. In fact from $R_{a*}U^* = (\mathrm{Ad}\,(a^{-1})U)^*$, $a \in G'(r)$, we have

$$[U^*, A^*] = -L_{A^*}U^* = (-L_AU)^* = [U, A]^* .$$

Therefore, from the structure equation, we get $\Omega(U^*, A^*) = 0$. Q.E.D.

We set $\tilde{\mathfrak{a}}_z(P) = \{X \in \tilde{\mathfrak{a}}(P) \mid \pi_{*_z}(X) = 0\}$. Then

LEMMA 2.3. *For $X, Y \in \tilde{\mathfrak{a}}(P)$, we have*

$$-\omega_z([X, Y]) = [-\omega_z(X), -\omega_z(Y)] - 2\Omega_z(X, Y) .$$

In particular if either X or Y belongs to $\tilde{\mathfrak{a}}_z(P)$, then we have

$$-\omega_z([X, Y]) = [-\omega_z(X), -\omega_z(Y)] .$$

Proof. From $L_X\omega = 0$, we have $X\omega(Z) = \omega([X, Z])$ for $Z \in \mathfrak{X}(P)$. Hence the assertion is clear from the structure equation and Lemma 2.2 (3). Q.E.D.

Let $A(S)$ be the group of all pseudo-conformal transformations of S. We consider the subset $\mathfrak{a}(S)$ of $\tilde{\mathfrak{a}}(S)$ consisting of complete vector fields in $\tilde{\mathfrak{a}}(S)$. Then $\mathfrak{a}(S)$ is a subalgebra of $\tilde{\mathfrak{a}}(S)$ which is naturally isomorphic with the Lie algebra of $A(S)$. Moreover $\mathfrak{a}(S)$ can be regarded as a subalgebra $\mathfrak{h}$ of $\tilde{\mathfrak{a}}(P)$ via $\pi_*: \tilde{\mathfrak{a}}(P) \to \tilde{\mathfrak{a}}(S)$. In fact $\mathfrak{h}$ coincides with the subalgebra $\mathfrak{a}(P)$ of $\tilde{\mathfrak{a}}(P)$ which consists of complete vector fields in $\tilde{\mathfrak{a}}(P)$.

Now let us fix a point p_0 of S and choose a point z_0 of the fibre $\pi^{-1}(p_0)$ over p_0. We set for each integer k

$$\mathfrak{h}_k = \mathfrak{h} \cap \omega_{z_0}^{-1}(\mathscr{L}_k(r)) .$$

Then $\mathfrak{h}_k = \mathfrak{h}$ $(k \leqq -2)$ and $\mathfrak{h}_k = \{0\}$ $(k \geqq 3)$. Note that the above definition is independent of the choice of z_0 in $\pi^{-1}(p_0)$, which is easily seen from Lemma 2.1 and the equalities $R_a^*\omega = \mathrm{Ad}\,(a^{-1})\omega$ and $R_{a*}X = X$, $a \in G'(r)$, $X \in \tilde{\mathfrak{a}}(P)$. Hence the above defines a filtration of $\mathfrak{a}(S)$ at p_0. From Lemma 2.2 and Lemma 2.3 we have

PROPOSITION 2.4. *With respect to the above filtration,* $\mathfrak{a}(S)$ *becomes a filtered Lie algebra. In particular* $(\mathfrak{a}(S))_{-1}$ *and* $(\mathfrak{a}(S))_0$ *are given by*

$$(\mathfrak{a}(S))_{-1} = \{X \in \mathfrak{a}(S) \mid X_{p_0} \in D_{p_0}\} ,$$
$$(\mathfrak{a}(S))_0 = \{X \in \mathfrak{a}(S) \mid X_{p_0} = 0\} .$$

Next we will consider the associated graded Lie algebra $\tilde{\mathfrak{h}}$ of the filtered Lie algebra $\mathfrak{h}$. Setting $\tilde{\mathfrak{h}}_k = \mathfrak{h}_k/\mathfrak{h}_{k+1}$ for each integer k (note $\tilde{\mathfrak{h}}_k = \{0\}$ for $|k| \geqq 3$), we define $\tilde{\mathfrak{h}}$ by

$$\tilde{\mathfrak{h}} = \sum_{k=-2}^{2} \tilde{\mathfrak{h}}_k \text{ (vector space direct sum)} .$$

The bracket operation of $\tilde{\mathfrak{h}}$ is defined in a natural manner. Obviously we have dim. $\mathfrak{h}$ = dim. $\tilde{\mathfrak{h}}$.

First observe that there exists an injective linear map $\nu_{z_0}^k$ of $\tilde{\mathfrak{h}}_k$ into $\mathfrak{g}_k(r)$ which satisfies the following commutative diagram

$$\begin{array}{ccc} \mathfrak{h}_k & \xrightarrow{-p_k\omega_{z_0}} & \mathfrak{g}_k(r) \\ {\scriptstyle \mu_k}\downarrow & \nearrow {\scriptstyle \nu_{z_0}^k} & \\ \tilde{\mathfrak{h}}_k & & \end{array}$$

where μ_k is the natural projection of $\mathfrak{h}_k$ onto $\tilde{\mathfrak{h}}_k = \mathfrak{h}_k/\mathfrak{h}_{k+1}$ and p_k is the projection of $\mathfrak{g}(r)$ onto $\mathfrak{g}_k(r)$ corresponding to $\mathfrak{g}(r) = \sum_{k=-2}^{2} \mathfrak{g}_k(r)$. We define an injective linear map ν_{z_0} of $\tilde{\mathfrak{h}}$ into $\mathfrak{g}(r)$ by setting

$$\nu_{z_0} = \nu_{z_0}^{-2} \times \nu_{z_0}^{-1} \times \cdots \times \nu_{z_0}^{2} .$$

LEMMA 2.5. *Notations being as above, the linear map* ν_{z_0} *is an injective homomorphism of* $\tilde{\mathfrak{h}}$ *into* $\mathfrak{g}(r)$.

Hence setting $\tilde{\mathfrak{h}}_{z_0} = \nu_{z_0}(\tilde{\mathfrak{h}})$, we see that $\tilde{\mathfrak{h}}_{z_0}$ is a graded subalgebra of $\mathfrak{g}(r)$ which is isomorphic with $\tilde{\mathfrak{h}}$ and satisfies dim. $\tilde{\mathfrak{h}}_{z_0} =$ dim. $\mathfrak{a}(S)$.

Proof of Lemma 2.5. It suffices to show $\nu_{z_0}([\tilde{X}_k, \tilde{Y}_l]) = [\nu_z(\tilde{X}_k), \nu_{z_0}(\tilde{Y}_l)]$ for $\tilde{X}_k \in \tilde{\mathfrak{h}}_k$ and $\tilde{Y}_l \in \tilde{\mathfrak{h}}_l$. Choose $X_k \in \mathfrak{h}_k$ (resp. $Y_l \in \mathfrak{h}_l$) such that $\tilde{X}_k = \mu_k(X_k)$ (resp. $\tilde{Y}_l = \mu_l(Y_l)$). Then

$$\nu_{z_0}([\tilde{X}_k, \tilde{Y}_l]) = -p_{k+l}\omega_{z_0}([X_k, Y_l]) .$$

Set $-\omega_{z_0}(X_k) = \sum_{i=k}^{2} \bar{X}_i, \bar{X}_i \in \mathfrak{g}_i(r)$ (resp. $-\omega_{z_0}(Y_l) = \sum_{i=l}^{2} \bar{Y}_i, \bar{Y}_i \in \mathfrak{g}_i(r)$). Then from the definition of ν_{z_0} and the graded structure of $\mathfrak{g}(r)$, we have

$$\nu_{z_0}(\tilde{X}_k) = \bar{X}_k , \qquad \nu_{z_0}(\tilde{Y}_l) = \bar{Y}_l$$

and

$$p_{k+l}([-\omega_{z_0}(X_k), -\omega_{z_0}(Y_l)]) = [\bar{X}_k, \bar{Y}_l]$$

(1) in case $k \geqq 0$ or $l \geqq 0$. From Lemma 2.3 we have $-\omega_{z_0}([X_k, Y_l]) = [-\omega_{z_0}(X_k), -\omega_{z_0}(Y_l)]$. Hence we get

$$\nu_{z_0}([\tilde{X}_k, \tilde{Y}_l]) = [\bar{X}_k, \bar{Y}_l] = [\nu_{z_0}(\tilde{X}_k), \nu_{z_0}(\tilde{Y}_l)] .$$

(2) otherwise. Non-trivial case is when $k = l = -1$. Form the above we have

$$\nu_{z_0}([\tilde{X}_{-1}, \tilde{Y}_{-1}]) = p_{-2}(-\omega_{z_0}([X_{-1}, Y_{-1}]) ,$$
$$[\nu_{z_0}(\tilde{X}_{-1}), \nu_{z_0}(\tilde{Y}_{-1})] = p_{-2}([-\omega_{z_0}(X_{-1}), -\omega_{z_0}(Y_{-1})]) .$$

In this case we have from Lemma 2.3

$$-\omega_{z_0}([X_{-1}, Y_{-1}]) = [-\omega_{z_0}(X_{-1}), -\omega_{z_0}(Y_{-1})] - 2\Omega_{z_0}(X_{-1}, Y_{-1}) .$$

But, due to Theorem A, the $\mathfrak{g}_{-2}(r)$-component Ω_{-2} of Ω vanishes identically. Hence we get $\nu_{z_0}([\tilde{X}_{-1}, \tilde{Y}_{-1}]) = [\nu_{z_0}(\tilde{X}_{-1}), \nu_{z_0}(\tilde{Y}_{-1})]$. Q.E.D.

Remark 2.6. Clearly the representation ν_{z_0} of $\tilde{\mathfrak{h}}$ into $\mathfrak{g}(r)$ is dependent on the choice of z_0 in $\pi^{-1}(p_0)$. Choose another point $z_1 = z_0 a$, if Ad(a) preserves the grading of $\mathfrak{g}(r)$, we get from $R_a^*\omega = \text{Ad}\,(a^{-1})\omega$

$$\tilde{\mathfrak{h}}_{z_0 a} = \text{Ad}\,(a^{-1})\tilde{\mathfrak{h}}_{z_0} .$$

Moreover if we define a vector subspace $\mathfrak{h}_{z_0}$ of $\mathfrak{g}(r)$ by $\mathfrak{h}_{z_0} = \omega_{z_0}(\mathfrak{h})$, we get similarly

$$\mathfrak{h}_{z_0 a} = \text{Ad}\,(a^{-1})\mathfrak{h}_{z_0} , \qquad a \in G'(r) .$$

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Remark 2.7. The discussion in this section can be well applied to a connected hypersurface S which is non-degenerate of index r *at a point*; Let S^* be the set of all points of S at which S is non-degenerate of index r. Obviously S^* is an open subset of S. Hence S^* is a non-degenerate (index r) hypersurface. Let $P^*(S^*, G'(r))$ be the corresponding $G'(r)$-bundle over S^*. We consider the restriction map *res* of $\mathfrak{a}(S)$ into $\tilde{\mathfrak{a}}(S^*)$. Since we are considering, exclusively, real analytic hypersurfaces, each infinitesimal pseudo-conformal transformation of S is a real analytic vector field on S. Hence the connectedness of S implies that $res;\ \mathfrak{a}(S) \to \tilde{\mathfrak{a}}(S^*)$ is an injective homomorphism. On the other hand $(\pi^*)_*$ is an isomorphism of $\tilde{\mathfrak{a}}(P^*)$ onto $\tilde{\mathfrak{a}}(S^*)$. Hence we can define a subalgebra $\mathfrak{h}$ of $\tilde{\mathfrak{a}}(P^*)$ by $\mathfrak{h} = (\pi^*)_*^{-1} \circ res\ (\mathfrak{a}(S))$. Then $\mathfrak{h}$ is isomorphic with $\mathfrak{a}(S)$. Therefore if we fix a point p_0 of S^*, we can define a filltration of $\mathfrak{h}$ (and consequently of $\mathfrak{a}(S)$) at p_0 similarly as in this section.

III. Relations between $A(S)$ $(S, A_{p_0}(S))$ and $P(S, G'(r))$.

Throughout this section we assume that S is a connected non-degenerate (index r) homogeneous (i.e. $A(S)$ acts transitively on S) hypersurface. Let $(P, \omega, \bar{l})$ be the normal pseudo-conformal connection over S. We denote by $\tilde{\sigma}$ the connection-preserving bundle isomorphism of $P(S, G'(r))$ induced by $\sigma \in A(S)$. Then from I. Theorem A, A(S) acts effectively on P as an automorphism group of the Cartan connection (P, ω).

Let us fix a point $p_0 \in S$ and take a point $z_0 \in \pi^{-1}(p_0)$. And we define $\iota_{z_0};\ A(S) \to P$ by $\iota_{z_0}(\sigma) = \tilde{\sigma}(z_0)$, $\sigma \in A(S)$. Then it is known ([4]) that ι_{z_0} is an imbedding of $A(S)$ as a closed submanifold of P.

Let $A_{p_0}(S)$ be the isotropy subgroup of $A(S)$ at $p_0 \in S$. Obviously we have

$$\iota_{z_0}(A_{p_0}(S)) \subset \pi^{-1}(p_0)\ .$$

On the other hand the fibre $\pi^{-1}(p_0)$ of $P(S, G'(r))$ is diffeomorphic with $G'(r)$ via a diffeomorphism γ_{z_0} of $G'(r)$ onto $\pi^{-1}(p_0)$, where $\gamma_{z_0}(a) = z_0 a$, $a \in G'(r)$. Therefore the composite map $\rho_{z_0} = \gamma_{z_0}^{-1} \circ \iota_{z_0}$ is an imbedding of $A_{p_0}(S)$ into $G'(r)$ and $\rho_{z_0}(A_{p_0}(S))$ is closed in $G'(r)$. Moreover we have

LEMMA 3.1. *The map $\rho_{z_0};\ A_{p_0}(S) \to G'(r)$ is an injective homomorphism. And $\rho_{z_0}(A_{p_0}(S))$ is a closed subgroup of $G'(r)$. Moreover $(\rho_{z_0*})_e = \omega_{z_0} \cdot (\iota_{z_0*})_e$, where e is the unit of $A_{p_0}(S)$.*

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Proof. Suppose $\rho_{z_0}(\sigma_i) = a_i$ $(i = 1, 2)$, that is, $\tilde{\sigma}_i(z_0) = z_0 \cdot a_i$, then

$$\iota_{z_0}(\sigma_1 \cdot \sigma_2) = \tilde{\sigma}_1 \cdot \tilde{\sigma}_2(z_0) = \tilde{\sigma}_1(z_0 \cdot a_2) = (z_0 \cdot a_1)a_2 = z_0(a_1 \cdot a_2) \,.$$

Hence we get $\rho_{z_0}(\sigma_1 \cdot \sigma_2) = a_1 \cdot a_2 = \rho_{z_0}(\sigma_1) \cdot \rho_{z_0}(\sigma_2)$. $\rho_{z_0}(A_{p_0}(S))$ is closed in $G'(r)$ since $A_{p_0}(S)$ is a closed subgroup of $A(S)$, $\iota_{z_0}(A(S))$ is a closed submanifold of P and $\pi^{-1}(p_0)$ is closed in P. In order to prove $(\rho_{z_0*})_e = \omega_{z_0} \cdot (\iota_{z_0*})_e$, it suffices to show $\omega_{z_0} = (\gamma_{z_0*})_{e'}^{-1}$, where e' is the unit element of $G'(r)$, which is clear from the definition of the fundamental vector field A^* corresponding to A and $\omega(A^*) = A$. Q.E.D.

Since $A(S)$ acts transitively on S, $A(S)$ is a principal $A_{p_0}(S)$-bundle over $S = A(S)/A_{p_0}(S)$. Then we have

PROPOSITION 3.2. *The imbedding ι_{z_0}; $A(S) \to P$ is an injective bundle homomorphism of $A(S)(S, A_{p_0}(S))$ into $P(S, G'(r))$ corresponding to ρ_{z_0}; $A_{p_0}(S) \to G'(r)$, which preserves the base space S.*

Hence $A(S)(S, A_{p_0}(S))$ can be regarded as a subbundle of $P(S, G'(r))$ via ι_{z_0}.

Proof of Proposition 3.2. Let τ be an element of $A_{p_0}(S)$. Let $\sigma \in A(S)$. Then we get easily the following commutative diagram

$$\begin{array}{ccc} A(S) & \xrightarrow{\iota_{z_0}} & P \\ {\scriptstyle R_\tau}\downarrow & & \downarrow{\scriptstyle R_{\rho_{z_0}(\tau)}}, \\ A(S) & \xrightarrow[\iota_{z_0}]{} & P \end{array} \qquad \tau \in A_{p_0}(S)\,.$$

Therefore ι_{z_0} is a bundle homomorphism corresponding to ρ_{z_0}. Moreover ι_{z_0} induces the identity transformation of S, which follows from $\pi \cdot \iota_{z_0}(\sigma) = \pi \cdot \tilde{\sigma}(z_0) = \sigma \cdot \pi(z_0) = \sigma(p_0)$ for $\sigma \in A(S)$. Q.E.D.

Now we will consider the relation between the Maurer-Cartan form on $A(S)$ and the normal pseudo-conformal connection form ω on P. First observe

LEMMA 3.3. *Let ω be the connection form on P and let Ω be its curvature form. Then $\iota_{z_0}^*\omega$ and $\iota_{z_0}^*\Omega$ are $\mathfrak{g}(r)$-valued left invariant forms on $A(S)$.*

Proof. Let $\sigma \in A(S)$. We denote by L_σ the left translation of $A(S)$ by σ. Then we get easily the following commutative diagram.

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$$\begin{array}{ccc} A(S) & \xrightarrow{\iota_{z_0}} & P \\ L_\sigma \downarrow & & \downarrow \tilde{\sigma} \\ A(S) & \xrightarrow{\iota_{z_0}} & P \end{array} \qquad \text{for } \sigma \in A(S) .$$

Therefore $\iota_{z_0}^*\omega$ is left invariant since $\tilde{\sigma}^*\omega = \omega$, $\sigma \in A(S)$. From the structure equation $d\omega + \frac{1}{2}[\omega \wedge \omega] = \Omega$, it is obvious that $\iota_{z_0}^*\Omega$ is also left invariant. Q.E.D.

In this section we denote by $\mathfrak{a}(S)$ the Lie algebra of $A(S)$. Then we have easily

$$\iota_{z_0}^*\omega(\mathfrak{a}(S)) = \omega_{z_0}(\mathfrak{h}) = \mathfrak{h}_{z_0} ,$$

where $\mathfrak{h} = \mathfrak{a}(P)$ (cf. II).

In case $\Omega = 0$ we have

PROPOSITION 3.4. *Suppose that the curvature form Ω of the normal pseudo-conformal connection vanishes identically. Then the linear map $\iota_{z_0}^*\omega$; $\mathfrak{a}(S) \to \mathfrak{g}(r)$ is a Lie algebra isomorphism of $\mathfrak{a}(S)$ into $\mathfrak{g}(r)$. Hence $\mathfrak{h}_{z_0}(= \iota_{z_0}^*\omega(\mathfrak{a}(S)))$ is a subalgebra of $\mathfrak{g}(r)$ which is isomorphic with $\mathfrak{a}(S)$. Moreover if we identify $\mathfrak{a}(S)$ with $\mathfrak{h}_{z_0}$, $\iota_{z_0}^*\omega$ is the Maurer-Cartan form of $A(S)$.*

Proof. From $\Omega = 0$ we get $d\iota_{z_0}^*\omega + \frac{1}{2}[\iota_{z_0}^*\omega \wedge \iota_{z_0}^*\omega] = 0$. Let $A, B \in \mathfrak{a}(S)$. Then we have

$$2\, d\iota_{z_0}^*\omega(A, B) = -\iota_{z_0}^*\omega([A, B]) ,$$

since $\iota_{z_0}^*\omega$ is left invariant. Hence we get $\iota_{z_0}^*\omega([A, B]) = [\iota_{z_0}^*\omega(A), \iota_{z_0}^*\omega(B)]$. Q.E.D.

Now we will consider an equivalence of two non-degenerate (index r) homogeneous hypersurfaces. Let M and M' be complex manifolds of dimension n. Let S (resp. S') be a connected non-degenerate (index r) homogeneous hypersurface of M (resp. M'). And let $(P, \omega, \bar{l})$ (resp. $(P', \omega', \bar{l}')$) be the normal pseudo-conformal connection over S (resp. S'). We denote by $A^0(S)$ the identity component of $A(S)$, and set $A_{p_0}^0(S) = A^0(S) \cap A_{p_0}(S)$. Note that the identity component $A^0(S)$ acts transitively on S.

PROPOSITION 3.5. *Notations being as above, let $p_0 \in S$ and $p_0' \in S'$. Suppose that for points, $z_0 \in \pi^{-1}(p_0)$, $z_0' \in \pi'^{-1}(p_0')$ suitably chosen, there exists a group isomorphism φ of $A^0(S)$ onto $A^0(S')$ satisfying* i), ii);

i) $\varphi(A^0_{p_0}(S)) = A^0_{p'_0}(S')$,

ii) $\varphi^* \iota^*_{z'_0} \omega' = \iota^*_{z_0} \omega$.

Then the bundle isomorphism φ *of* $A^0(S)$ $(S, A^0_{p_0}(S))$ *onto* $A^0(S')$ $(S', A^0_{p'_0}(S'))$ *induces a pseudo-conformal homeomorphism of* S *onto* S'.

Proof. From i) it is obvious that φ induces a bundle isomorphism of $A^0(S)(S, A^0_{p_0}(S))$ onto $A^0(S')(S', A^0_{p'_0}(S'))$. Since $A^0(S)(S, A^0_{p_0}(S))$ (resp. $A^0(S')(S', A^0_{p'_0}(S'))$) is a subbundle of P $(S, G'(r))$ (resp. $P'(S', G'(r))$), φ induces a bundle isomorphism $\tilde{\varphi}$ of $P(S, G'(r))$ onto $P'(S', G'(r))$ which satisfies the following commutative diagram

$$\begin{array}{ccc} A^0(S) & \xrightarrow{\varphi} & A^0(S') \\ \iota_{z_0}\big\downarrow & & \big\downarrow \iota_{z'_0} \\ P & \xrightarrow{\tilde{\varphi}} & P' \end{array} .$$

From ii) we get $\iota^*_{z_0} \tilde{\varphi}^* \omega' = \iota^*_{z_0} \omega$. Moreover, since $\tilde{\varphi}$ is a bundle isomorphism, we have $\tilde{\varphi}^* \omega' = \omega$. Therefore, from I. Theorem A, $\tilde{\varphi}$ induces a pseudo-conformal homeomorphism of S onto S'. Q.E.D.

IV. Graded subalgebras of $\mathfrak{g}(r)$.

First we will go into details about the structure of the graded Lie algebra $\mathfrak{g}(r) = \sum_{k=-2}^{2} \mathfrak{g}_k(r)$.

Identifying $\mathfrak{g}(r)$ with $\mathfrak{su}(\tilde{I}_r)$ we represent each element X of $\mathfrak{g}(r)$ as a matrix of the following form

$$X = \begin{pmatrix} -\bar{u} & -\sqrt{-1}\,{}^t\bar{w} I_r & w_n \\ \xi & v & w \\ \xi_n & \sqrt{-1}\,{}^t\bar{\xi} I_r & u \end{pmatrix},$$

where $\xi_n, w_n \in \boldsymbol{R}$, $u \in \boldsymbol{C}$ (and $\bar{u}$ is the complex conjugate of u), $\xi, w \in \boldsymbol{C}^{n-1}$, $v \in \mathfrak{u}(I_r)$ and $u - \bar{u} + \text{trace } v = 0$. For $\xi \in \boldsymbol{C}^{n-1}$, we define an element $\underset{\sim}{\xi} \in \mathfrak{g}_{-1}(r)$ and an element $\tilde{\xi} \in \mathfrak{g}_1(r)$ by

$$\underset{\sim}{\xi} = \begin{pmatrix} 0 & 0 & 0 \\ \xi & 0 & 0 \\ 0 & \sqrt{-1}\,{}^t\bar{\xi} I_r & 0 \end{pmatrix}, \qquad \tilde{\xi} = \begin{pmatrix} 0 & -\sqrt{-1}\,{}^t\bar{\xi} I_r & 0 \\ 0 & 0 & \xi \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover for $a \in \boldsymbol{R}$, we define an element $\underset{\approx}{a} \in \mathfrak{g}_{-2}(r)$ and an element $\tilde{\tilde{a}} \in \mathfrak{g}_2(r)$ by

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$$\underset{\approx}{a} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}, \qquad \tilde{\tilde{a}} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $\xi, w \in C^{n-1}$, we set $\langle \xi, w \rangle = {}^t\bar{\xi} I_r w$. $\langle\ ,\ \rangle$ is an indefinite hermitian inner product of C^{n-1} of type $(r, n-r-1)$. Then for $\tilde{\tilde{a}} \in \mathfrak{g}_2(r)$, $\tilde{w} \in \mathfrak{g}_1(r)$, $\underset{\sim}{\xi} \in \mathfrak{g}_{-1}(r)$ and $X_0 = \begin{pmatrix} -\bar{u} & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & u \end{pmatrix} \in \mathfrak{g}_0(r)$, we have

$$[\underset{\sim}{\xi}, \tilde{\tilde{a}}] = \widetilde{a\xi} \in \mathfrak{g}_1(r) \tag{4.1}$$

$$[\underset{\sim}{\xi}, \tilde{w}] = \begin{pmatrix} \sqrt{-1}\langle w, \xi \rangle & 0 & 0 \\ 0 & -\sqrt{-1}(\xi\, {}^t\bar{w} + w\, {}^t\bar{\xi}) I_r & 0 \\ 0 & 0 & \sqrt{-1}\langle \xi, w \rangle \end{pmatrix} \in \mathfrak{g}_0(r) \tag{4.2}$$

$$[X_0, \tilde{w}] = \widetilde{vw - uw} \in \mathfrak{g}_1(r) \tag{4.3}$$

$$[\tilde{w}_1, \tilde{w}_2] = \widetilde{\widetilde{\sqrt{-1}(\langle w_2, w_1 \rangle - \langle w_1, w_2 \rangle)}} \in \mathfrak{g}_2(r)\ . \tag{4.4}$$

From the above we easily obtain

LEMMA 4.1.

$$[\mathfrak{g}_{-1}(r), \mathfrak{g}_2(r)] = \mathfrak{g}_1(r)\ , \quad [\mathfrak{g}_1(r), \mathfrak{g}_1(r)] = \mathfrak{g}_2(r)\ , \quad [\mathfrak{g}_{-1}(r), \mathfrak{g}_1(r)] = \mathfrak{g}_0(r)\ .$$

Now we will consider a graded subalgebra $\mathfrak{k} = \sum_{k=-2}^{2} \mathfrak{k}_k$ of $\mathfrak{g}(r)$ which satisfies

$$\mathfrak{k}_{-2} = \mathfrak{g}_{-2}(r) \quad \text{and} \quad \mathfrak{k}_{-1} = \mathfrak{g}_{-1}(r)\ .$$

First we have

LEMMA 4.2. *If* $\mathfrak{k}_2 \neq \{0\}$, *then* $\mathfrak{k} = \mathfrak{g}(r)$.

Proof. Since $\dim. \mathfrak{g}_2(r) = 1$, we have $\mathfrak{k}_2 = \mathfrak{g}_2(r)$. Hence from $\mathfrak{k}_{-2} = \mathfrak{g}_{-2}(r)$, $\mathfrak{k}_{-1}(r) = \mathfrak{g}_{-1}(r)$, and Lemma 4.1 we get $\mathfrak{k} = \mathfrak{g}(r)$. Q.E.D.

Therefore from now on we further assume $\mathfrak{k}_2 = \{0\}$. Let δ_r be a linear isomorphism of C^{n-1} onto $\mathfrak{g}_1(r)$ defined by $\delta_r(\xi) = \tilde{\xi}$, $\xi \in C^{n-1}$. Then we have

LEMMA 4.3. $\mathfrak{k}_1$ *is an abelian subalgebra of* $\mathfrak{g}(r)$; $\delta_r^{-1}(\mathfrak{k}_1)$ *is a complex isotropic vector subspace of the (indefinite) hermitian space* $(C^{n-1}, \langle\ \rangle)$. *In particular* $\dim. \mathfrak{k}_1 = 2s \leqq 2r$.

Proof. Let $\tilde{w} \in \mathfrak{k}_1$ and $\underset{\sim}{\xi} \in \mathfrak{k}_{-1} = \mathfrak{g}_{-1}(r)$. Then we have from (4.2) and (4.3)

$$\operatorname{ad}(\tilde{w})^2(\underset{\sim}{\xi}) = [\tilde{w}, [\tilde{w}, \underset{\sim}{\xi}]] = \widetilde{-\sqrt{-1}\langle w, w\rangle\xi - 2\sqrt{-1}\langle \xi, w\rangle w} \in \mathfrak{k}_1 .$$

Moreover from (4.4) we have

$$\operatorname{ad}(\tilde{w})^3(\underset{\sim}{\xi}) = \widetilde{\widetilde{-3(\langle \xi, w\rangle + \langle w, \xi\rangle)\langle w, w\rangle}} \in \mathfrak{k}_2 .$$

Since $\langle\ ,\ \rangle$ is a non-degenerate hermitian form, we can find $\xi_1 \in \boldsymbol{C}^{n-1}$ such that $\langle \xi_1, w\rangle = -\frac{1}{2}$. Hence from $\mathfrak{k}_2 = \{0\}$, we have

$$\operatorname{ad}(\tilde{w})^3(\underset{\sim}{\xi}_1) = \widetilde{\widetilde{3\langle w, w\rangle}} = 0 \quad (\text{i.e. } \langle w, w\rangle = 0) \quad \text{for any } \tilde{w} \in \mathfrak{k}_1 .$$

Moreover we have $\operatorname{ad}(\tilde{w})^2(\underset{\sim}{\xi}_1) = \widetilde{\sqrt{-1}w} \in \mathfrak{k}_1$. Therefore $\delta_r^{-1}(\mathfrak{k}_1)$ is a complex vector subspace of $\boldsymbol{C}^{n-1}$. On the other hand let $w_1, w_2 \in \delta_r^{-1}(\mathfrak{k}_1)$. Then from

$$\begin{cases} \tilde{w}_1 + \tilde{w}_2 = \widetilde{w_1 + w_2} \in \mathfrak{k}_1 , \\ [\tilde{w}_1, \tilde{w}_2] = \widetilde{\widetilde{\sqrt{-1}(\langle w_2, w_1\rangle - \langle w_1, w_2\rangle)}} \in \mathfrak{k}_2 , \end{cases}$$

we get $[\tilde{w}_1, \tilde{w}_2] = 0$ (i.e. $\langle w_1, w_2\rangle = \langle w_2, w_1\rangle$) and $\langle w_1 + w_2, w_1 + w_2\rangle = 0$. Hence we get $\langle w_1, w_2\rangle = 0$. Q.E.D.

Let $\{e_i\}_{1 \leq i \leq n-1}$ be the natural base of $\boldsymbol{C}^{n-1}$. Setting $w_i = e_i + e_{n-i}$ $(i = 1, 2, \cdots, s)$, we consider a complex vector subspace of $\boldsymbol{C}^{n-1}$ spanned by the s vectors $w_1, \cdots, w_s$. This subspace is an s-dimensional complex isotropic subspace of the (indefinite) hermitian space $(\boldsymbol{C}^{n-1}, \langle\ ,\ \rangle)$. We denote by $\mathfrak{c}_s(r)$ its image under δ_r. Then $\mathfrak{c}_s(r)$ is an abelian subalgebra of $\mathfrak{g}(r)$ of dimension $2s$ contained in $\mathfrak{g}_1(r)$.

Now recall the following which is a direct consequence of Witt's theorem (cf. [1, p. 121]).

LEMMA B. *Let V_1 and V_2 be s-dimensional complex isotropic vector subspaces of the indefinite hermitian space $(\boldsymbol{C}^{n-1}, \langle\ ,\ \rangle)$. Then there exists an element σ of $U(I_r)$ which sends V_1 onto V_2.*

Then we have

LEMMA 4.4. *Let s be the complex dimension of $\delta_r^{-1}(\mathfrak{k}_1)$. Then there exists $\tau_1 \in G'(r)$ such that* $\operatorname{Ad}(\tau_1)$ *preserves the grading of $\mathfrak{g}(r)$ and satisfies* $\operatorname{Ad}(\tau_1)\mathfrak{k}_1 = \mathfrak{c}_s(r)$.

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Proof. $\delta_r^{-1}(\mathfrak{k}_1)$ and $\delta_r^{-1}(\mathfrak{c}_s(r))$ are s-dimensional complex isotropic subspaces of $(C^{n-1}, \langle\ ,\ \rangle)$. Hence from Lemma B we can find $\sigma_1 \in U(I_r)$ such that $\sigma_1(\delta_r^{-1}(\mathfrak{k}_1)) = \delta_r^{-1}(\mathfrak{c}_s(r))$. Set $\sigma_1' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then σ_1' belongs to $U(\tilde{I}_r) \subset \hat{G}(r)$. Hence $\tau_1 = \chi(\sigma_1')$ is an element of $G'(r)$. In fact τ_1 belongs to the analytic subgroup of $G'(r)$ corresponding to the subalgebra $\mathfrak{g}_0(r)$ of $\mathfrak{g}'(r)$. In particular $\mathrm{Ad}\,(\tau_1)$ preserves the grading of $\mathfrak{g}(r)$. On the other hand

$$\mathrm{Ad}\,(\tau_1)\tilde{w} = \widetilde{\sigma_1 w} \qquad \text{for } \tilde{w} \in \mathfrak{g}_1(r)\ ,$$

so we can conclude $\mathrm{Ad}\,(\tau_1)\mathfrak{k}_1 = \mathfrak{c}_s(r)$. Q.E.D.

Next we will consider $\mathfrak{k}_0$. We define a subalgebra $\mathfrak{b}_s(r)$ of $\mathfrak{g}_0(r)$ by

$$\mathfrak{b}_s(r) = \{X \in \mathfrak{g}_0(r) \mid \mathrm{ad}\,(X)(\mathfrak{c}_s(r)) \subset \mathfrak{c}_s(r)\}\ .$$

Then we have

LEMMA 4.5. *Notations being the same as in Lemma* 4.4, *we have*
(i) $\mathrm{Ad}\,(\tau_1)\mathfrak{k}_0 \subset \mathfrak{b}_s(r)$ and $[\mathfrak{g}_{-1}(r), \mathfrak{c}_s(r)] \subset \mathfrak{b}_s(r)$
(ii) $\dim.\ \mathfrak{b}_s(r) = \dim.\ \mathfrak{g}_0(r) - s(2(n-1) - 3s)$.

Proof. (i) is clear from $\mathrm{Ad}\,(\tau_1)\mathfrak{k}_1 = \mathfrak{c}_s(r)$, $[\mathfrak{k}_0, \mathfrak{k}_1] \subset \mathfrak{k}_1$, (4.2) and (4.3). In order to prove (ii) we first note that $\mathfrak{g}_0(r)$ can be decomposed into the direct sum of $\langle\{E_0\}\rangle_R$ and $\mathfrak{u}(I_r)$, where $\langle\{E_0\}\rangle_R$ is the line spanned by

$$E_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and $\mathfrak{u}(I_r)$ is identified with the subalgebra of $\mathfrak{g}_0(r)$ which consists of matrices of the form

$$\begin{pmatrix} -\frac{1}{2}\,\mathrm{trace}\, v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -\frac{1}{2}\,\mathrm{trace}\, v \end{pmatrix} \quad \text{with} \quad {}^t\bar{v}I_r + I_r v = 0\ .$$

For $X = \begin{pmatrix} -\bar{u} & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & u \end{pmatrix} \in \mathfrak{g}_0(r)$, we have from (4.3)

$$\mathrm{ad}\,(X)(\tilde{w}) = \widetilde{vw - uw} \qquad \tilde{w} \in \mathfrak{c}_s(r)\ .$$

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Since $\delta_r^{-1}(\mathfrak{c}_s(r))$ is a complex vector subspace of C^{n-1}, we have $\widetilde{uw} \in \mathfrak{c}_s(r)$. Hence X belongs to $\mathfrak{b}_s(r)$ if and only if $v(\delta_r^{-1}(\mathfrak{c}_s(r)) \subset \delta_r^{-1}(\mathfrak{c}_s(r))$. Obviously E_0 belongs to $\mathfrak{b}_s(r)$. Therefore in order to calculate the dimension of $\mathfrak{b}_s(r)$, we have only to calculate the dimension of a subalgebra of $\mathfrak{u}(I_r)$ which consists of all elements leaving the subspace $\delta_r^{-1}(\mathfrak{c}_s(r))$ invariant. A direct computation shows the above equality (ii). Q.E.D.

We set $\mathfrak{g}^*(r, s) = \mathfrak{g}_{-2}(r) \oplus \mathfrak{g}_{-1}(r) \oplus \mathfrak{b}_s(r) \oplus \mathfrak{c}_s(r)$. In the case $s = 0$, we write $\mathfrak{g}^*(r)$ instead of $\mathfrak{g}^*(r, 0)$, that is, $\mathfrak{g}^*(r) = \mathfrak{g}_{-2}(r) \oplus \mathfrak{g}_{-1}(r) \oplus \mathfrak{g}_0(r)$. Then from the above lemmas we have

PROPOSITION 4.6. *Let $\mathfrak{k}$ be a proper graded subalgebra of $\mathfrak{g}(r)$ satisfying $\mathfrak{k}_{-2} = \mathfrak{g}_{-2}(r)$ and $\mathfrak{k}_{-1} = \mathfrak{g}_{-1}(r)$. Then there exists $\tau \in G'(r)$ such that* Ad (τ) *preserves the grading of $\mathfrak{g}(r)$ and* Ad $(\tau)\mathfrak{k} \subset \mathfrak{g}^*(r, s)$, *where* $2s =$ dim. $\mathfrak{k}_1(\leqq 2r)$.

From this we obtain dim. $\mathfrak{k} \leqq$ dim. $\mathfrak{g}^*(r, s) = n^2 + 1 - s(2(n-2) - 3s)$. Since s is an integer satisfying $0 \leqq s \leqq r$, from the above considerations we obtain

PROPOSITION 4.7. *Let $\mathfrak{k}$ be a proper graded subalgebra of $\mathfrak{g}(r)$ satisfying $\mathfrak{k}_{-2} = \mathfrak{g}_{-2}(r)$ and $\mathfrak{k}_{-1} = \mathfrak{g}_{-1}(r)$. Then we have*

(1) *The case $n = 3$ and $r = 1$*

We have dim. $\mathfrak{k} \leqq n^2 + 2 = 11$. *The equality holds if and only if there exists $\tau \in G'(1)$ such that* Ad (τ) *preserves the grading of $\mathfrak{g}(1)$ and*

$$\mathrm{Ad}\,(\tau)\mathfrak{k} = \mathfrak{g}^*(1, 1)\ .$$

(2) *The case $n = 5$ and $r = 2$*

We have dim. $\mathfrak{k} \leqq n^2 + 1 = 26$. *The equality holds if and only if there exists $\tau \in G'(2)$ such that* Ad (τ) *preserves the grading of $\mathfrak{g}(2)$ and*

$$\mathrm{Ad}\,(\tau)\mathfrak{k} = \mathfrak{g}^*(2, 2) \quad \textit{or} \quad \mathfrak{g}^*(2)\ .$$

(3) *Otherwise*

We have dim. $\mathfrak{k} \leqq n^2 + 1$. *The equality holds if and only if there exists $\tau \in G'(r)$ such that* Ad (τ) *preserves the grading of $\mathfrak{g}(r)$ and*

$$\mathrm{Ad}\,(\tau)\mathfrak{k} = \mathfrak{g}^*(r)\ .$$

Remark 4.8. Let $D(r)$ be an $(n-2)$-dimensional complex vector subspace of C^{n-1} spanned by the $n-2$ vectors $w_1, e_2, \cdots$, and e_{n-2}, where $w_1 = e_1 + e_{n-1}$. We set $\mathfrak{d}^1(r) = \{\tilde{\xi} \in \mathfrak{g}_1(r) \mid \xi \in D(r)\}$, $\mathfrak{d}^{-1}(r) = \{\underset{\sim}{\xi} \in \mathfrak{g}_{-1}(r) \mid \xi \in D(r)\}$,

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$\mathfrak{e}(r) = \{X \in \mathfrak{g}_0(r) \,|\, \mathrm{ad}\,(X)(\mathfrak{d}^i(r)) \subset \mathfrak{d}^i(r)\ i = 1, 2\}$, $\mathfrak{c}_s^*(r) = \{\xi \in \mathfrak{g}_{-1}(r) \,|\, \xi \in \delta_r^{-1}(\mathfrak{c}_s(r))\}$ and $\mathfrak{b}_s^*(r) = \{X \in \mathfrak{g}_0(r) \,|\, \mathrm{ad}\,(X)(\mathfrak{c}_s^*(r)) \subset \mathfrak{c}_s^*(r)\}$ $(=\mathfrak{b}_s(r))$. Moreover we set

$$\begin{cases} \mathfrak{g}^0(r) = \mathfrak{g}_{-2}(r) + \mathfrak{d}^{-1}(r) + \mathfrak{e}(r) + \mathfrak{d}^1(r) + \mathfrak{g}_2(r)\ , \\ \mathfrak{g}^{**}(r, s) = \mathfrak{c}_s^*(r) + \mathfrak{b}_s^*(r) + \mathfrak{g}_1(r) + \mathfrak{g}_2(r)\ . \end{cases}$$

Then without the homogeneity assumption we have

PROPOSITION 4.9. *Let* $\mathfrak{k}$ *be a proper graded subalgebra of* $\mathfrak{g}(r)$. *Then we have*

(1) *The case* $n = 3$ *and* $r = 1$; $\dim.\, \mathfrak{k} \leqq n^2 + 2 = 11$. *The equality holds if and only if there exists* $\tau \in G'(1)$ *such that* $\mathrm{Ad}\,(\tau)$ *preserves the grading of* $\mathfrak{g}(1)$ *and*

$$\mathrm{Ad}\,(\tau)\mathfrak{k} = \mathfrak{g}^*(1, 1) \quad \textit{or} \quad \mathfrak{g}^{**}(1, 1)\ .$$

(2) *The case* $n = 5$ *and* $r = 2$; $\dim.\, \mathfrak{k} \leqq n^2 + 1 = 26$. *The equality holds if and only if there exists* $\tau \in G'(2)$ *such that* $\mathrm{Ad}\,(\tau)$ *preserves the grading of* $\mathfrak{g}(2)$ *and*

$$\mathrm{Ad}\,(\tau)\mathfrak{k} = \mathfrak{g}^*(2, 2),\ \mathfrak{g}^{**}(2, 2),\ \mathfrak{g}^*(2),\ \mathfrak{g}'(2)\ , \quad \textit{or} \quad \mathfrak{g}^0(2)\ .$$

(3) *The case* $n \geqq 2$ *and* $r = 0$; $\dim.\, \mathfrak{k} \leqq n^2 + 1$, *the equality holds if and only if there exists* $\tau \in G'(0)$ *such that* $\mathrm{Ad}\,(\tau)$ *preserves the grading of* $\mathfrak{g}(0)$ *and*

$$\mathrm{Ad}\,(\tau)\mathfrak{k} = \mathfrak{g}^*(0) \quad \textit{or} \quad \mathfrak{g}'(0)\ .$$

(4) *Otherwise*; $\dim.\, \mathfrak{k} \leqq n^2 + 1$. *The equality holds if and only if there exists* $\tau \in G'(r)$ *such that* $\mathrm{Ad}\,(\tau)$ *preserves the grading of* $\mathfrak{g}(r)$ *and*

$$\mathrm{Ad}\,(\tau)\mathfrak{k} = \mathfrak{g}^*(r), \mathfrak{g}'(r) \quad \textit{or} \quad \mathfrak{g}^0(r)\ .$$

V. Determination of $(\mathfrak{a}(S), \mathfrak{a}_{p_0}(S))$.

Throughout this section we assume that S is a connected non-degenerate (index r) homogeneous hypersurface. Let $(P, \omega, \bar{l})$ be the normal pseudo-conformal connection over S. Moreover we naturally identify the Lie algebra $\mathfrak{a}(S)$ of $A(S)$ with the Lie algebra of all infinitesimal pseudo-conformal transformations of S which generate (global) 1-parameter groups of pseudo-conformal transformations.

Now let us fix a point p_0 of S. As in the section II, we introduce the filtration of $\mathfrak{a}(S)$ at p_0 through the connection form ω. Notations being as in the section II, we first consider the associated graded Lie algebra $\tilde{\mathfrak{h}}$ of $\mathfrak{h}$.

LEMMA 5.1. *Let* $z_0 \in \pi^{-1}(p_0)$. *Suppose that* $A(S)$ *has the largest dimension* $n^2 + 2n$, *then* $\nu_{z_0}; \tilde{\mathfrak{h}} \to \mathfrak{g}(r)$ *is a Lie algebra isomorphism of* $\tilde{\mathfrak{h}}$ *onto* $\mathfrak{g}(r)$.

This lemma is clear from Lemma 2.5 and dim. $\mathfrak{g}(r) =$ dim. $\tilde{\mathfrak{h}}$ $(= n^2 + 2n)$.

Let z be an arbitrary point of $\pi^{-1}(p_0)$. Since $A(S)$ acts transitively on S, $\tilde{\mathfrak{h}}_z = \nu_z(\tilde{\mathfrak{h}})$ satisfies $(\tilde{\mathfrak{h}}_z)_{-2} = \mathfrak{g}_{-2}(r)$ and $(\tilde{\mathfrak{h}}_z)_{-1} = \mathfrak{g}_{-1}(r)$. Therefore from Proposition 4.7 and Remark 2.6 we get

LEMMA 5.2. *Suppose that* $A(S)$ *has the second largest dimension, then there exists* $z_1 \in \pi^{-1}(p_0)$ *such that*

(1) $\tilde{\mathfrak{h}}_{z_1} = \mathfrak{g}^*(1, 1)$ *if* $n = 3$ *and* $r = 1$,

(2) $\tilde{\mathfrak{h}}_{z_1} = \mathfrak{g}^*(2, 2)$ *or* $\mathfrak{g}^*(2)$ *if* $n = 5$ *and* $r = 2$,

(3) $\tilde{\mathfrak{h}}_{z_1} = \mathfrak{g}^*(r)$ *otherwise* .

As for $\mathfrak{h}_z = \omega_z(\mathfrak{h})$, we have

LEMMA 5.3. *Let* $z_0 \in \pi^{-1}(p_0)$. *Suppose that* $A(S)$ *has the largest dimension* $n^2 + 2n$, *then* $-\omega_{z_0}; \mathfrak{h} \to \mathfrak{g}(r)$ *is a linear isomorphism of* $\mathfrak{h}$ *onto* $\mathfrak{g}(r)$.

This lemma is also clear from dim. $\mathfrak{g}(r) =$ dim. $\mathfrak{h}$.

LEMMA 5.4. *Suppose that* $A(S)$ *has the second largest dimension, then there exists* $z_0 \in \pi^{-1}(p_0)$ *such that*

(1) $\mathfrak{h}_{z_0} = \mathfrak{g}^*(1, 1)$ *if* $n = 3$ *and* $r = 1$,

(2) $\mathfrak{h}_{z_0} = \mathfrak{g}^*(2, 2)$ *or* $\mathfrak{g}^*(2)$ *if* $n = 5$ *and* $r = 2$,

(3) $\mathfrak{h}_{z_0} = \mathfrak{g}^*(r)$ *otherwise*,

as vector subspaces of $\mathfrak{g}(r)$.

In order to prove Lemma 5.4, it suffices to show the following lemma. (Note that $\mathfrak{g}^*(r, s)(0 \leqq s \leqq r)$ contains E_0).

LEMMA 5.5. *If* $\tilde{\mathfrak{h}}_{z_1}$ *contains* $E_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ *for some point* z_1 *of* $\pi^{-1}(p_0)$, *then there exists a point* z_0 *of* $\pi^{-1}(p_0)$ *such that* $\mathfrak{h}_{z_0}$ *coincides with* $\tilde{\mathfrak{h}}_{z_1}$ *as a vector subspace of* $\mathfrak{g}(r)$.

Proof. Since the filtration of $\mathfrak{h}_z$ is given by $(\mathfrak{h}_z)_k = \mathfrak{h}_z \cap \mathscr{L}_k(r)$ ($\mathscr{L}_k(r) = \sum_{l=k}^{2} \mathfrak{g}_l(r)$), we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{h}_k & \xrightarrow{-\omega_z} & (\mathfrak{h}_z)_k \subset \mathfrak{g}(r) \\ {\scriptstyle \mu_k}\big\downarrow & & \big\downarrow{\scriptstyle p_k} \\ \tilde{\mathfrak{h}}_k & \xrightarrow{\nu_z} & (\tilde{\mathfrak{h}}_z)_k \subset \mathfrak{g}_k(r) , \end{array}$$

where p_k is the projection of $\mathfrak{g}(r)$ onto $\mathfrak{g}_k(r)$ corresponding to the decomposition $\mathfrak{g}(r) = \sum_{k=-2}^{2} \mathfrak{g}_k(r)$. From the assumption $(\tilde{\mathfrak{h}}_{z_1})_0$ contains E_0. Hence there exists $E \in (\mathfrak{h}_{z_1})_0$ such that $p_0(E) = E_0$. Since E belongs to $\mathscr{L}_0(r) = \sum_{k=0}^{2} \mathfrak{g}_k(r)$, there exist $\tilde{w}_0 \in \mathfrak{g}_1(r)$ and $\tilde{\tilde{c}}_0 \in \mathfrak{g}_2(r)$ such that $E = E_0 + \tilde{w}_0 + {}_0\tilde{\tilde{c}}$. Now we set $A_0 = \tilde{w}_0 + \frac{1}{2}\tilde{\tilde{c}}_0$. Then A_0 belongs to $\mathscr{L}_1(r)$ and satisfies $\mathrm{Ad}\,(\exp A_0)(E) = E_0$. Moreover $a_0 = \exp A_0$ is an element of $G'(r)$. Set $z_0 = z_1 a_0^{-1}$, then from Remark 2.6 we have $\mathfrak{h}_{z_0} = \mathrm{Ad}\,(a_0)\mathfrak{h}_{z_1}$. In particular $\mathfrak{h}_{z_0}$ contains E_0.

First we will see that $\tilde{\mathfrak{h}}_{z_0}$ coincides with $\tilde{\mathfrak{h}}_{z_1}$. From the above diagram we have $(\tilde{\mathfrak{h}}_{z_i})_k = p_k(\mathfrak{h}_{z_i} \cap \mathscr{L}_k(r))$ $(i = 0, 1)$. For $X \in \mathfrak{h}_{z_1} \cap \mathscr{L}_k(r)$, $\mathrm{Ad}\,(a_0)(X) = \exp \mathrm{ad}\,(A_0)(X)$ lies in $\mathfrak{h}_{z_0} \cap \mathscr{L}_k(r)$. This is obvious from $\mathfrak{h}_{z_0} = \mathrm{Ad}\,(a_0)\mathfrak{h}_{z_1}$ and Lemma 2.1. Moreover, since $A_0 \in \mathscr{L}_1(r)$, we have $\mathrm{ad}\,(A_0)(\mathscr{L}_k(r)) \subset \mathscr{L}_{k+1}(r)$. Hence we get $p_k(\mathrm{Ad}\,(a_0)(X)) = p_k(X)$. Therefore $(\tilde{\mathfrak{h}}_{z_0})_k = (\tilde{\mathfrak{h}}_{z_1})_k$.

Next we will see that $\mathfrak{h}_{z_0}$ coincides with $\tilde{\mathfrak{h}}_{z_0}$ as a vector subspace of $\mathfrak{g}(r)$. First one should note that Lemma 2.3 implies $[(\mathfrak{h}_{z_0})_0, \mathfrak{h}_{z_0}] \subset \mathfrak{h}_{z_0}$ and that $\mathfrak{h}_{z_0}$ contains E_0. Let X be an arbitrary element of $\mathfrak{h}_{z_0}$, and X_k $(k = -2, -1, \cdots, 2)$ be the $\mathfrak{g}_k(r)$-component of X. From $[(\mathfrak{h}_{z_0})_0, \mathfrak{h}_{z_0}] \subset \mathfrak{h}_{z_0}$ and $(\mathfrak{h}_{z_0})_0 \ni E_0$, we obtain

$$\begin{cases} -X_{-2} + X_2 = \frac{1}{6}(\mathrm{ad}\,(E_0)^3(X) - \mathrm{ad}\,(E_0)(X)) \in \mathfrak{h}_{z_0} \\ X_{-2} + X_2 = \frac{1}{12}(\mathrm{ad}\,(E_0)^4(X) - \mathrm{ad}\,(E_0)^2(X)) \in \mathfrak{h}_{z_0} \\ -X_{-1} + X_1 = \frac{1}{3}(4\mathrm{ad}\,(E_0)(X) - \mathrm{ad}\,(E_0)^3(X)) \in \mathfrak{h}_{z_0} \\ X_{-1} + X_1 = \frac{1}{3}(4\mathrm{ad}\,(E_0)^2(X) - \mathrm{ad}\,(E_0)^4(X)) \in \mathfrak{h}_{z_0} . \end{cases}$$

Hence we get $X_{-2}, X_{-1}, X_1, X_2 \in \mathfrak{h}_{z_0}$. Therefore X_k $(k = -2, -1, 0, 1, 2)$ lies in $\mathfrak{h}_{z_0}$, that is, $\mathfrak{h}_{z_0}$ decomposes as follows

$$\mathfrak{h}_{z_0} = \sum_{k=-2}^{2} \mathfrak{h}_{z_0} \cap \mathfrak{g}_k(r) .$$

In other words, $\mathfrak{h}_{z_0}$ is a graded subspace of $\mathfrak{g}(r)$. Then from the construction of the assoicated graded Lie algebra, we have $(\tilde{\mathfrak{h}}_{z_0})_k = \mathfrak{h}_{z_0} \cap \mathfrak{g}_k(r)$.

Therefore we obtain $\mathfrak{h}_{z_0} = \tilde{\mathfrak{h}}_{z_0}$. Q.E.D.

Next we will see that the curvature form Ω of the normal pseudo-conformal connection of S vanishes identically if $A(S)$ has either the largest dimension $n^2 + 2n$ or the second largest dimension. First we will show the following proposition.

PROPOSITION 5.6. *If* $\mathfrak{h}_{z_0}$ *contains* $E_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ *for some point* z_0 *of* $\pi^{-1}(p_0)$, *then* $\Omega_z = 0$ *for any* $z \in \pi^{-1}(p_0)$.

Proof. The proof is quite analogous to that of IV. Theorem 3.2 of [4]. Recall that $\mathfrak{h} = \mathfrak{a}(P) = \{X \in \mathfrak{X}(P) \mid L_X\omega = 0,\ R_{a*}X = X,\ a \in G'(r)$, and X is complete$\}$ (see II). Since $\mathfrak{h}_{z_0} = \omega_{z_0}(\mathfrak{a}(P))$, there exists $X_0 \in \mathfrak{a}(P)$ such that $(X_0)_{z_0} = \omega_{z_0}^{-1}(E_0) = (E_0)^*_{z_0}$. First we know

LEMMA C (cf. [5; p. 233]). *For the curvature form* $\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$, *we have*

(1) $A^*(\Omega(\xi^*, \eta^*)) = -[A, \Omega(\xi^*, \eta^*)] + \Omega([A, \xi]^*, \eta^*) + \Omega(\xi^*, [A, \eta]^*)$ *for* $\xi, \eta \in \mathfrak{g}(r)$, $A \in \mathfrak{g}'(r)$,

(2) $L_X\Omega = 0$ *and* $X(\Omega(\xi^*, \eta^*)) = 0$ *for* $X \in \mathfrak{a}(P)$, $\xi, \eta \in \mathfrak{g}(r)$.

Applying the above lemma to $(X_0)_{z_0} = (E_0)^*_{z_0}$, we obtain

$$(5.1) \qquad [E_0, \Omega_{z_0}(\xi^*, \eta^*)] = \Omega_{z_0}([E_0, \xi]^*, \eta^*) + \Omega_{z_0}(\xi^*, [E_0, \eta]^*) .$$

Since $\Omega(U^*, A^*) = 0$ for $U \in \mathfrak{g}(r)$ and $A \in \mathfrak{g}'(r)$ (cf. II. Lemma 2.2), we have only to show $\Omega(\xi^*, \eta^*) = 0$ for $\xi, \eta \in \mathfrak{m}(r) = \mathfrak{g}_{-2}(r) \oplus \mathfrak{g}_{-1}(r)$. For the sake of simplicity we show the above equality in the case $\xi, \eta \in \mathfrak{g}_{-1}(r)$. Let Ω_k $(k = -2, -1, \cdots, 2)$ be the $\mathfrak{g}_k(r)$-component of Ω. From I. Theorem A, we have $\Omega_{-1} = 0$ and $\Omega_{-2} = 0$. Hence from (5.1) we get

$$(\Omega_1)_{z_0}(\xi^*, \eta^*) + 2(\Omega_2)_{z_0}(\xi^*, \eta^*) = -2(\Omega_0 + \Omega_1 + \Omega_2)_{z_0}(\xi^*, \eta^*) , \qquad \xi, \eta \in \mathfrak{g}_{-1}(r) .$$

From this it follows $(\Omega_k)_{z_0}(\xi^*, \eta^*) = 0$ $(k = 0, 1, 2)$. Therefore we obtain $\Omega_{z_0} = 0$. For any $z \in \pi^{-1}(p_0)$, there exists $a \in G'(r)$ such that $z_0 = za$. Then from $R_a^*\omega = \mathrm{Ad}\,(a^{-1})\omega$, we have $\Omega_z = \mathrm{Ad}\,(a)R_a^*\Omega_{z_0} = 0$. Q.E.D.

From Lemma 5.3, Lemma 5.4 and Proposition 5.6 we get

PROPOSITION 5.7. *Let S be a non-degenerate homogeneous hyper-surface. If $A(S)$ has either the largest dimension $n^2 + 2n$ or the second largest dimension, then S is flat, that is, the curvature form of the* *normal pseudo-conformal connection vanishes identically.*

Hence from Proposition 3.4, $\iota_z^*\omega$ is a Lie algebra isomorphism of $\mathfrak{a}(S)$ into $\mathfrak{g}(r)$ for any $z \in P$.

Summarizing the results of this section we obtain.

THEOREM 5.8. *Let M be a complex manifold of dimension n. Let S be a connected non-degenerate (index r) homogeneous hypersurface of M. Let p_0 be an arbitrary point of S.*

(1) *If* $\dim. A(S) = n^2 + 2n$, *then $\iota_{z_0}^*\omega$ is a Lie algebra isomorphism of $\mathfrak{a}(S)$ onto $\mathfrak{g}(r)$ for any $z_0 \in \pi^{-1}(p_0)$.*

(2) *If* $\dim. A(S) < n^2 + 2n$, *we have the following three cases.*

(i) *The case $n = 3$ and $r = 1$; We have* $\dim. A(S) \leqq n^2 + 2 = 11$. *The equality holds if and only if there exists $z_0 \in \pi^{-1}(p_0)$ such that $\iota_{z_0}^*\omega$ is a Lie algebra isomorphism of $\mathfrak{a}(S)$ onto $\mathfrak{g}^*(1,1)$.*

(ii) *The case $n = 5$ and $r = 2$; We have* $\dim. A(S) \leqq n^2 + 1 = 26$. *The equality holds if and only if there exists $z_0 \in \pi^{-1}(p_0)$, such that $\iota_{z_0}^*\omega$ is a Lie algebra isomorphism of $\mathfrak{a}(S)$ onto $\mathfrak{g}^*(2,2)$ or $\mathfrak{g}^*(2)$.*

(iii) *Otherwise; We have* $\dim. A(S) \leqq n^2 + 1$. *The equality holds if and only if there exists $z_0 \in \pi^{-1}(p_0)$ such that $\iota_{z_0}^*\omega$ is a Lie algebra isomorphism of $\mathfrak{a}(S)$ onto $\mathfrak{g}^*(r)$.*

VI. Model spaces.

We consider the analytic subgroups (i.e. connected Lie subgroups) of $G(r)$ corresponding to $\mathfrak{g}(r)$ and $\mathfrak{g}^*(r,s)$ $(0 \leqq s \leqq r)$. The identity component $G^0(r)$ of $G(r)$ corresponds to $\mathfrak{g}(r)$. We denote by $G^*(r,s)$ the analytic subgroup of $G(r)$ corresponding to $\mathfrak{g}^*(r,s)$. In particular we set $G^*(r) = G^*(r,0)$.

First we will characterize $G^*(r,s)$ geometrically. Let χ be the natural homomorphism of $U(\tilde{I}_r)$ onto $G^0(r)\,(= U(\tilde{I}_r)/U(1))$. We set $\hat{G}^*(r,s) = \chi^{-1}(G^*(r,s))$. Take the natural base $\{e_i\}_{0 \leqq i \leqq n}$ of C^{n+1} and set $w_i = e_i + e_{n-i}$ $(i = 1, 2, \cdots, s)$. We denote by $C_s(r)$ the $(s+1)$-dimensional complex vector subspace of C^{n+1} spanned by the $(s+1)$ vectors $w_1, w_2, \cdots, w_s$ and e_n. Then $C_s(r)$ is an $(s+1)$-dimensional complex isotropic subspace of the indefinite hermitian space $(C^{n+1}, \tilde{I}_r)$.

LEMMA 6.1.

$$\hat{G}^*(r,s) = \{\sigma \in U(\tilde{I}_r) \,|\, \sigma(C_s(r)) = C_s(r)\}\,.$$

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Proof. Since we are identifying $\mathfrak{g}(r)$ with $\mathfrak{su}(\tilde{I}_r)$, χ_* is identified with the projection of $\mathfrak{u}(\tilde{I}_r)$ onto $\mathfrak{su}(\tilde{I}_r)$ corresponding to the decomposition $\mathfrak{u}(\tilde{I}_r) = \mathfrak{u}(1) \oplus \mathfrak{su}(\tilde{I}_r)$, where $\mathfrak{u}(1)$ is the center of $\mathfrak{u}(\tilde{I}_r)$. For $X \in \mathfrak{u}(\tilde{I}_r)$;

$$X = \begin{pmatrix} -\bar{u} & -\sqrt{-1}\,{}^t\bar{w}I_r & w_n \\ \xi & v & w \\ \xi_n & \sqrt{-1}\,{}^t\bar{\xi}I_r & u \end{pmatrix} \qquad \xi_n, w_n \in \boldsymbol{R},\ \xi, w \in \boldsymbol{C}^{n-1},\ v \in \mathfrak{u}(I_r), \text{ we note}$$

$$\mathfrak{g}^*(r,s) \ni \chi_*(X) \text{ if and only if} \begin{cases} w_n = 0 \\ w \in \delta_r^{-1}(\mathfrak{c}_s(r))\ , \\ v(\delta_r^{-1}(\mathfrak{c}_s(r)) \subset \delta_r^{-1}(\mathfrak{c}_s(r))\ . \end{cases}$$

On the other hand for $(0, \eta, z_n) \in C_s(r)$ we have

$$X\begin{pmatrix} 0 \\ \eta \\ z_n \end{pmatrix} = \begin{pmatrix} -\sqrt{-1}\langle w, \eta\rangle + w_n z_n \\ v\eta + z_n w \\ \sqrt{-1}\langle \xi, \eta\rangle + u z_n \end{pmatrix}.$$

Hence we have

$$X(C_s(r)) \subset C_s(r) \text{ if and only if} \begin{cases} -\sqrt{-1}\langle w, \eta\rangle + w_n z_n = 0 \\ v\eta + z_n w \in \delta_r^{-1}(\mathfrak{c}_s(r)) \end{cases}$$
$$\text{for } z_n \in \boldsymbol{C},\ \eta \in \delta_r^{-1}(\mathfrak{c}_s(r))\ .$$

From the above $\mathfrak{g}^*(r,s) \ni \chi^*(X)$ if and only if $X(C_s(r)) \subset C_s(r)$. We set $K = \{\sigma \in U(\tilde{I}_r) \mid \sigma(C_s(r)) = C_s(r)\}$. From $G^*(r,s) = \hat{G}^*(r,s)/U(1)$, we see that $\hat{G}^*(r,s)$ is connected. In fact, $\hat{G}^*(r,s)$ is the analytic subgroup of $U(\tilde{I}_r)$ corresponding to $\chi_*^{-1}(\mathfrak{g}^*(r,s))$. Therefore $\hat{G}^*(r,s)$ coincides with the identity component of K.

In order to prove $\hat{G}^*(r,s) = K$, we have only to show that K is connected. For this we take a base $\{f_i\}_{0 \leq i \leq n}$ of $\boldsymbol{C}^{n+1}$ such that $\{f_i\}_{0 \leq i \leq s}$ forms a base of $C_s(r)$ and with respect to this base the hermitian form $\tilde{I}_r$ is represented as a matrix of the following form

$$\tilde{I}_r = \begin{pmatrix} 0 & E_{s+1} & 0 \\ E_{s+1} & 0 & 0 \\ 0 & 0 & I_s^* \end{pmatrix}, \qquad I_s^* = \begin{pmatrix} -E_{r-s} & 0 \\ 0 & E_{n-(r+s+1)} \end{pmatrix}.$$

(The existence of such a base is guaranteed by the Witt's theorem). Then each $\sigma \in K$ is represented as a matrix of the form

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$$\begin{pmatrix} A & -\frac{1}{2}A(C + {}^t\bar{K}I_s^*K) & -A\,{}^t\bar{K}I_s^*B \\ 0 & {}^t\bar{A}^{-1} & 0 \\ 0 & K & B \end{pmatrix};$$

$$A \in GL(s+1, C),\ B \in U(I_s^*),\ {}^t\bar{C} + C = 0\ .$$

From this we see that K is homeomorphic with $GL(s+1, C) \times U(I_s^*) \times \mathfrak{u}(s+1) \times M(n-2s-1, s+1; C)$, where $M(n-2s-1, s+1; C)$ is the set of all complex $(n-2s-1) \times (s+1)$ matrices. In particular K is connected. Q.E.D.

Now we consider the orbit of $G^0(r)$ or $G^*(r, s)$ passing through o of Q_r as the model space corresponding to $\mathfrak{g}(r)$ or $\mathfrak{g}^*(r, s)$.

Since $G^0(r)$ acts transitively on Q_r, the model space corresponding to $\mathfrak{g}(r)$ is Q_r itself. We denote by $Q_r^*(s)$ the model space corresponding to $\mathfrak{g}^*(r, s)$. In particular we set $Q_r^* = Q_r^*(0)$.

LEMMA 6.2.

$$Q_r^* = \{(z_0, z_1, \cdots, z_n) \in Q_r \,|\, z_0 \neq 0\}\ ,$$

and

$$Q_r^*(s) = \{(z_0, z_1, \cdots, z_n) \in Q_r \,|\, |z_0| + |z_1 - z_{n-1}| + \cdots + |z_s - z_{n-s}| \neq 0\} \quad (s \geqq 1)\ .$$

Proof. We consider the orbital decomposition of Q_r by $G^*(r, s)$. We denote by (,) the indefinite hermitian inner product of C^{n+1} defined by $\tilde{I}_r$. And set $(C_s(r))^\perp = \{\zeta \in C^{n+1} \,|\, (\zeta, \eta) = 0 \text{ for } \eta \in C_s(r)\}$. Then from Lemma 6.1 we see that each $\sigma \in \hat{G}^*(r, s)$ leaves $(C_s(r))^\perp$ invariant as well. On the other hand we have $Q_r = \{\zeta = (\zeta_0, \cdots, \zeta_n) \,|\, (\zeta, \zeta) = 0\}$ in homogeneous coordinate. Then using the arguments in the proof of the Witt's theorem ([1, p. 121]), we easily see that Q_r is decomposed by $G^*(r, s)$ into the following three orbits;

$$R_r^0(s) = \{\kappa(\zeta) \in Q_r \,|\, \zeta \notin (C_s(r))^\perp\}\ ,$$
$$R_r^1(s) = \{\kappa(\zeta) \in Q_r \,|\, \zeta \in C_s(r)\}\ ,$$
$$R_r^2(s) = \{\kappa(\zeta) \in Q_r \,|\, \zeta \in (C_s(r))^\perp \backslash C_s(r)\}\ ,$$

where κ is the projection of $C^{n+1}\backslash\{0\}$ onto $P^n(C)$. From $o = \kappa(e_0)$, $e_n \in C_s(r)$ and $(e_0, e_n) = \sqrt{-1} \neq 0$, we see $o \in R_r^0(s)$. Hence we have $Q_r^*(s) = R_r^0(s)$. Q.E.D.

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Remark 6.3. From the above we have the orbital decomposition of Q_r by $G^*(r, s)$;

$$Q_r = Q_r^*(s) \cup R_r^1(s) \cup R_r^2(s) .$$

Note that

(1) $R_r^1(s) = \{\tilde{o}\}$ if and only if $s = 0$, where $\tilde{o} = \kappa(e_n)$,

(2) $R_r^2(s) = \emptyset$ if and only if $s = r$.

Hence we have

$$Q_r = Q_r^* \cup \{\tilde{o}\} \cup R_r^2(0) \qquad \left(1 \leqq r \leqq \left[\frac{n-1}{2}\right]\right),$$

$$Q_0 = Q_0^* \cup \{\tilde{o}\} ,$$

$$Q_r = Q_r^*(r) \cup R_r^1(r) .$$

From Lemma 6.2 we see that $Q_r^*(s)$ is a connected open subset of Q_r, hence it is a connected non-degenerate (index r) homogeneous flat hypersurface of $P^n(C)$.

Next we will determine the groups $A(Q_r), A(Q_r^*(s))$ of all pseudo-conformal transformations of $Q_r, Q_r^*(s)$.

PROPOSITION 6.4 ([6]). $A(Q_r) = G(r)$.

Proof. Let us fix a frame $x_0 \in F(Q_r, \tilde{G}(r))$ at o. For $\tau \in G(r)$ we set $\bar{l}_0(\tau) = \tau_*(x_0)$. Then $\bar{l}_0$ is a bundle homomorphism of $G(r)$ $(Q_r, G'(r))$ onto $\tilde{F}(Q_r, \tilde{G}(r))$ corresponding to l, $G'(r) \to \tilde{G}(r)$, which preserves the base space Q_r. It is known ([6; Theorem 6]) that $G(r)$ $(Q_r, G'(r))$ together with $\bar{l}_0$ is the pseudo-conformal $G'(r)$-bundle over Q_r and that the Maurer-Cartan form on $G(r)$ coincides with the normal pseudo-conformal connection form. Hence we have $A(Q_r) = G(r)$ as a Lie transformation group. Q.E.D.

PROPOSITION 6.5

(1) *In the case* $r \neq \dfrac{n-1}{2}$, $A(Q_r^*(s)) = G^*(r, s)$,

(2) *In the case* $r = \dfrac{n-1}{2}(n\,;\text{odd})$, $A(Q_r^*(s)) = G^*(r, s) \cup \tau_s(G^*(r, s))$,

where $\tau_s = \chi(\sigma_s)$;

$$\sigma_s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_s^* & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad I_s^* = \begin{pmatrix} 0 & 0 & E_s \\ 0 & I_s^{**} & 0 \\ E_s & 0 & 0 \end{pmatrix}, \quad I_s^{**} = \begin{pmatrix} 0 & E_{r-s} \\ E_{r-s} & 0 \end{pmatrix}.$$

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Proof. Let π_r be the projection of $G(r)$ onto Q_r (i.e. $\pi_r(\tau) = \tau(o)$ for $\tau \in G(r)$). Since $Q_r^*(s)$ is an open subset of Q_r, the restriction $\pi_r^{-1}(Q_r^*(s))$ $(Q_r^*(s), G'(r))$ of $G(r)$ $(Q_r, G'(r))$ is the pseudo-conformal $G'(r)$-bundle over $Q_r^*(s)$ and the restriction ω_s of the Maurer-Cartan form of $G(r)$ coincides with the normal pseudo-conformal connection form. Hence we get $A(Q_r^*(s)) = \{\tau \in G(r) \mid \tau(Q_r^*(s)) = Q_r^*(s)\}$. On the other hand we have $Q_r^*(s) = \{\zeta = (\zeta_0, \cdots, \zeta_n) \in Q_r \mid \zeta \notin (C_s(r))^\perp\}$ and $G^*(r, s) = \{\chi(\sigma) \in G^0(r) \mid \sigma(C_s(r)) = C_s(r)\}$. From these we see easily $A(Q_r^*(s)) \cap G^0(r) = G^*(r, s)$. In case $G(r)$ is not connected $\left(\text{i.e. in case } r = \dfrac{n-1}{2}\right)$, we can find an element $\tau_s \in A(Q_r^*(s))$ which does not belong to $G^0(r)$. Q.E.D.

From the above we have $P(Q_r^*(s), G'(r)) = \pi_r^{-1}(Q_r^*(s))$ $(Q_r^*(s), G'(r))$ and $A^0(Q_r^*(s)) = G^*(r, s)$. Let $e \in \pi_r^{-1}(o)$ be the unit element of $G(r)$. Then the natural inclusion ι_e of $G^*(r, s)$ into $G(r)$ induces the imbedding ι_e of $A^0(Q_r^*(s))$ into $P(Q_r^*(s), G'(r))$ in the sense of Proposition 3.2. In fact, letting z_0 and ρ_{z_0} be the same as in Proposition 3.2 we may take e as z_0, then ρ_{z_0} coincides with the natural inclusion of the isotropy subgroup of $G^*(r, s)$ at o into $G'(r)$. Moreover $\iota_e^*\omega_s$ is just the Maurer-Cartan form on $G^*(r, s)$. In particular we have $\mathfrak{h}_e = \mathfrak{g}^*(r, s)$, where the notation $\mathfrak{h}_e$ is the same as in Proposition 3.4.

Now we will investigate in detail the model spaces $Q_r, Q_r^*(s)$ and their groups $G^0(r), G^*(r, s)$ of pseudo-conformal transformations.

First we have

PROPOSITION 6.6. *Let us fix an integer r with $0 \leqq r \leqq \left[\dfrac{n-1}{2}\right]$ $(n \geqq 2)$. Then $P^n(C) \supset Q_r$, $Q_r^*(s)$ $(0 \leqq s \leqq r)$ are all simply connected.*

Proof. (1) Simply connectedness of Q_r; We consider

$$Q'_r = \left\{(z_0, \cdots, z_n) \in P^n(C) \,\middle|\, -\sum_{i=0}^{r} z_i\bar{z}_i + \sum_{i=r+1}^{n} z_i\bar{z}_i = 0\right\}.$$

Then Q'_r and Q_r are projectively equivalent (hence they are pseudo-conformally equivalent). One should note that Q'_0 is the $(2n-1)$-dimensional unit sphere in $C^n = \{(z_0, \cdots, z_n) \in P^n(C) \mid z_0 \neq 0\}$. We will show the simply connectedness of Q'_r $(r \geqq 1)$. From Proposition 6.4 we know $A^0(Q'_r) = U(r+1, n-r)/U(1)$. Moreover it is easily seen that the maximal compact subgroup $K = U(r+1) \times U(n-r)$ of $U(r+1, n-r)$ acts transitively on Q'_r, where

$$K = \left\{ \sigma \in U(r+1, n-r) \middle| \sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \sigma_1 \in U(r+1), \quad \sigma_2 \in U(n-r) \right\}.$$

Let o' be a point of Q'_r with homogeneous coordinate $(1, 0, \cdots, 0, 1)$. Then the isotropy subgroup L of K at o' is given by

$$L = \left\{ \sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \in K \middle| \sigma_1 = \begin{pmatrix} \exp(\sqrt{-1}\theta) & 0 \\ 0 & \tilde{\sigma}_1 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} \tilde{\sigma}_2 & 0 \\ 0 & \exp(\sqrt{-1}\theta) \end{pmatrix} \right.$$
$$\left. \tilde{\sigma}_1 \in U(r), \ \tilde{\sigma}_2 \in U(n-r-1) \right\}.$$

Hence L is isomorphic with $U(1) \times U(r) \times U(n-r-1)$. From the above Q'_r is homeomorphic with K/L. Then the following homotopy exact sequence of the principal fibre bundle $K(Q'_r, L)$ shows the simply connectedness of Q'_r;

$$\longrightarrow \pi_1(L, e) \xrightarrow{i_*} \pi_1(K, e) \xrightarrow{p_*} \pi_1(Q'_r, o') \xrightarrow{\Delta} \pi_0(L, e).$$

In fact, the arcwise connectedness of L implies $\pi_0(L, e) = \{0\}$. Hence we have only to check that i_* is onto. Since we suppose $r \geqq 1$, we have

$$\begin{cases} \pi_1(K, e) = \pi_1(U(r+1), e) \times \pi_1(U(n-r), e) \ (\cong \mathbf{Z} \oplus \mathbf{Z}), \\ \pi_1(L, e) = \pi_1(U(1), e) \times \pi_1(U(r), e) \times \pi_1(U(n-r-1), e) \ (\cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}). \end{cases}$$

Moreover the generator of $\pi_1(U(r), e) \subset \pi_1(L, e)$ is also the generator of $\pi_1(U(r+1), e) \subset \pi_1(K, e)$ and similarly the generator of $\pi_1(U(n-r-1, e))$ $\subset \pi_1(L, e)$ is also the generator of $\pi_1(U(n-r), e) \subset \pi_1(K, e)$. Hence i_* is onto.

(2) Simply connectedness of Q^*_r; We identify C^n with the set of points of $P^n(C)$ for which $z_0 \neq 0$. Then from $Q^*_r = Q_r \cap C^n$, we have

$$Q^*_r = \left\{ (z'_1, \cdots, z'_n) \in C^n \middle| \operatorname{Im} z'_n = \frac{1}{2}\left(-\sum_{i=1}^{r} |z'_i|^2 + \sum_{i=r+1}^{n-1} |z'_i|^2 \right) \right\},$$

where $\operatorname{Im} z'_n$ is the imaginary part of z'_n. Hence it is clear that Q^*_r is diffeomorphic with R^{2n-1}. In particular Q^*_r is simply connected.

(3) Simply connectedness of $Q^*_r(s)$ $(1 \leqq s \leqq r)$; From Lemma 6.2 we have the orbital decomposition of Q_r by $G^*(r, s)$; $Q_r = Q^*_r(s) \cup R^1_r(s)$ $\cup R^2_r(s)$. From $\dim._C C_s(r) = s + 1$ we have $\dim. R^1_r(s) = 2s \leqq 2r$. Moreover from $\dim._C (C_s(r))^\perp = n - s$, we have $\dim. R^2_r(s) = 2(n-s) - 3$ provided that $s < r$ (if $s = r$, $R^2_r(r) = \emptyset$). Hence if $s \geqq 1$, both $R^1_r(s)$ and $R^2_r(s)$ are regular submanifolds of Q_r of codimension greater than or

equal to 3. Obviously $R_r^1(s)$ is closed in Q_r and $R_r^2(s)$ is closed in $Q_r \backslash R_r^1(s)$. Therefore the simply connectedness of $Q_r^*(s)$ follows from (1) and the next proposition.

PROPOSITION D (cf. [3; VII Proposition 9.6]). *Let M be a connected manifold, and let S be a closed submanifold of M with* $\dim. S \leqq \dim. M - 3$. *Then $M \backslash S$ is connected and $\pi_1(M)$ is isomorphic with $\pi_1(M \backslash S)$.*

Q.E.D.

Next we consider $G^0(r)$ and $G^*(r, s)$. We set $G_0'(r) = G^0(r) \cap G'(r)$. Since $Q_r = G^0(r)/G_0'(r)$ is simply connected, $G_0'(r)$ is connected.

PROPOSITION 6.7. *$G^0(r)$ satisfies the following;*

(1) *There exists an element τ_0 of $G^0(r)$ such that o is the only fixed point of τ_0 in Q_r.*

(2) *The center $Z(G^0(r))$ of $G^0(r)$ is reduced to the unit.*

(3) *The normalizer $N_{G^0(r)}(G_0'(r))$ of $G_0'(r)$ in $G^0(r)$ coincides with $G_0'(r)$.*

Proof. (1) Let κ be the projection of $C^{n+1} \backslash \{0\}$ onto $P^n(C)$. Let $\sigma \in U(\tilde{I}_r)$ and $p = \kappa(\zeta) \in Q_r$ (i.e. $(\zeta, \zeta) = 0$). Then for $\chi(\sigma) \in G^0(r)$ we have

$$\chi(\sigma)(p) = p \text{ if and only if } \sigma(\zeta) = \lambda\zeta \qquad \text{for some } \lambda \in C \backslash \{0\} .$$

Hence $\chi(\sigma)$ fixes a point $p = \kappa(\zeta)$ of Q_r if and only if ζ is an isotropic eigenvector of σ. Therefore finding an element of $G^0(r)$ having $o = \kappa(e_0)$ as the only fixed point in Q_r is equivalent to finding an element of $U(\tilde{I}_r)$ having $\langle e_0 \rangle_C$ as the only isotropic eigenline. Here we mean by an eigenline of σ a 1-dimensional subspace invariant by σ. Using the Witt's theorem one can easily construct such an element $\sigma \in U(\tilde{I}_r)$.

(2) Let $\tau \in Z(G^0(r))$ and let τ_0 be as in (1). From $\tau_0 \cdot \tau = \tau \cdot \tau_0$ we have $\tau_0(\tau(o)) = \tau(\tau_0(o)) = \tau(o)$. Hence $\tau(o)$ is a fixed point of τ_0. But τ_0 fixes o alone. Therefore $\tau(o) = o$. Since $G^0(r)$ acts transitively on Q_r, we see easily τ fixes every point of Q_r. Then since $G^0(r)$ acts effectively on Q_r, τ is the unit of $G^0(r)$.

(3) Let $\tau \in G^0(r)$. Since $G_0'(r)$ is the isotropy subgroup of $G^0(r)$ at $o \in Q_r$, $\tau(G_0'(r))\tau^{-1}$ is the isotropy subgroup of $G^0(r)$ at $\tau(o)$. Hence each element of $\tau(G_0'(r))\tau^{-1}$ fixes $\tau(o)$. Now let $\tau_1 \in N_{G^0(r)}(G_0'(r))$, and let τ_0 be as in (1). Since $\tau_1(G_0'(r))\tau_1^{-1} = G_0'(r)$, each element of $G_0'(r)$ fixes $\tau_1(o)$. In particular $G_0'(r) \ni \tau_0$ fixes $\tau_1(o)$. Hence we have $\tau_1(o) = o$, that is, $\tau_1 \in G_0'(r)$. Therefore we get $N_{G^0(r)}(G_0'(r)) \subset G_0'(r)$. The opposite inclusion is obvious.

Q.E.D.

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Let $G_o^*(r, s)$ be the isotropy subgroup of $G^*(r, s)$ at $o \in Q_r^*(s)$. Since $Q_r^*(s) = G^*(r, s)/G_o^*(r, s)$ is simply connected, $G_o^*(r, s)$ is connected.

PROPOSITION 6.8. *$G^*(r, s)$ $(0 \leqq s \leqq r)$ satisfies the following*

(1) *There exists an element τ_0^s of $G^*(r, s)$ such that o is the only fixed point of τ_0^s in $Q_r^*(s)$.*

(2) *The center $Z(G^*(r, s))$ of $G^*(r, s)$ is reduced to the unit.*

(3) *The normalizer $N_{G^*(r,s)}(G_o^*(r, s))$ of $G_o^*(r, s)$ in $G^*(r, s)$ coincides with $G_o^*(r, s)$.*

Proof. (1) Since $Q_r^*(s) = \{\zeta = (\zeta_0, \cdots, \zeta_n) \in Q_r \mid \zeta \notin (C_s(r))^{\perp}\}$ and $\hat{G}^*(r, s) = \{\sigma \in U(\tilde{I}_r) \mid \sigma(C_s(r)) = C_s(r)\}$, we have only to find an element σ_0^s of $U(\tilde{I}_r)$ which satisfies

(i) $\sigma_0^s(C_s(r)) = C_s(r)$

(ii) $\langle e_0 \rangle_C$ is the only isotropic eigenline of σ_0^s that is not included in $(C_s(r))^{\perp}$.

(cf. the proof of (1) Proposition 6.7). Using the Witt's theorem one can easily construct such an element $\sigma_0^s \in U(\tilde{I}_r)$.

Since $G^*(r, s)$ acts effectively and transitively on $Q_r^*(s)$, in view of (1), (2) and (3) can be proved similarly as in Proposition 6.7. Q.E.D.

VII. Determination of $(A(S), A_{p_0}(S))$.

In this section let $\mathfrak{g}$ be $\mathfrak{g}(r)$ or $\mathfrak{g}^*(r, s)$ $(s = 0, 1, \cdots, r)$. Let G be the analytic subgroup of $G(r)$ with Lie algebra $\mathfrak{g}$, and let Q be the model space corresponding to $\mathfrak{g}$ which is defined in VI. Moreover let G' be the isotropy subgroup of G at $o \in Q$, and let $\mathfrak{g}'$ be its Lie algebra. Hence in the case $\mathfrak{g} = \mathfrak{g}(r)$ (resp. $\mathfrak{g}^*(r, s)$), we have $G = G^0(r)$ (resp. $G^*(r, s)$), $Q = Q_r$ (resp. $Q_r^*(s)$) and $G' = G_0'(r)$ (resp. $G_o^*(r, s)$). From Propositions 6.6, 6.7 and 6.8 we have

(1) *$Q = G/G'$ is connected and simply connected.*

(2) *The center $Z(G)$ of G is reduced to the unit.*

(3) *The normalizer $N_G(G')$ of G' in G coincides with G'.*

(4) *$\mathfrak{g}'$ contains $E_0 \in \mathfrak{g}(r)$ which defines the grading of $\mathfrak{g}(r)$.*

As we see in VI, Q is a connected non-degenerate (index r) homogeneous flat hypersurface of $P^n(C)$ for which G is the identity component of $A(Q)$.

Now we have

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PROPOSITION 7.1. *Let* $\mathfrak{g}, \mathfrak{g}', Q, G$ *and* G' *be as above. Let* S *be a connected non-degenerate (index* r*) homogeneous hypersurface, and let* $(P, \omega, \bar{l})$ *be the normal pseudo-conformal connection over* S. *For* $p_0 \in S$ *we suppose that there exists a point* $z_1 \in \pi^{-1}(p_0)$ *such that* $\tilde{\mathfrak{h}}_{z_1} = \mathfrak{g}$. *Then* S *is pseudo-conformally equivalent to* Q.

Proof. Since $\mathfrak{g}'$ contains E_0, we see from Lemma 5.5, Proposition 5.6 and Proposition 3.4 that there exists a point $z_0 \in \pi^{-1}(p_0)$ such that $\iota_{z_0}^*\omega$ is a Lie algebra isomorphism of $\mathfrak{a}(S)$ onto $\mathfrak{g}$. In particular we have $\iota_{z_0}^*\omega(\mathfrak{a}_{p_0}(S)) = \mathfrak{g}'$. On the other hand, from Lemma 3.1 we have $(\rho_{z_0*})_e = \omega_{z_0}(\iota_{z_0*})_e$, that is, $\rho_{z_{0*}} = \iota_{z_0}^*\omega$ as a Lie algebra homomorphism. Let $(A_{p_0}(S))^0$ be the identity component of $A_{p_0}(S)$. Then ρ_{z_0} is a group isomorphism of $(A_{p_0}(S))^0$ onto G'.

Next we compare $A^0(S)$ with G. Since G is connected and $Z(G) = \{e\}$, the adjoint representation Ad_G of G is an isomorphism of G onto the adjoint group $\mathrm{Int}(\mathfrak{g})$. Hence the adjoint representation $\mathrm{ad}_{\mathfrak{g}}$ of $\mathfrak{g}$ is also faithful. On the other hand the adjoint representation $\mathrm{Ad}_{A^0(S)}$ of $A^0(S)$ is a homomorphism of $A^0(S)$ onto $\mathrm{Int}(\mathfrak{a}(S))$. Set $h = \iota_{z_0}^*\omega$. Then since h is a Lie algebra isomorphism of $\mathfrak{a}(S)$ onto $\mathfrak{g}, h$ naturally induces a group isomorphism $\tilde{h}$ of $\mathrm{Int}(\mathfrak{a}(S))$ onto $\mathrm{Int}(\mathfrak{g})$. More precisely we set $(\tilde{h}(\tau))(X) = h\cdot\tau\cdot h^{-1}(X)$ for $\tau \in \mathrm{Int}(\mathfrak{a}(S))$, $X \in \mathfrak{g}$. Then we have $\tilde{h}_*\cdot\mathrm{ad}_{\mathfrak{a}(S)} = \mathrm{ad}_{\mathfrak{g}}\cdot h$.

Now we set $\varphi = (\mathrm{Ad}_G)^{-1}\cdot\tilde{h}\cdot\mathrm{Ad}_{A^0(S)}$. Then φ is a homomorphism of $A^0(S)$ onto G such that $\varphi_* = h$. Moreover we consider a mapping ψ of $A^0(S)/\varphi^{-1}(G')$ onto Q which satisfies the following commutative diagram

$$\begin{array}{ccc} A^0(S) & \xrightarrow{\varphi} & G \\ \downarrow & & \downarrow \\ A^0(S)/\varphi^{-1}(G') & \xrightarrow{\psi} & Q = G/G'. \end{array}$$

Then ψ is a C^ω-homeomorphism of $A^0(S)/\varphi^{-1}(G')$ onto Q. Since $\varphi_* = h$, we have $\varphi_*(\mathfrak{a}_{p_0}(S)) = \mathfrak{g}'$. Hence the Lie algebra of $\varphi^{-1}(G')$ coincides with $\mathfrak{a}_{p_0}(S)$. On the other hand $\varphi^{-1}(G')$ is connected since Q (therefore $A^0(S)/\varphi^{-1}(G')$) is simply connected. Hence we have $\varphi^{-1}(G') = (A_{p_0}(S))^0$. From $N_G(G') = G'$ and the connectedness of G', we see that G' is the only Lie subgroup of G with Lie algebra $\mathfrak{g}'$. On the other hand $\varphi(A_{p_0}^0(S))$ is a Lie subgroup of G with Lie algebra $\varphi_*(\mathfrak{a}_{p_0}(S)) = \mathfrak{g}'$. Hence we have $\varphi(A_{p_0}^0(S)) = G'$. In particular $A_{p_0}^0(S) \subset \varphi^{-1}(G') = (A_{p_0}(S))^0$. Therefore we conclude $A^0_{p_0}(S) = (A_{p_0}(S))^0$, that is, $A^0_{p_0}(S)$ is connected. Moreover comparing the restriction of φ to $A^0_{p_0}(S)$ with ρ_{z_0}, we have $\varphi_* = \rho_{z_0*} = h$. Hence we get $\varphi|_{A^0_{p_0}(S)} = \rho_{z_0}$. In particular $\varphi|_{A^0_{p_0}(S)}$ is an isomorphism of $A^0_{p_0}(S)$ onto G'.

Now from $\varphi^{-1}(G') = A^0_{p_0}(S)$ and $S = A^0(S)/A^0_{p_0}(S)$, the above diagram can be rewritten as follows

$$\begin{array}{ccc} A^0(S) & \xrightarrow{\varphi} & G \\ \downarrow & & \downarrow \\ S & \xrightarrow{\psi} & Q . \end{array}$$

Since ψ is a C^ω-homeomorphism of S onto Q and the restriction of φ to $A^0_{p_0}(S)$ is an isomorphism of $A^0_{p_0}(S)$ onto G', φ becomes a bundle isomorphism of $A^0(S)$ $(S, A^0_{p_0}(S))$ onto $G(Q, G')$. Hence φ is a group isomorphism of $A^0(S)$ onto G.

Now we compare two (connected non-degenerate (index r) homogeneous) hypersurface S and Q. Let $(\pi_r^{-1}(Q), \omega_Q, \bar{l}_0)$ be the normal pseudo-conformal connection over Q (for the notations see Proposition 6.5). If we choose points $z_0 \in \pi^{-1}(p_0)$ and $e \in \pi_r^{-1}(o)$, then φ satisfies the assumption of Proposition 3.5 since $\varphi(A^0_{p_0}(S)) = G'$, $\varphi_* = \iota^*_{z_0}\omega$ (as Lie algebra isomorphisms) and $\iota^*_e\omega_Q$ is the Maurer-Cartan form of G. Therefore ψ is a pseudo-conformal homeomorphism of S onto Q. Q.E.D.

From Theorem 5.8 and the above proposition, we have the main theorem of this paper.

THEOREM 7.2. *Let M be a complex manifold of dimension n. Let S be a connected non-degenerate (index r) homogeneous hypersurface of M.*

(1) *If* $\dim. A(S) = n^2 + 2n$, *then S is pseudo-conformally equivalent to*

$$Q_r = \left\{(z_0, \cdots, z_n) \in P^n(C) \middle| -\sqrt{-1}z_0\bar{z}_n - \sum_{i=1}^{r} z_i\bar{z}_i + \sum_{i=r+1}^{n-1} z_i\bar{z}_i + \sqrt{-1}z_n\bar{z}_0 = 0\right\} .$$

(2) *If* $\dim. A(S) < n^2 + 2n$, *we have the following three cases.*

(i) *the case $n = 3$ and $r = 1$; We have* $\dim. A(S) \leqq n^2 + 2 = 11$. *The equality holds if and only if S is pseudo-conformally equivalent to*

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$$Q_1^*(1) = \{(z_0, \cdots, z_3) \in Q_1 \,|\, |z_0| + |z_1 - z_2| \neq 0\} .$$

(ii) *the case* $n = 5$ *and* $r = 2$; *We have* $\dim. A(S) \leqq n^2 + 1 = 26$. *The equality holds if and only if* S *is pseudo-conformally equivalent to*

$$Q_2^*(2) = \{(z_0, \cdots, z_5) \in Q_2 \,|\, |z_0| + |z_1 - z_4| + |z_2 - z_3| \neq 0\}$$

or

$$Q_2^* = \{(z_0, \cdots, z_5) \in Q_2 \,|\, z_0 \neq 0\} .$$

(iii) *otherwise*; *We have* $\dim. A(S) \leqq n^2 + 1$. *The equality holds if and only if* S *is pseudo-conformally equivalent to*

$$Q_r^* = \{(z_0, \cdots, z_n) \in Q_r \,|\, z_0 \neq 0\} .$$

In Theorem 7.2, if we specify the ambient space M, then the question arises whether a hypersurface S with $\dim. A(S) = n^2 + 2n$ (or $n^2 + 1$) exists in M, in other words, whether Q_r (or Q_r^*) can be pseudo-conformally imbedded in M or not. In general this is a very hard problem. Concerning with this we observe

COROLLARY 7.3. *Let* C^n *be the complex number space of dimension* n. *Let* S *be a connected non-degenerate (index* r*) homogeneous hypersurface of* C^n. *Then we have*

(1) *In the case* $r = 0$ *(i.e. in the case* S *is strongly pseudo-convex)* $A(S)$ *has the largest dimension* $n^2 + 2n$, *if and only if* S *is pseudo-conformally equivalent to the unit sphere* S^{2n-1}. *And* $A(S)$ *has the second largest dimension* $n^2 + 1$, *if and only if* S *is pseudo-conformally equivalent to the hyperconic*

$$Q_0^* = \left\{(z_1, \cdots, z_n) \in C^n \,\middle|\, \operatorname{Im} z_n = \frac{1}{2} \sum_{i=1}^{n-1} |z_i|^2 \right\} .$$

(2) *In the case* $1 \leqq r < \left[\dfrac{n-1}{2}\right]$

$A(S)$ *has the largest dimension* $n^2 + 1$, *if and only if* S *is pseudo-conformally equivalent to*

$$Q_r^* = \left\{(z_1, \cdots, z_n) \in C^n \,\middle|\, \operatorname{Im} z_n = \frac{1}{2}\left(-\sum_{i=1}^{r} |z_i|^2 + \sum_{i=r+1}^{n-1} |z_i|^2\right) \right\} .$$

(3) *In the case* $r = \left[\dfrac{n-1}{2}\right]$, *we have the following three cases.*

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(i) $n = 3$ *We have* dim. $A(S) \leqq n^2 + 2 = 11$.

(ii) $n = 5$ *We have* dim. $A(S) \leqq n^2 + 1 = 26$.

(iii) *otherwise*; $A(S)$ *has the largest dimension* $n^2 + 1$, *if and only if* S *is pseudo-conformally equivalent to* Q_r^*.

Before the proof, recall the following

PROPOSITION E (cf. [5; VII Proposition 4.6], [6; Corollary to Theorem 5]). *Let* S *be a compact hypersurface of* $\boldsymbol{C}^n$. *Then there exists a point* p_0 *of* S *such that* S *is strongly pseudo-convex at* p_0.

Proof of Corollary 7.3. If dim. $A(S) = n^2 + 2n$, S is pseudo-conformally equivalent to Q_r from Theorem 7.2. Hence S is compact. Then r must be zero as the above proposition shows. In other words, if $r \geqq 1$, Q_r cannot be realized as a hypersurface of $\boldsymbol{C}^n$. On the other hand from the proof of Proposition 6.6, we know that Q_0 is projectively equivalent to S^{2n-1}. Other assertions of the corollary is obvious from Theorem 7.2. Q.E.D.

We don't know whether $Q_1^*(1)$ (resp. $Q_2^*(2)$) can be pseudo-conformally imbedded into $\boldsymbol{C}^3$ (resp. $\boldsymbol{C}^5$).

Finally we will see that in the case dim. $A(S) = n^2 + 2n$, the homogeneity assumption is dispensable. In fact we have

THEOREM 7.4. *Let* M *be a complex manifold of dimension* n. *Let* S *be a connected hypersurface of* M *which is non-degenerate of index* r *at a point* $p_0 \in S$. *If* dim. $A(S) = n^2 + 2n$, *then* S *is pseudo-conformally equivalent to* Q_r.

Proof. We denote by $\mathfrak{a}(S)$ the Lie algebra of all infinitesimal pseudo-conformal transformations of S which generate global 1-parameter groups of transformations. Then $\mathfrak{a}(S)$ is naturally isomorphic with the Lie algebra of $A(S)$. Let S^* be the set of points of S at which S is non-degenerate of index r. Obviously S^* is an open subset of S containing p_0. Hence S^* is a non-degenerate (index r) hypersurface. Let (P^*, ω^*, l^*) be the normal pseudo-conformal connection over S^*. We consider the Lie algebra $\tilde{\mathfrak{a}}(S^*)$ of all infinitesimal pseudo-conformal transformations of S^*. Since S^* is an open subset of S and each element of $\mathfrak{a}(S)$ is a real analytic vector field on S, the restriction map *res* of $\mathfrak{a}(S)$ into $\tilde{\mathfrak{a}}(S^*)$ is an injective homomorphism. Set $\tilde{\mathfrak{a}}(P^*) = \{X \in \mathfrak{X}(P^*) \mid L_X\omega^* = 0,\ R_{a*}X = X\ a \in G'(r)\}$. Since $(\pi^*)_*$ is an isomorphism of $\tilde{\mathfrak{a}}(P^*)$ onto $\tilde{\mathfrak{a}}(S^*)$, we have dim. $\tilde{\mathfrak{a}}(S^*) \leqq n^2 + 2n$. On the other hand from the assumption we have dim. $\mathfrak{a}(S) = n^2 + 2n$. Hence *res* is an isomorphism of $\mathfrak{a}(S)$ onto $\tilde{\mathfrak{a}}(S^*)$. In particular *res* maps the isotropy subalgebra $\mathfrak{a}_{p_0}(S)$ of $\mathfrak{a}(S)$ at p_0 onto the isotropy subalgebra $\tilde{\mathfrak{a}}_{p_0}(S^*)$ of $\tilde{\mathfrak{a}}(S^*)$ at p_0. Then from dim. $\tilde{\mathfrak{a}}_{p_0}(S^*) = n^2 + 1$, we have dim. $\mathfrak{a}_{p_0}(S) = n^2 + 1$.

Now we consider the orbit S^{**} of $A^0(S)$ passing through p_0. Then as is easily seen from dim. $\mathfrak{a}(S) = n^2 + 2n$ and dim. $\mathfrak{a}_{p_0}(S) = n^2 + 1$, $S^{**} = A^0(S)/A^0_{p_0}(S)$ is an open submanifold of S. Hence S^{**} is a connected non-degenerate (index r) homogeneous hypersurface. Moreover we have dim. $A(S^{**}) = n^2 + 2n$. In fact we have only to show that $A^0(S)$ acts effectively on S^{**}, which is clear since S^{**} is an open subset of S and pseudo-conformal transformations of S are C^ω-homeomorphisms of S. Therefore from Theorem 7.2 S^{**} is pseudo-conformally equivalent to Q_r. In particular S^{**} is compact. On the other hand S^{**} is an open subset of a connected hypersurface S. Hence we must have $S = S^{**}$. Therefore S is pseudo-conformally equivalent to Q_r. Q.E.D.

COROLLARY 7.5. *Let S be a compact connected hypersurface of $\boldsymbol{C}^n$. If* dim. $A(S) = n^2 + 2n$, *then S is pseudo-conformally equivalent to the unit sphere S^{2n-1}.*

This is clear from the above theorem and Proposition E.

Remark 7.6. In the case of second largest dimension ($r \geqq 1$), the homogeneity assumption is indispensable. In fact $Q_r \backslash \{\tilde{o}\} = Q_r^* \cup R_r^2(0)$ ($r \geqq 1$) is a connected (inhomogeneous) hypersurface of $P^n(\boldsymbol{C})$ for which $G^*(r)$ is the identity component of $A(Q_r \backslash \{\tilde{o}\})$. We will treat the inhomogeneous second largest dimension case in a forthcoming paper.

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